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Partial differential equations

p -Laplacian Keller–Segel equation: Fair competition and diffusion-dominated cases

Équation d'agrégation et diffusion avec un p -Laplacien : cas de la compétition équitable et de la diffusion dominante

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ABSTRACT

This work deals with the aggregation diffusion equation

$$\partial_t \rho = \Delta_p \rho + \lambda \operatorname{div}((K_a * \rho) \rho),$$

where $K_a(x) = \frac{x}{|x|^a}$ is an attraction kernel and Δ_p is the so called p -Laplacian. We show that the domain $a < p(d+1) - 2d$ is subcritical with respect to the competition between the aggregation and diffusion by proving the existence of a solution unconditionally with respect to the mass. In the critical case, we show the existence of a solution in a small mass regime for an $L \ln L$ initial condition.

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R É S U M É

Ce travail concerne l'étude d'une famille d'équations d'agrégation diffusion

$$\partial_t \rho = \Delta_p \rho + \lambda \operatorname{div}((K_a * \rho) \rho),$$

où $K_a(x) = \frac{x}{|x|^a}$ est un champ d'attraction et Δ_p est le p -Laplacien. On démontre que le domaine $a < p(d+1) - 2d$ est sous-critique du point de vue de la compétition entre l'agrégation et la diffusion en montrant l'existence d'une solution, quelle que soit la masse. Dans le cas critique, on montre l'existence d'une solution dans un régime de petite masse pour une condition $L \ln L$.

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Version française abrégée

On entend par équation d'agrégation–diffusion une equation aux dérivées partielles non linéaire sur \mathbb{R}^d de la forme

$$\partial_t \rho = \mathcal{D}(\rho) + \lambda \operatorname{div}((K_a * \rho)\rho),$$

où, pour $a \in (0, d)$, $K_a(x) = \frac{x}{|x|^a}$ est un noyau d'attraction, $\lambda > 0$ indique l'intensité de cette interaction et \mathcal{D} est un opérateur de diffusion. Cette équation décrit, par exemple, l'évolution de la densité d'une population de bactéries ou d'astres en gravitation (voir par exemple [6]).

Ce modèle a été largement étudié dans le cas de l'opérateur de diffusion non linéaire $\mathcal{D}(\rho) = \Delta(\rho^m)$ pour $m > 0$ (voir [4]). Grâce à la structure algébrique conférée par ce choix de diffusion, on montre que l'EDP est en fait un flot de gradient pour la distance de Wasserstein d'ordre 2 d'une certaine fonctionnelle. De l'étude de cette fonctionnelle découle le fait que la ligne $a = 2 - d(m - 1)$ (dans le plan (m, a)) est critique du point de vue de la compétition entre l'agrégation et la diffusion. Le demi-plan situé au-dessus de cette droite correspond au régime d'agrégation dominante, et celui au-dessous à celui de diffusion-dominante.

Lorsque la diffusion est fractionnaire, i.e. lorsque $\mathcal{D} = \Delta^{\alpha/2}$ est le Laplacien fractionnaire d'exposant $\alpha \in (0, 2)$, on montre que la ligne critique est la première bissectrice $a = \alpha$ (voir [13,8]) et qu'elle délimite dans ce cas également deux régimes opposés.

Cette note poursuit cette étude, dans le cas du p -Laplacien $\mathcal{D}(\rho) = \Delta_p(\rho) = \operatorname{div}(|\nabla \rho|^{p-2} \nabla \rho)$ pour $p \in (\frac{2d}{d+1}, \frac{3d}{d+1})$. On montre ici que le domaine $a < p(d + 1) - 2d$ est sous-critique et qu'il y a existence pour une petite masse dans le cas d'égalité. Au passage, on établit une estimation de moments pour la p -équation de la chaleur.

1. Introduction

Aggregation diffusion equations play an important role in the modeling of collective behaviors and, more specially, in the case of the motion of cells and bacteria (see for instance [6]). The (parabolic-elliptic) Keller–Segel equation, which has been extensively studied (see [3]), is a typical example. In generality, we mean by aggregation equation the class of the mean-field nonlinear conservation equation of the form

$$\partial_t \rho = \mathcal{D}(\rho) + \lambda \operatorname{div}((K_a * \rho)\rho), \tag{1}$$

where K_a is an aggregation kernel defined by $K_a(x) = \frac{x}{|x|^a}$, $\lambda > 0$ is a parameter encoding the intensity of the aggregation and \mathcal{D} is some diffusion operator. Equation (1) can then be interpreted as the evolution of the probability density of particles attracting each other through K_a and diffusing through \mathcal{D} . Then, depending on the result of the competition between these two phenomena, the equation may yield to global existence or finite time blow-up.

The case of the power-law diffusion $\mathcal{D}(\rho) = \Delta(\rho^m)$ for some $m > 0$ has been studied in [4], where the line $a = 2 - d(m - 1)$ is shown to be critical. In that case, Eq. (1) can be seen as the gradient flow of some suitable functional with respect to the Wasserstein-2 distance and the criticality appears from the asymptotic study of this functional.

The case of fractional diffusion $\mathcal{D}(\rho) = \Delta^{\alpha/2} \rho$ for some $\alpha \in (0, 2)$ has been studied in [13,8], where it was shown that the critical line is the first bisector $\alpha = a$, above which blow-up of solutions may occur in finite time, and under which global well-posedness and propagation of chaos hold.

In order to complete this study, this note investigates the case where the diffusion operator is the p -Laplacian $\mathcal{D} = \Delta_p$ (see, e.g., [10]), which is defined for any $\rho \in W_{loc}^{1,p-1}$ by

$$\forall \varphi \in C_c^\infty(\mathbb{R}^d), \langle \Delta_p \rho, \varphi \rangle = - \int_{\mathbb{R}^d} |\nabla \rho|^{p-2} \nabla \rho \cdot \nabla \varphi,$$

and appears, for example, in the diffusion equations for sand piles (see, e.g., [1,5]).

2. Main results

The aggregation equation (1) with $\mathcal{D} = \Delta_p$,

$$\partial_t \rho = \operatorname{div}(|\nabla \rho|^{p-2} \nabla \rho) + \lambda \operatorname{div}((K_a * \rho)\rho), \tag{2}$$

has not been much studied, to the best of the author's knowledge. The only reference at this matter is [11], which concerns the case $a = d$ and $p \in (2, \frac{3d}{d+1})$ (Fig. 1).

Denoting $\|\rho\|_{L^p_k} := \|\rho m\|_{L^p}$ with $m(x) = \langle x \rangle^k$ and $L \ln L = \{\rho \geq 0, \rho \in L^1, \rho \ln \rho \in L^1\}$, we state the main result of this note.

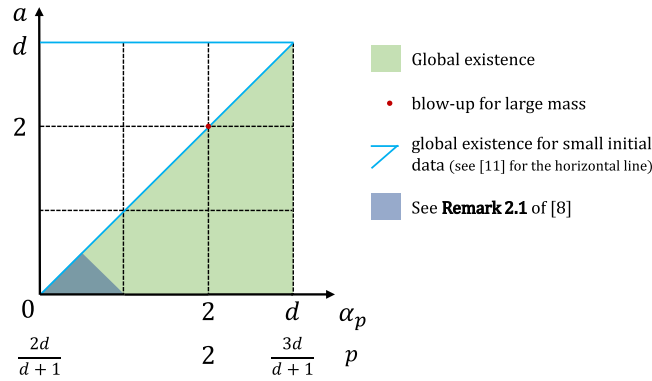


Fig. 1. Graph of results about Eq. (2). The case $a + \alpha_p < 1$ has to be a priori excluded, since moments of order at least $1 - a$ on ρ are needed for the term $(K_a * \rho)$ to make sense, and Δ_p propagates at most moments of order α_p (see Lemma 3.1). We refer to Remark 2.1 of [8].

Theorem 2.1. Let $d \geq 2$, $\lambda > 0$ and $(a, p) \in (0, d) \times (\frac{2d}{d+1}, \frac{3d}{d+1})$. Denote $\alpha_p := p(d + 1) - 2d$ and assume $\alpha_p + a > 1$. Let $\rho^{\text{in}} \in L \ln L \cap L^1_k$ for some $k \in ((1 - a)_+, \alpha_p \wedge 1)$. Then in

- **the diffusion-dominated case**, $a < \alpha_p$,
- **the fair competition case**, $a = \alpha_p$, if ρ^{in} satisfies

$$M_0 := \|\rho^{\text{in}}\|_{L^1} < C_{d,p} \lambda^{-\frac{1}{3-p}},$$

there exists a solution $\rho \in L^{p/p'}_{\text{loc}}(\mathbb{R}_+, L^{p^*/p'}) \cap L^\infty_{\text{loc}}(\mathbb{R}_+, L^1_k)$ to Eq. (2) with initial condition ρ^{in} .

Remark 1. The constant $C_{d,p}$ is given by

$$C_{d,p} = \left((d - \alpha_p) C_{d,\alpha_p,\frac{2d}{2d-\alpha_p}}^{\text{HLS}} \left(p'^{-1} C_{d,p}^S \right)^p \right)^{-\frac{1}{3-p}},$$

where for $a \in (0, d)$ and $2 - \frac{a}{d} = \frac{2}{q}$, $C_{d,a,q}^{\text{HLS}}$ is the best constant for Hardy–Littlewood–Sobolev’s inequality,

$$\iint_{\mathbb{R}^{2d}} |x - y|^{-a} \rho(x) \rho(y) \, dx \, dy \leq C_{d,a,q}^{\text{HLS}} \|\rho\|_{L^q}^2,$$

and for $q \in (0, d)$, and $q^* = dq/(d - q)$, $C_{d,q}^S$ is the best constant for Sobolev’s embeddings,

$$\|\rho\|_{L^{q^*}} \leq C_{d,q}^S \|\nabla \rho\|_{L^q}.$$

The explicit value for these constants are known (see [2,14,9]).

Note that, for $d = 2$, the point $(a, m) = (2, 1)$ in the context of power-law diffusion, $(a, \alpha) = (2, 2)$ in the notations of fractional diffusion and $(a, p) = (2, 2)$ in the notations of the present paper all correspond to the classical Keller–Segel equation, and the three different definitions of the fair competition case coincide for this equation.

3. Proof of Theorem 2.1

We begin this section by introducing, for $p > 1$, the p -Fisher information I_p on $(W^{1,p})^{p'} := \{\rho, \rho^{1/p'} \in W^{1,p}\}$ as

$$I_p(\rho) = \int_{\mathbb{R}^d} \frac{|\nabla \rho|^p}{\rho} = (p')^p \|\nabla(\rho^{1/p'})\|_{L^p}^p,$$

which is a generalization of the classical Fisher information (i.e. the case $p = 2$). First remark that a straightforward computation using Hölder’s and Sobolev’s inequalities shows that $(W^{1,p})^{p'} \subset W_{\text{loc}}^{1,p-1}$ so that $\Delta_p \rho$ is well-defined for ρ with finite p -Fisher information. Then for any $p \in (\frac{2d}{d+1}, \frac{3d}{d+1})$, $q \in [1, r]$ with $r = \frac{p^*}{p}$ and $\rho \in (W^{1,p})^{p'} \cap L^1$ it holds

$$\|\rho\|_{L^q} \leq ((p')^{-1} C_{d,p}^S)^{\frac{r'}{q}} \|\rho\|_{L^1}^{1-\frac{r'}{q}} I_p(\rho)^{\frac{r'}{q p}}. \tag{3}$$

Indeed by Sobolev’s embeddings, it holds

$$\|\rho\|_{L^{p'}}^{p/p'} = \|\rho^{1/p'}\|_{L^{p^*}}^p \leq (C_{d,p}^S)^p \|\nabla(\rho^{1/p'})\|_{L^p}^p \leq ((p')^{-1} C_{d,p}^S)^p I_p(\rho),$$

and using interpolation inequality between L^1 and L^r yields the result. Then we need some tools in order to provide some moment estimate.

Lemma 3.1. *There is $C > 0$ such that for any $\rho \in L^1 \cap (W^{1,p})^{p'}$ and $k > 0$, it holds*

$$\int_{\mathbb{R}^d} (\Delta_p \rho) m \leq C \begin{cases} \|\rho\|_{L^1}^{\frac{p-1}{\alpha_p}} I_p(\rho)^{\frac{\alpha_p-1}{\alpha_p}}, & \text{if } p \geq 2 \text{ and } k \in [0, 1], \\ (\int_{\mathbb{R}^d} \rho m)^{\frac{1}{p'}} I_p(\rho)^{\frac{1}{p'}}, & \text{if } p \in (\frac{2d}{d+1}, 2) \text{ and } k \in (0, \alpha_p). \end{cases} \tag{4}$$

Proof. First in the case $p \geq 2$ and $k \in [0, 1]$, since $k \leq 1$, by Hölder’s inequality, it holds

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla \rho|^{p-2} \nabla \rho \cdot \nabla m &\leq k \int_{\mathbb{R}^d} \rho^{\frac{1}{p'}} \rho^{-\frac{1}{p'}} |\nabla \rho|^{p-1} \langle x \rangle^{k-1} \\ &\leq k \left(\int_{\mathbb{R}^d} \rho^{\frac{p}{p'}} \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^d} \frac{|\nabla \rho|^p}{\rho} \right)^{\frac{1}{p'}} = k \left(\|\rho\|_{L^{\frac{p}{p'}}} I_p(\rho) \right)^{\frac{1}{p'}}. \end{aligned}$$

Then, using inequality (3), we obtain

$$\|\rho\|_{L^{\frac{p}{p'}}} \leq C \|\rho\|_{L^1}^{\frac{1}{p'}(1-\frac{r'(p-2)}{(p-1)})} I_p(\rho)^{\frac{r'p'}{(p-1)p}},$$

and the result follows, since

$$\left(\frac{r'p'}{(p-1)p} + 1 \right) \frac{1}{p'} = \frac{\alpha_p - 1}{\alpha_p} \text{ and } \frac{1}{p'} \left(1 - \frac{r'(p-2)}{(p-1)} \right) = \frac{p-1}{\alpha_p}.$$

Then in the case $p \in (\frac{2d}{d+1}, 2)$ and $k \in (0, \alpha_p)$, by Hölder’s inequality, since $p \leq 2$,

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla \rho|^{p-2} \nabla \rho \cdot \nabla m &\leq k \int_{\mathbb{R}^d} |\nabla \rho|^{p-1} \langle x \rangle^{k-1} \\ &\leq k \int_{\mathbb{R}^d} \rho^{\frac{1}{p'}} \langle x \rangle^{\frac{k}{p'}} \rho^{-\frac{1}{p'}} |\nabla \rho|^{p-1} \langle x \rangle^{\frac{k}{p}-1} \\ &\leq k \left(\int_{\mathbb{R}^d} \rho m \right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^d} \frac{|\nabla \rho|^p}{\rho} \right)^{\frac{1}{p'}} \left(\int_{\mathbb{R}^d} \langle x \rangle^{\frac{k-p}{2-p}} \right)^{\frac{2}{p}-1}, \end{aligned}$$

and the result follows since, by assumption, $\frac{k-p}{2-p} < -d$. \square

Proof of Theorem 2.1. We only provide the a priori estimate necessary to the rigorous proof. Following the claim of [11, Proof of Theorem 5.2, Step 1], we can retrieve well-posedness for the regularized problem (1) with K_a replaced with $K_a^\varepsilon(x) = \mathbb{1}_{|x| \geq \varepsilon} K(x) + \mathbb{1}_{|x| \leq \varepsilon} \varepsilon^{-a} x$. The preservation of positivity is a consequence of Kato’s inequality for the p -Laplacian (see [7,12]). Then letting ε go to 0 and using the a priori estimate which we are about to prove, together with a standard compactness argument (similarly as what is done in [3, Section 2.5]) provides the rigorous proof.

Step 1. Entropy dissipation. We first estimate the dissipation of entropy using together Hardy–Littlewood–Sobolev’s inequality and (3) with $q = \frac{2d}{2d-a}$ as

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} \rho \log \rho &= - \int_{\mathbb{R}^d} (|\nabla \rho|^{p-2} \nabla \rho) \cdot \nabla \log \rho + \lambda \int_{\mathbb{R}^d} \operatorname{div}((K_a * \rho)\rho) (\log \rho + 1) \\ &= -I_p(\rho) + \lambda \int_{\mathbb{R}^d} (\operatorname{div}(K_a) * \rho)\rho \end{aligned}$$

$$\begin{aligned}
 &= -I_p(\rho) + \lambda(d-a) \iint_{\mathbb{R}^{2d}} \frac{\rho(x)\rho(y)}{|x-y|^a} dx dy \\
 &\leq -I_p(\rho) + \lambda(d-a) C_{d,a,q}^{\text{HLS}} \left(p'-1 C_{d,p}^S \right)^{2\frac{r'p'}{q}} M_0^{2-2\frac{r'}{q}} I_p(\rho)^{2\frac{r'p'}{q}}.
 \end{aligned}$$

And since $2\frac{r'p'}{q} = \frac{a}{\alpha_p}$, $2 - 2\frac{r'}{q} = 2\left(1 - (p-1)\frac{a}{2\alpha_p}\right)$ and $2\frac{r'p'}{q} = p\frac{a}{\alpha_p}$, defining

$$C_{d,p}^{p-3} := (d - \alpha_p) C_{d,\alpha_p,\frac{2d}{2d-\alpha_p}}^{\text{HLS}} \left(p'-1 C_{d,p}^S \right)^p,$$

we conclude this step with

$$\frac{d}{dt} \int_{\mathbb{R}^d} \rho \log \rho \leq \begin{cases} -\left(1 - \lambda C_{d,p}^{p-3} M_0^{3-p}\right) I_p(\rho), & \text{if } a = \alpha_p \\ -\frac{1}{2} I_p(\rho) + C & \text{if } a < \alpha_p. \end{cases} \tag{5}$$

Step 2. Moment estimate. First, in the case $p \geq 2$, we choose $k \in ([1-a]_+, 1)$ and use Lemma 3.1, symmetry and Young’s inequality for any $\varepsilon > 0$ to obtain

$$\begin{aligned}
 \frac{d}{dt} \int_{\mathbb{R}^d} \rho m &\leq C M_0^{\frac{p-1}{\alpha_p}} I_p(\rho)^{\frac{\alpha_p-1}{\alpha_p}} - \frac{\lambda}{2} \iint_{\mathbb{R}^{2d}} K_a(x-y) \cdot (\nabla m(x) - \nabla m(y)) \rho(dx) \rho(dy) \\
 &\leq C_\varepsilon M_0^{\frac{p-1}{\alpha_p} \left(\frac{\alpha_p}{\alpha_p-1}\right)'} + \varepsilon I_p(\rho) - \frac{\lambda}{2} \iint_{\mathbb{R}^{2d}} K_a(x-y) \cdot (\nabla m(x) - \nabla m(y)) \rho(dx) \rho(dy).
 \end{aligned}$$

Then, in the case $p \in \left(\frac{2d}{d+1}, 2\right)$, we choose $k \in ([1-a]_+, \alpha_p)$, use Lemma 3.1 and obtain

$$\begin{aligned}
 \frac{d}{dt} \int_{\mathbb{R}^d} \rho m &\leq C \left(\int_{\mathbb{R}^d} \rho m \right)^{\frac{1}{p'}} I_p(\rho)^{\frac{1}{p'}} - \frac{\lambda}{2} \iint_{\mathbb{R}^{2d}} K_a(x-y) \cdot (\nabla m(x) - \nabla m(y)) \rho(dx) \rho(dy) \\
 &\leq C_\varepsilon \left(\int_{\mathbb{R}^d} \rho m \right)^{\frac{p}{p'}} + \varepsilon I_p(\rho) - \frac{\lambda}{2} \iint_{\mathbb{R}^{2d}} K_a(x-y) \cdot (\nabla m(x) - \nabla m(y)) \rho(dx) \rho(dy).
 \end{aligned}$$

The last term in the r.h.s. is dealt similarly as in [8, Proof of Proposition 3.1] and in any case, we end up with

$$\frac{d}{dt} \int_{\mathbb{R}^d} \rho m \leq \varepsilon I_p(\rho) + C \left(1 + \int_{\mathbb{R}^d} \rho m \right), \tag{6}$$

where C only depends on $d, p, a, \lambda, \varepsilon, k$ and M_0 .

Step 3. Conclusion. We will now only treat the case $a = \alpha_p$ and $\lambda M_0^{3-p} C_{d,p}^{p-3} < 1$, since the case $a < \alpha_p$ can be treated even more straightforwardly. For $k \geq 0$ denote $\nu_k > 0$ such that $\int_{\mathbb{R}^d} e^{-\nu_k m(x)} dx = 1$, and recall that, with $h(u) = u \ln u - u + 1 \geq 0$, it holds

$$\int_{\mathbb{R}^d} \frac{\rho}{M_0} \ln \frac{\rho}{M_0} = \int_{\mathbb{R}^d} h\left(\frac{\rho}{M_0} e^{\nu_k m}\right) e^{-\nu_k m} + \int_{\mathbb{R}^d} \frac{\rho}{M_0} \ln(e^{-\nu_k m}) \geq -\nu_k \int_{\mathbb{R}^d} \frac{\rho}{M_0} m,$$

and then

$$\int_{\mathbb{R}^d} \rho \ln \rho \geq M_0 \ln M_0 - \nu_k \int_{\mathbb{R}^d} \rho m,$$

which yields, for a fixed $\nu > \nu_k$, combining linearly (5) and (6),

$$\begin{aligned}
 (v - v_k) \int_{\mathbb{R}^d} \rho m &\leq -M_0 \ln M_0 + \int_{\mathbb{R}^d} \rho \log \rho + v \int_{\mathbb{R}^d} \rho m \\
 &\leq \int_{\mathbb{R}^d} \rho^{\text{in}} \log \rho^{\text{in}} + v \int_{\mathbb{R}^d} \rho^{\text{in}} m + vC \int_0^t \left(\int_{\mathbb{R}^d} \rho(s) m + 1 \right) ds \\
 &\quad - \left(1 - \lambda C_{d,p}^{p-3} M_0^{3-p} - \varepsilon v \right) \int_0^t I_p(\rho)(s) ds.
 \end{aligned}$$

Therefore, for $\varepsilon > 0$ small enough, $\rho \in L_{\text{loc}}^\infty(\mathbb{R}_+, L_k^1)$ by Gronwall's inequality. We emphasize that this estimate also applies to the p -heat equation, i.e. (2) with $\lambda = 0$. Finally coming back to (5) yields

$$\begin{aligned}
 \left(1 - \lambda C_{d,p}^{p-3} M_0^{3-p} \right) \int_0^t I_p(\rho)(s) ds &\leq \int_{\mathbb{R}^d} \rho^{\text{in}} \log \rho^{\text{in}} - \int_{\mathbb{R}^d} \rho \log \rho \\
 &\leq \int_{\mathbb{R}^d} \rho^{\text{in}} \log \rho^{\text{in}} + v_k \int_{\mathbb{R}^d} \rho m - M_0 \ln M_0,
 \end{aligned}$$

and we conclude the proof using inequality (3). \square

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