



Algebraic geometry

Complex surfaces of general type with $K^2 = 3, 4$ and $p_g = q = 0$



Surfaces complexes de type général avec $K^2 = 3, 4$ et $p_g = q = 0$

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ABSTRACT

We construct complex surfaces of general type with $p_g = 0$ and $K^2 = 3, 4$ as double covers of Enriques surfaces (called Keum–Naie surfaces) with a different way to the original constructions of Keum and Naie. As a result, we show that there is a (-4) -curve on the example with $K^2 = 3$, which might imply a special relation between Keum–Naie surfaces with $K^2 = 3$ and $K^2 = 4$.

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R É S U M É

Nous construisons des surfaces complexes de type général avec $p_g = 0$ et $K^2 = 3, 4$ (appelées surfaces de Keum–Naie), comme revêtements doubles de surfaces d'Enriques. Notre construction diffère de celle utilisée originellement par Keum–Naie. Comme application, nous montrons qu'il existe une (-4) -courbe sur une telle surface avec $K^2 = 3$, ce qui suggère l'existence d'une relation particulière entre les surfaces de Keum–Naie satisfaisant $K^2 = 3$ et $K^2 = 4$.

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1. Introduction

In this paper, we construct minimal complex surfaces of general type with $p_g = 0$ and $K^2 = 3, 4$, so-called *Keum–Naie surfaces*, which may be in a special relationship to each other.

Keum [3] and Naie [5] construct minimal complex surfaces of general type with $K^2 = 4$ and $p_g = 0$ as double covers of a Enriques surface with eight disjoint (-2) -curves branched along certain special divisors. Such Enriques surface is constructed as a quotient of the product $E_1 \times E_2$ of two elliptic curves E_1 and E_2 by a certain group action of \mathbb{Z}_2^2 ; cf. Mendes Lopes–Pardini [4, Example 1] for details. By choosing a similar branch divisor (to the $K^2 = 4$ case) but with special property, they construct also complex surfaces with $K^2 = 3$.

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Following a similar strategy in Keum [3] and Naie [5] but constructing such Enriques surface in a different standard way, we construct again Keum–Naie surfaces with $K^2 = 3, 4$. Furthermore, we show that there is a special relation between Keum–Naie surfaces with $K^2 = 3$ and Keum–Naie surfaces with $K^2 = 4$.

Theorem 1.1 (Theorem 2.4, Theorem 3.1). *We construct complex surfaces of general type with $p_g = 0$ and $K^2 = 4$. We also construct complex surfaces of general type with $p_g = 0$ and $K^2 = 3$ on which there is a (-4) -curve.*

We briefly summarize how to construct such examples. We first construct an Enriques surface with disjoint eight nodal curves in a classical way. That is, we choose, on $\mathbb{P}^1 \times \mathbb{P}^1$, four distinct fibers F_1, \dots, F_4 of bidegree $(1, 0)$ and four distinct fibers G_1, \dots, G_4 of bidegree $(0, 1)$. Then the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along eight fibers $F_1 \cup \dots \cup F_4 \cup G_1 \cup \dots \cup G_4$ is a singular K3 surface with 16 A_1 -singularities. Resolving the singularities, we get a K3 surface Q . There are two involutions on the K3 surface Q : The first one, say $j_Q : Q \rightarrow Q$, is an involution given as a lifting of a specially chosen involution j of $\mathbb{P}^1 \times \mathbb{P}^1$. We will describe the involution j in the next section more carefully. The other involution $k : Q \rightarrow Q$ is the one coming from the double cover; that is, k reverses two sheets of the double cover. Then the quotient by $k \circ j_Q$, $Y = Q / (k \circ j_Q)$, is the Enriques surface with eight nodal curves A_1, \dots, A_8 .

Then the examples in the above theorem are given as double covers of the Enriques surface Y branched along special branch divisors B on Y . For this, we follow a similar method of Rito [6]. We choose a smooth curve D of bidegree $(1, 1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ satisfying certain conditions. Then we will show that the image of $D + j(D)$ on the Enriques surface Y is an irreducible reduced curve H with a certain singular point. If we take $B = A_1 \cup \dots \cup A_8 \cup H$, then we get a Keum–Naie surface with $K^2 = 4$. If we impose one more condition on D (which will be described in Section 3) and if we take $B = A_1 \cup \dots \cup A_7 \cup H$, then we get an example with $K^2 = 3$ containing a (-4) -curve.

It would be nice to know whether the singular surface obtained by contracting the (-4) -curve in the example with $K^2 = 3$ would have a smoothing. We leave it as a further research topic.

In Section 2, we construct a minimal complex surface of general type with $K^2 = 4$ and $p_g = q = 0$. In Section 3, we construct a minimal complex surface of general type with $K^2 = 3$ and $p_g = q = 0$.

2. Keum–Naie surfaces with $K^2 = 4$

First, let us introduce the classical construction of the Enriques surface Y with eight nodal curves from $\mathbb{P}^1 \times \mathbb{P}^1$.

Let $j : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be the involution defined by $[x_0 : x_1, y_0 : y_1] \mapsto [x_1 : x_0, y_1 : y_0]$. And let us choose four distinct fibers of the first and second projections, respectively. The union of these eight fibers is invariant under j and does not have any fixed points for j . Then we can construct a K3 surface Q as a minimal resolution of a double covering of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along the union of eight fibers above.

Since the branch locus is invariant under j , the involution $j : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ extends on Q , $j_Q : Q \rightarrow Q$. Let $k : Q \rightarrow Q$ be the fixed-point free involution that reverses two sheets of the covering space Q . Since the branch locus has no fixed points, we get the fixed-point free involution $k \circ j_Q$ again. Then the quotient of Q by $(k \circ j_Q)$ is the Enriques surface $Y = Q / (k \circ j_Q)$ with eight nodal curves.

From now on, let us write the union of eight fibers, the branch locus of $\mathbb{P}^1 \times \mathbb{P}^1$, as C for convenience.

Let $D = (f)$ be a curve on $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(1, 1)$ passing through the two points $a_1, a_2 = j(a_1)$ that are not on C . And let $j(D) = (g)$ be the image of D for j -involution. Since $D \cdot j(D) = 2$, we can find the defining polynomials f, g so that $D, j(D)$ meet transversally at the two points $a_1, a_2 = j(a_1)$.

Let $p : Q' \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be a double covering branched along C . And consider the divisor $D + j(D)$. Let (x, y) be the local coordinate of $\mathbb{P}^1 \times \mathbb{P}^1$ around a_i for each $i = 1, 2$. The local equation of $D + j(D)$ at a_i is of the type $(x - y)(x + y)$. And since p is étale outside $D + j(D)$, $p^*(D + j(D))$ has four singularities of type $(z^2 - w^2)$ at $p^{-1}(a_1), p^{-1}(a_2)$. By $(k \circ j)$ -action p^*D and $p^{-1}(a_1)$ are identified with $p^*(j(D))$ and $p^{-1}(a_2)$, respectively. Let $\rho : Q \rightarrow Q'$ be a minimal resolution of Q' and $q : Q \rightarrow Y$ be the quotient map by $(k \circ j_Q)$. Then the image of $(\rho \circ p)^*(D + j(D))$ on the Enriques surface Y becomes a reduced curve H having one ordinary double point of type $z^2 - w^2$, where $H \in |F + G|$ and $F = 2F^+ + (A_1 + A_2 + A_3 + A_4)$, $G = 2G^+ + (A_1 + A_2 + A_5 + A_6)$ are multiple fibers of Y (Fig. 2.1).

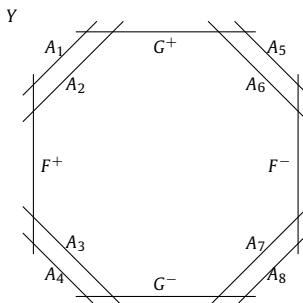


Fig. 2.1. An Enriques surface Y .

For instance, we can take $f(x_0, x_1, y_0, y_1) = 52x_0y_0 - 188x_0y_1 + 32x_1y_0 + 32x_1y_1$ and $g(x_0, x_1, y_0, y_1) = 52x_1y_1 - 188x_1y_0 + 32x_0y_1 + 32x_0y_0$ as polynomials defining D and $j(D)$, respectively. Then these polynomials satisfy the above conditions and determine a branch divisor on Y .

Remark 2.1. Another possibility is that the two curves, D and $j(D)$, meet on C , where C is the branch divisor of a double covering $p : Q' \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$. If we choose a curve D passing through two smooth points $a_1, a_2 = j(a_1)$ of C . Then $j(D)$ also passes two smooth points transversally. That is, the local equation of $D + j(D)$ at each a_i is also of type $(x - y)(x + y)$. But, since $p : (z, w) \mapsto (x, y) = (z^2, w)$ on the branch divisor, $p^*(D + j(D))$ has two singularities of type $(z^4 - w^2)$ at $p^{-1}(a_1), p^{-1}(a_2)$.

So the image of $p^*(D + j(D))$ on the Enriques surface Y becomes a reduced curve H having one tacnode of type $z^4 - w^2$, where $H \in |2F + 2G|$. However, this case also has the same result.

Before introducing the main theorem, we will refer to some lemma in order to calculate some invariants.

Lemma 2.2 ([1], (7.2) Theorem, p. 108). *Let $\pi : X \rightarrow Y$ be a double covering with X normal and Y nonsingular, ramified over the (reduced) curve $B \subset Y$. Let \mathcal{L} be the line bundle on Y , satisfying $\mathcal{L}^{\otimes 2} = \mathcal{O}_Y(B)$, which determines the covering. Consider the canonical resolution diagram where β is a sequence of blow ups and \bar{X} is nonsingular (Fig. 2.2):*

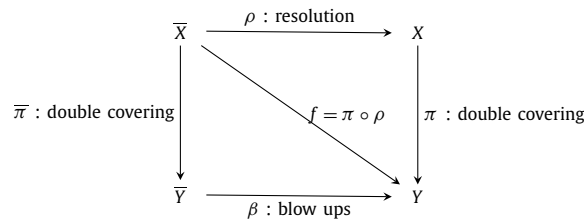


Fig. 2.2. A canonical resolution diagram for $\pi : X \rightarrow Y$.

Then there is a divisor $Z \geq 0$ on \bar{X} with $\text{supp}(Z)$ contained in the union of the exceptional curves for ρ such that

$$\mathcal{K}_{\bar{X}} = f^*(\mathcal{K}_Y \otimes \mathcal{L}) \otimes \mathcal{O}_{\bar{X}}(-Z).$$

Furthermore, $Z = 0$ if and only if the singularities of B (hence of X) are simple. That is, if B has at most simple singularities, then

$$\mathcal{K}_{\bar{X}} = f^*(\mathcal{K}_Y \otimes \mathcal{L}).$$

From Lemma 2.2, Leray spectral sequence, Riemann–Roch and Serre duality, we obtain the following equations.

Lemma 2.3 ([1], Chapter V.22, pp. 236–238). *All settings are the same as Lemma 2.2.*

(1) *If B has at most simple singularities:*

$$\begin{cases} \chi(\bar{X}) = 2\chi(Y) + \frac{1}{2}(\mathcal{L} \cdot \mathcal{K}_Y + \mathcal{L}^2) = 2\chi(Y) + \frac{1}{4}B \cdot K_Y + \frac{1}{8}B^2, \\ c_1^2(\bar{X}) = 2(c_1^2(Y) + B \cdot K_Y + \frac{1}{4}B^2), \\ p_g(\bar{X}) = p_g(Y) + h^0(Y, \mathcal{K}_Y \otimes \mathcal{L}). \end{cases}$$

(2) *If B has ordinary d -fold points, the result is divided into two cases where d is either even or odd.*

Without loss of generality, we may assume that the curve B has r singular points y_j of multiplicity d_j for each $j = 1, \dots, r$. Let $\beta : \bar{Y} \rightarrow Y$ be the blowing-up of Y at $y_1 \cup \dots \cup y_r$ and let \bar{B} be the proper transform of B . Then there exists a double covering $\bar{\pi} : \bar{X} \rightarrow \bar{Y}$ branched along a smooth 2-divisible divisor B_1 where $B_1 = \bar{B}$ if $d_j = 2m_j$ for each j and $B_1 = \bar{B} + \sum_{j=1}^r E_j$ if $d_j = 2m_j + 1$ ($E_j = \beta^{-1}(y_j)$) for each $j = 1, \dots, r$.

Then application of (1) yields:

$$\left\{ \begin{array}{l} \chi(\bar{X}) = 2\chi(Y) + \frac{1}{4}B \cdot K_Y + \frac{1}{8}B^2 - \sum_{j=1}^r \frac{1}{2}m_j(m_j - 1), \\ c_1^2(\bar{X}) = 2(c_1^2(Y) + B \cdot K_Y + \frac{1}{4}B^2) - 2 \sum_{j=1}^r (m_j - 1)^2, \\ \text{if } p_g(Y) = 0 : \\ p_g(\bar{X}) = \text{dimension of the subspace of } \Gamma(\mathcal{K}_Y \otimes \mathcal{L}) \text{ consisting of those sections vanishing} \\ \text{of order at least } m_j - 1 \text{ in } y_j \text{ for each } j = 1, \dots, r. \end{array} \right.$$

Theorem 2.4. Let X be a double covering of the Enriques surface Y branched along $B = \sum_{i=1}^8 A_i + H$ where A_i are eight nodal curves on Y . And $H \in |F + G|$ is the image (on Y) of $p^*(D + j(D))$ where F and G are multiple fibers of Y . Then the minimal resolution V of a double covering X is a minimal surface of general type with $K_V^2 = 4$ and $p_g = q = 0$.

Proof. Since a double covering and a blowing up commute, let us first resolve the singularity of H . Let $\beta : \bar{Y} \rightarrow Y$ be a resolution of Y at an ordinary double point. And let \bar{H} be the strict transform of H . Consider a double covering $\bar{\pi} : \bar{X} \rightarrow \bar{Y}$ branched over $\bar{B} = \sum_{i=1}^8 \bar{A}_i + \bar{H}$, where \bar{A}_i are eight nodal curves on \bar{Y} . Note that every elliptic fibration on an Enriques surface has two multiple fibers. If we say those multiple fibers as F and G (Fig. 2.1), then $H \in |F + G|$. Hence B is 2-divisible and so does \bar{B} .

Since H has the simple singularity; applying Lemma 2.3 (1) we get the following invariants.

$K_{\bar{X}}^2 = -4$, and after contracting eight (-1) curves via blow down $\beta_d : \bar{X} \rightarrow V$, we get the minimal surface V with $K_V^2 = 4$ that we want.

The reason why V is minimal as follows. Suppose that E is a (-1) curve on V , and hence E is also (-1) curve on \bar{X} . There are two possibilities, $\pi^*E_{\bar{Y}} = E$ or $\pi^*E_{\bar{Y}} = 2E$ for some curve $E_{\bar{Y}}$ on \bar{Y} . That is, $E_{\bar{Y}} \cap \bar{B} = \emptyset$ or $E_{\bar{Y}} \cap \bar{B} \neq \emptyset$, respectively. By definition, $E^2 = -1$ and $K_{\bar{X}} \cdot E = -1$ by adjunction formula. If $E_{\bar{Y}} \cap \bar{B} = \emptyset$ then $E_{\bar{Y}} \cdot \bar{B} = 0$ and $-1 = 2K_{\bar{Y}} \cdot E_{\bar{Y}}$. It is a contradiction. If $E_{\bar{Y}} \cap \bar{B} \neq \emptyset$ then $E_{\bar{Y}}^2 = -2$ and $K_{\bar{Y}} \cdot E_{\bar{Y}} = 0$. This means that every (-1) curve comes from the nodal curve on \bar{Y} and those nodal curves are contained in \bar{B} .

$$\chi(\bar{X}) = 2\chi(Y) + \frac{1}{4}B \cdot K_Y + \frac{1}{8}B^2 = 2 + 0 + \frac{-8}{8} = 1.$$

Since $p_g(\bar{X}) = p_g(\bar{Y}) + h^0(\mathcal{O}(K_{\bar{Y}} + \frac{1}{2}\bar{B})) = h^0(\mathcal{O}(\beta^*(K_Y + \frac{1}{2}B))) = h^2(\mathcal{O}_Y(-\frac{1}{2}B))$, consider the divisor $K_Y + \frac{1}{2}B$ on Y . We know that the following exact sequences for this divisor:

$$\begin{aligned} 0 &\rightarrow \mathcal{O}(-K_Y - \frac{1}{2}B) \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_{K_Y + \frac{1}{2}B} \rightarrow 0 \\ \implies 0 &\rightarrow \mathcal{O}_Y(-\frac{1}{2}B) \rightarrow \mathcal{O}_Y(K_Y) \rightarrow \mathcal{O}_{K_Y + \frac{1}{2}B}(K_Y) \rightarrow 0 \\ \implies 0 &\rightarrow H^0(Y, \mathcal{O}_Y(-\frac{1}{2}B)) \rightarrow H^0(Y, \mathcal{O}_Y(K_Y)) \rightarrow H^0(Y, \mathcal{O}_{K_Y + \frac{1}{2}B}(K_Y)) \\ &\rightarrow H^1(Y, \mathcal{O}_Y(-\frac{1}{2}B)) \rightarrow H^1(Y, \mathcal{O}_Y(K_Y)) \rightarrow \dots \end{aligned}$$

Since $h^0(Y, \mathcal{O}_Y(K_Y)) = p_g(Y) = 0$ and $h^1(Y, \mathcal{O}_Y(K_Y)) = q(Y) = 0$, $h^0(\mathcal{O}_Y(-\frac{1}{2}B)) = 0$. And $h^1(\mathcal{O}_Y(-\frac{1}{2}B)) = h^0(Y, \mathcal{O}_{K_Y + \frac{1}{2}B}(K_Y)) = (K_Y + \frac{1}{2}B) \cdot K_Y = 0$. Here, $h^2(\mathcal{O}_Y(-\frac{1}{2}B)) = \chi(\mathcal{O}_Y(-\frac{1}{2}B)) - h^0(\mathcal{O}_Y(-\frac{1}{2}B)) + h^1(\mathcal{O}_Y(-\frac{1}{2}B)) = 0$. So $p_g(\bar{X}) = 0$. Recall that $\chi(\bar{X}) = 1$, then $q(\bar{X}) = p_g(\bar{X}) - \chi(\bar{X}) + 1 = 0$.

Since both χ , p_g and q are birational invariants, $\chi(V) = 1$ and $p_g(V) = q(V) = 0$.

By Enriques–Kodaira classification, positivity of $K_V^2 = 4$ guarantees that V is rational or of general type. Since the surface \bar{X} is the resolution of a double covering of Enriques surface, V is not rational by Castelnuovo’s rationality criterion. Therefore, the minimal surface V is of general type with $K_V^2 = 4$ and $p_g = q = 0$. \square

3. A complex minimal surface of $K^2 = 3$ and $p_g = q = 0$

Let us add a condition to drop one of K_V^2 . Choose D passing through p , which is one of the sixteen intersection points of eight fibers. And the other assumptions for D are the same with Section 2. Then $j(D)$ passes through $j(p) \neq p$, which is one of the sixteen intersection points. These imply that the image H of $p^*(D + j(D))$ has also one double point on Y and meets one nodal curve at two points. That is, let A_8 be a such nodal curve on Y then $H \in |F + G - A_8|$ where F and G are multiple fibers of Y . Let $B = \sum_{i=1}^7 A_i + H$ so as to maintain 2-divisibility of branch divisor B .

In this case, $f(x_0, x_1, y_0, y_1) = 34x_0y_0 - 96x_0y_1 - 33x_1y_0 + 52x_1y_1$ and $g(x_0, x_1, y_0, y_1) = 34x_1y_1 - 96x_1y_0 - 33x_0y_1 + 52x_0y_0$ are such polynomials.

Theorem 3.1. *Let X be a double covering of Y branched along $B = \sum_{i=1}^7 A_i + H$ where A_i are nodal curves that do not meet H and H is the image of $p^*(D + j(D))$ on Enriques surface Y . Then the minimal resolution V of a double covering X is a surface of general type with $K_V^2 = 3$ and $p_g(V) = q(V) = 0$.*

Proof. Same here as Section 2, let $\beta : \bar{Y} \rightarrow Y$ be a resolution of Y at a double point and \bar{H} be a strict transform of H . Then there is a double covering $\bar{\pi} : \bar{X} \rightarrow \bar{Y}$ branched along the 2-divisible smooth divisor \bar{B} . Apply again Lemma 2.2 to this situation, then we are able to compute the following invariants.

$K_{\bar{X}}^2 = \{\bar{\pi}^*(K_{\bar{Y}} + \frac{1}{2}\bar{B})\}^2 = 2(K_{\bar{Y}}^2 + K_{\bar{Y}} \cdot \bar{B} + \frac{1}{4}\bar{B}^2) = 2(-1 + 2 + \frac{-12}{4}) = -4$. After contracting seven (-1) curves via blow down $\beta_d : \bar{X} \rightarrow V$, we get the surface V with $K_V^2 = 3$. And V is also minimal for the same reason as in Theorem 2.4.

Other invariants and classification can be obtained in the same way as in Section 2, and the results are the same. That is, $\chi(V) = 1$, $p_g(V) = q(V) = 0$, and V is a surface of general type. \square

Proposition 3.2. *There is a (-4) -curve on the surface in Theorem 3.1.*

Proof. Recall that $H \cdot A_8 = 2$ and 2-divisible branch divisor $B = \sum_{i=1}^7 A_i + H$: these facts are right above Theorem 3.1. We wrote $\bar{\pi} : \bar{X} \rightarrow \bar{Y}$ as a branched double cover in the proof of Theorem 3.1. And let $\bar{a}_8 \mapsto \bar{A}_8$, where \bar{A}_8 is the pullback of A_8 on the resolution \bar{Y} of Y . Since A_8 is not a branch divisor $\bar{\pi}^*(\bar{A}_8) = \bar{a}_8$. So, \bar{a}_8 becomes a (-4) -curve on \bar{X} . And its image on the minimal resolution V of \bar{X} is also a (-4) -curve. \square

Remark 3.3. The construction of the original Keum–Naie surface with $K^2 = 3$ ([2] Section 3.2) is a little bit different from ours. To facilitate the comparison, we will write the notations used in his paper as follows: R_i ($i = 2, 4, 6, 8, 9, 10, 11, 12$) to A_i ($i = 1, \dots, 8$), in particular R_{11} to A_8 . That is, $B = R_2 + R_4 + R_6 + R_8 + R_9 + R_{10} + R_{11} + R_{12} + H$ in [2] means that $B = \sum_{i=1}^8 A_i + H$ in our notations.

Note that $B = \sum_{i=1}^8 A_i + H$ and $A_8 \cdot H = 4$. Since A_8 is contained in the branch divisor B , we first consider the blowing-up at these four intersection points of A_8 and H [3] ((3.2.2) proposition). Let $\beta : \bar{Y} \rightarrow Y$ be blowing ups, then $\bar{A}_8^2 = -6$, where \bar{A}_8 is a proper transform of A_8 . For all i , $\bar{\pi}^*A_i = 2\bar{a}_i$ via the branched double covering $\bar{\pi} : \bar{X} \rightarrow \bar{Y}$ $\bar{a}_i \mapsto \bar{A}_i$. Hence, $\bar{a}_i^2 = -1$ ($i = 1, \dots, 7$), $\bar{a}_8^2 = -3$ on \bar{X} , and four exceptional curves on \bar{Y} become (-2) -curves on \bar{X} . After contracting seven (-1) -curves all, he got the minimal resolution V of \bar{X} with $K^2 = 3$. And the image of \bar{a}_8 and (-2) -curves are also (-3) -curve and (-2) -curves on V , respectively.

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