C. R. Acad. Sci. Paris, Ser. I 357 (2019) 278–283

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Functional analysis/Complex analysis

Dynamics of weighted composition operators in the unit ball ☆

Dynamiques des opérateurs de composition pondérés dans la boule unité

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A R T I C L E I N F O

Article history:
Received 1 December 2017
Accepted after revision 18 February 2019
Available online 21 March 2019

Presented by the Editorial Board

A B S T R A C T

In the present paper, we investigate the dynamic behavior of weighted composition operators acting on the space of holomorphic functions on the unit ball in $\mathbb{C}^N$.

R É S U M É

Nous étudions dans cette Note le comportement dynamique des opérateurs de composition pondérés agissant sur l'espace des fonctions holomorphes sur la boule unité de $\mathbb{C}^N$.

1. Introduction

Let $\mathbb{B}_N$ be the unit ball of a N-dimensional complex space $\mathbb{C}^N$ with the boundary $\partial\mathbb{B}_N$. The class of all holomorphic functions on $\mathbb{B}_N$ will be denoted by $H(\mathbb{B}_N)$. For any $z=(z_1,z_2,\cdots,z_N)$ and $w=(w_1,w_2,\cdots,w_N)$ in $\mathbb{C}^N$, the inner product is defined by $\langle z,w\rangle=\sum_{j=1}^N z_i\overline{w_i}$ and $\|z\|^2=\langle z,z\rangle$.

Let $\varphi(z)=(\varphi_1(z),\cdots,\varphi_n(z))$ be a holomorphic self-map of $\mathbb{B}_N$ and $u\in H(\mathbb{B}_N)$, the weighted composition operator $uC_\varphi : H(\mathbb{B}_N)\to H(\mathbb{B}_N)$, is defined by

$$uC_\varphi(f)(z):=u(z)(f\circ\varphi)(z).$$

For general references on the theory of weighted composition operators, we refer the interested readers to the books [3,6].

Definition 1.1. An operator T on a topological vector space X is called hypercyclic provided that there is some vector f in X whose orbit $Orb(T,f)=\{T^nf : n=0,1,\cdots\}$ is dense in X. Such f is called a hypercyclic vector for T.

☆ The work was supported in part by the National Natural Science Foundation of China (Grant No. 11771323), and also supported in part by the Science and Technology Development Fund of Tianjin Commission for Higher Education (Grant No. 2017KJ095).

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https://doi.org/10.1016/j.crma.2019.02.003

Definition 1.2. An operator T on a topological vector space X is topologically transitive provided that, for each pair of non-empty open subsets U and V of X, there exists an integer n such that $T^n(U)\bigcap V \neq \emptyset$.

Note that when X is a separable Fréchet space, Birkhoff's Transitivity Theorem ensures that T is hypercyclic if and only if it is topologically transitive.

Definition 1.3. An operator T on a locally convex space X is weakly hypercyclic provided that it is hypercyclic with respect to the weak topology of X.

Definition 1.4. A map φ will be called a linear fractional map if

$$\varphi(z) = (Az + B)(\langle z, C\rangle + D)^{-1}$$

where A is an $N \times N$ matrix, B and C are (column) vectors in $\mathbb{C}^N$, and D is a complex number.

We introduce the Siegal upper half-plane $\mathbb{H}_N$ defined by

$$\mathbb{H}_N = \{(w_1, \cdots, w_n) = (w_1, w') \in \mathbb{C}^N, \operatorname{Im}(w_1) > \|w'\|^2\}.$$

Let $e_1 = (1, 0, \cdots, 0) = (1, 0')$. The Cayley transform, defined by $\mathcal{C}(z) = \mathrm{i}(e_1 + z)/(1 - z_1)$ is a biholomorphic map of $\mathbb{B}_N$ onto $\mathbb{H}_N$.

We say that Φ is a generalized Heisenberg translation of $\mathbb{H}_N$ if it may be written

$$\Phi(w_1, w') = (w_1 + 2\mathrm{i}\langle w', \gamma\rangle + b, w' + \gamma),$$

with $b \in \mathbb{C}$, $\gamma \in \mathbb{C}^{N-1}\backslash\{0\}$ and $\operatorname{Im}(b) \geq \|\gamma\|^2$. A generalized Heisenberg translation is an automorphism if and only if $\operatorname{Im}(b) = \|\gamma\|^2$.

Definition 1.5. Let $H^\infty(\mathbb{B}_N)$ denote the space of bounded holomorphic functions f on the unit ball with the supremum norm $\|f\|_\infty = \sup_{z\in\mathbb{B}_N} |f(z)|$.

Definition 1.6. Given $\tau \in \partial\mathbb{B}_N$ and $\theta > 0$, we let $Lip_\theta(\tau)$ denote the set of weights $u \in H(\mathbb{B}_N)$ that are continuous on $\mathbb{B}_N \cup \{\tau\}$ and for which there exists $0 < \delta < 1$ such that

$$\sup_{\{z\in\mathbb{B}_N:0<\|z-\tau\|<\delta\}} \frac{|u(z) - u(\tau)|}{\|z - \tau\|^\theta} < \infty.$$

For example, if $u \in H(\mathbb{B}_N)$ is holomorphic at τ, then $u \in Lip_\theta(\tau)$ for $\theta \in (0, 1]$. Moreover, if $D^\alpha u(\tau) = 0$, for $0 < |\alpha| \leq n$, then $u \in Lip_\theta(\tau)$ for $\theta \in (0, n + 1]$, where $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $D_j = \partial/\partial z_j$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

In the past two decades, many authors focused on the dynamics of the weighted composition operator. Recently, in [2], Bès showed that a weighted composition operator on $H(\Omega)$ is weakly supercyclic if and only if it is mixing, whenever $\Omega \subset \mathbb{C}$ is a simply connected plane domain. In this paper, we will discuss the higher dimensional case, our theorems generalize part of the results obtained in [2] and [7].

2. Some lemmas

In this section, we present some lemmas that will be used in the proofs of our main results in the next section. We first give the generalization of the Runge theorem to the several variables.

Lemma 2.1. *[5, Corollary 5.2.9] Let Ω be a Stein manifold and let $K \subset \Omega$ be a compact holomorphically convex set. Then every function that is holomorphic in a neighborhood of K can be approximated uniformly on K by holomorphic functions on Ω.*

Lemma 2.2. *If T is weakly hypercyclic, then T^* does not have any eigenvectors.*

Proof. From the definition of 1-weakly hypercyclic operators in [4], it is obvious that if T is weakly hypercyclic, then it is also 1-weakly hypercyclic. The lemma follows from Proposition 3.2 in [4], which shows that if T is 1-weakly hypercyclic, then T^* does not have any eigenvectors. □

Lemma 2.3. *Assume $\varphi : \mathbb{B}_N \to \mathbb{B}_N$ is a linear fractional map, with no fixed points in $\mathbb{B}_N$. Then there exists a unique point $\tau \in \partial\mathbb{B}_N$ such that $\varphi(\tau) = \tau$ and $\langle \mathrm{d}\varphi_\tau(\tau), \tau\rangle = \alpha(\varphi)$ with $0 < \alpha(\varphi) \leq 1$.*

The point $\tau \in \partial\mathbb{B}_N$ in Lemma 2.3 is called the Denjoy–Wolff point of φ and $\alpha(\varphi)$ is the boundary dilation coefficient of φ. We say that φ is hyperbolic if $\alpha(\varphi) < 1$, while we say that it is parabolic if $\alpha(\varphi) = 1$.

Lemma 2.4. *[1, Theorem 3.1] Let φ be a parabolic automorphism of $\mathbb{B}_N$. Then $\Phi := \mathcal{C} \circ \varphi \circ \mathcal{C}^{-1}$ is a generalized Heisenberg translation.*

Lemma 2.5. *Assume that $\varphi : \mathbb{B}_N \to \mathbb{B}_N$ is univalent with no fixed points in $\mathbb{B}_N$, then for any compact subset $K \subset \mathbb{B}_N$, there is some sufficiently large n_0 such that $\varphi^{[n_0]}(K) \cap K = \emptyset$, where $\varphi^{[n_0]}$ denotes the n-fold composition of φ with itself.*

Proof. By Theorem 2.83 in [3], there exists a point ξ of norm 1 such that the iterates $\varphi^{[n]}$ of φ converge to ξ uniformly on compact subsets of $\mathbb{B}_N$. Given any compact set $K \subset \mathbb{B}_N$, let ρ denote the distance from K to $\partial\mathbb{B}_N$. And let $U := \{z \in \mathbb{B}_N : \|z - \xi\| < \rho/2\}$. Then there is some n_0 such that $\varphi^{[n_0]}(K) \subset U$. Hence, $\varphi^{[n_0]}(K) \cap K = \emptyset$. □

3. Main theorems

Proposition 3.1. *Assume that the operator uC_φ is weakly hypercyclic on $H(\mathbb{B}_N)$, then $u(z) \neq 0$ for every $z \in \mathbb{B}_N$ and φ is univalent and without fixed points in $\mathbb{B}_N$.*

Proof. Suppose that uC_φ is weakly hypercyclic on $H(\mathbb{B}_N)$. Note that, for each $z_0 \in \mathbb{B}_N$, the point evaluation linear functional $\delta_{z_0} : H(\mathbb{B}_N) \to \mathbb{C}$ by $\delta_{z_0}(f) = f(z_0)$ is continuous on $H(\mathbb{B}_N)$ and

$$(uC_\varphi)^*(\delta_{z_0})(f) = \delta_{z_0} uC_\varphi(f) = u(z_0) f(\varphi(z_0)) = u(z_0)\delta_{\varphi(z_0)}(f).$$

Thus, if $\varphi(z_0) = z_0$ or $u(z_0) = 0$ for some $z_0 \in \mathbb{B}_N$, the adjoint of uC_φ has an eigenvalue and, by Lemma 2.2, uC_φ is not weakly hypercyclic. Now we prove that φ must be univalent. Let $\varphi(p) = \varphi(q)$ and consider a weakly hypercyclic vector f of uC_φ. Notice that $(uC_\varphi)^n f = \prod_{j=0}^{n-1} C_\varphi^j(u) C_\varphi^n f$, then

$$\frac{(uC_\varphi)^n f(p)}{u(p)} = \prod_{j=1}^{n-1} C_\varphi^j(u)(p) C_\varphi^n(f)(p) = \prod_{j=1}^{n-1} C_\varphi^j(u)(q) C_\varphi^n(f)(q) = \frac{(uC_\varphi)^n f(q)}{u(q)}.$$

We may pick a net $\{(uC_\varphi)^{n_l}\}$ in $Orb(uC_\varphi f)$ such that $(uC_\varphi)^{n_l}(f)$ weakly converges to g, where $g \in H(\mathbb{B}_N)$. Since the point evaluation functional is weakly continuous on $H(\mathbb{B}_N)$, then both $\frac{\delta_p}{u(p)}$ and $\frac{\delta_q}{u(q)}$ are also weakly continuous, and setting $g = 1$, we have

$$\frac{\delta_p}{u(p)}(uC_\varphi)^{n_l}(f) \to \frac{\delta_p}{u(p)}1$$

and

$$\frac{\delta_q}{u(q)}(uC_\varphi)^{n_l}(f) \to \frac{\delta_q}{u(q)}1.$$

Since the left sides are the same, we have $u(p) = u(q)$. Taking $g(z) = z_i$ for $1 \le i \le n$, respectively, and arguing as above, we obtain $p_i = q_i$, and consequently $p = q$. □

Theorem 3.2. *Assume that φ is a holomorphic self-map of $\mathbb{B}_N$, and let $u \in H(\mathbb{B}_N)$. Then The operator uC_φ is hypercyclic on $H(\mathbb{B}_N)$ if and only if the following conditions hold:*

(a) for every $z \in \mathbb{B}_N$, $u(z) \neq 0$ and φ is univalent without fixed points in $\mathbb{B}_N$;

(b) for every holomorphic convex compact subset $K \subset \mathbb{B}_N$, there exists n such that $K \bigcap \varphi^{[n]}(K) = \phi$ and the set $K \bigcup \varphi^{[n]}(K)$ is holomorphic convex.

Proof. First we show the sufficiency. Note that the compact-open topology on $H(\mathbb{B}_N)$ is independent of the chosen exhaustion. We set $K_n := \{z \in \mathbb{B}_N : \|z\| \le 1 - 1/n\}$, $n \in \mathbb{N}$, which is an exhaustion of $\mathbb{B}_N$, then we endow $H(\mathbb{B}_N)$ with the topology induced by the seminorms $p_n(f) := \sup_{z \in K_n} |f(z)|$, $f \in H(\mathbb{B}_N)$. Let U, V be non-empty open subsets of $H(\mathbb{B}_N)$, and fix $f \in U$, $g \in V$. By the definition of the topology on $H(\mathbb{B}_N)$, there is a closed ball K centered at 0 and an $\epsilon > 0$ such that an holomorphic function h belongs to U (or to V) whenever $\sup_{z\in K} |f(z) - h(z)| < \epsilon$ (or $\sup_{z\in K} |g(z) - h(z)| < \epsilon$, respectively). Let $\widetilde{K}$ be a closed ball in $\mathbb{B}_N$ such that $K \subset \widetilde{K}^\circ \subset \widetilde{K}$. Since φ is univalent and without fixed points in $\mathbb{B}_N$, then the function f is holomorphic on some neighborhood of $\widetilde{K}$, and function $\frac{g\circ(\varphi^{[n_0]})^{-1}}{\prod_{k=1}^{n_0}(u\circ(\varphi^{[k]})^{-1})}$ is holomorphic on some neighborhood of $\varphi^{[n_0]}(\widetilde{K})$. By assumption (b), there exists n_0 such that $\varphi^{[n_0]}(\widetilde{K}) \cap \widetilde{K} = \emptyset$ and the compact set $\mathcal{K} := \widetilde{K} \bigcup \varphi^{[n_0]}(\widetilde{K})$ is holomorphically convex; from Lemma 2.1, there exists a holomorphic function $h \in H(\mathbb{B}_N)$ such that

$$\sup_{z\in\widetilde{K}}|f(z)-h(z)|<\epsilon \text{ and } \sup_{y\in\varphi^{[n_0]}(\widetilde{K})}\left|\frac{g\circ(\varphi^{[n_0]})^{-1}}{\prod_{k=1}^{n_0}(u\circ(\varphi^{[k]})^{-1})}(y)-h(y)\right|<\frac{\epsilon}{M},$$

where

$$M:=\max_{y\in\varphi^{[n_0]}(\widetilde{K})}\left|\prod_{k=1}^{n_0}(u\circ(\varphi^{[k]})^{-1})(y)\right|.$$

Hence

$$\sup_{z\in K}|f(z)-h(z)|<\epsilon$$

and

$$\begin{aligned}&\sup_{z\in K}\left|g(z)-(uC_\varphi)^{n_0}h(z)\right|\\ &=\sup_{z\in K}\left|\prod_{k=1}^{n_0}(u\circ(\varphi^{[k]})^{-1})(y)\left(\frac{g\circ(\varphi^{[n_0]})^{-1}}{\prod_{k=1}^{n_0}(u\circ(\varphi^{[k]})^{-1})}(y)-h(y)\right)\right|<\epsilon,\end{aligned}$$

where $y:=\varphi^{[n_0]}(z)$. This shows that $h\in U$ and $(uC_\varphi)^{n_0}h\in V$, so that uC_φ is topologically transitive. Since $H(\mathbb{B}_N)$ is a separable Fréchet space, uC_φ is hypercyclic.

Next we show the necessity. Assume that uC_φ is hypercyclic, then it is obviously weakly hypercyclic and, by Proposition 3.1, condition (a) follows. Now we show that condition (b) holds. Since uC_φ is hypercyclic, then it is topologically transitive. Hence, for every $\epsilon>0$, f,g holomorphic on $\mathbb{B}_N$ and k compact and holomorphically convex, there exists $n_0\in N$ and a holomorphic function h such that

$$|f-g|<\epsilon \text{ and } |(uC_\varphi)^{n_0}(h)-g|<\epsilon \text{ on } K.$$

Since φ is injective, we have

$$\sup_{z\in K}|f(z)-h(z)|<\epsilon \text{ and } \sup_{y\in\varphi^{[n_0]}(K)}\left|\frac{g\circ(\varphi^{[n_0]})^{-1}}{\prod_{k=1}^{n_0}(u\circ(\varphi^{[k]})^{-1})}(y)-h(y)\right|<\frac{\epsilon}{M},$$

where M is defined as above. We have two cases.

- **Case 1:** $M\geq 1$ Take $f=0$, $g=M$, $\epsilon=1/2$, then $h(K)\subset\frac{1}{2}\mathbb{D}$ and $h(\varphi^{[n_0]}(K))\subset\mathbb{C}-(1-\frac{1}{2M})\mathbb{D}$.
- **Case 2:** $0<M<1$ Take $f=0$, $g=M$, $\epsilon=M/2$, then $h(K)\subset\frac{M}{2}\mathbb{D}$ and $h(\varphi^{[n_0]}(K))\subset\mathbb{C}-\frac{1}{2}\mathbb{D}$.
  Hence, the set K and $\varphi^{[n_0]}(K)$ are separate. Therefore, using Lemma 2 in [8], we have $K\bigcap\varphi^{[n_0]}(K)=\phi$ and $K\bigcup\varphi^{[n_0]}(K)$ is holomorphic convex. □

Proposition 3.3. *Let φ be a holomorphic linear fractional self map of $\mathbb{B}_N$ without fixed points in $\mathbb{B}_N$. Then $\sum_{n=1}^{\infty}(1-\|\varphi^{[n]}(z)\|^2)^\alpha$ converges locally uniformly on $\mathbb{B}_N$ whenever φ is hyperbolic and $\alpha>0$ or φ is a parabolic automorphism and $\alpha>1/2$.*

Proof. Let τ be the Denjoy–Wolff fixed point of φ. Without loss of generality, we assume that $\tau=e_1\in\partial\mathbb{B}_N$. We first suppose that φ is hyperbolic with $\alpha>0$. By Julia's Lemma (Lemma 2.77 in [3]), we have

$$\frac{|1-\varphi_1(z)|^2}{1-\|\varphi(z)\|^2}\leq\alpha(\varphi)\frac{|1-z_1|^2}{1-\|z\|^2}.$$

By substituting $\varphi^{[n]}(z)$ for $\varphi(z)$ in the above inequality, we obtain

$$\frac{|1-\varphi_1^{[n]}(z)|^2}{1-\|\varphi^{[n]}(z)\|^2}\leq(\alpha(\varphi))^n\frac{|1-z_1|^2}{1-\|z\|^2}$$

for every $z\in\mathbb{B}_N$ and all $n\geq 0$. Moreover,

$$\begin{aligned}&\frac{|1-\varphi_1^{[n]}(z)|^2}{1-\|\varphi^{[n]}(z)\|^2}\geq\frac{1}{2}\frac{(1-|\varphi_1^{[n]}(z)|)^2}{1-\|\varphi^{[n]}(z)\|}\geq\frac{1}{2}(1-|\varphi_1^{[n]}(z)|)\\ &\geq\frac{1}{2}(1-\|\varphi^{[n]}(z)\|)\geq\frac{1}{4}(1-\|\varphi^{[n]}(z)\|^2).\end{aligned}$$

Note that if K is a compact subset of $\mathbb{B}_N$, then there exists a constant $M > 0$ such that $\frac{|1-z_1|^2}{1-\|z\|^2} < M$ for all $z \in K$. Therefore,

$$1 - \|\varphi^{[n]}(z)\|^2 \leq 4M(\alpha(\varphi))^n,$$

and consequently,

$$(1 - \|\varphi^{[n]}(z)\|^2)^\alpha \leq (4M)^\alpha (\alpha(\varphi))^{n\alpha}.$$

Thus the hyperbolic case follows since $0 < \alpha(\varphi) < 1$.

Now assume that φ is a parabolic automorphism and $\alpha > 1/2$. Let

$$\mathcal{C}(z) = \mathrm{i}(e_1 + z)/(1 - z_1) \text{ and } \Phi := \mathcal{C} \circ \varphi \circ \mathcal{C}^{-1}.$$

Note that $\mathcal{C}^{-1}(w) = \left(\frac{w_1 - \mathrm{i}}{w_1 + \mathrm{i}}, \frac{2w'}{w_1 + \mathrm{i}}\right)$. By Lemma 2.4, for each $w = (w_1, w') \in \mathbb{H}_N$, a straightforward computation shows that

$$1 - |\mathcal{C}_1^{-1}(w)|^2 = \frac{4\mathrm{Im}(w_1)}{|w_1 + \mathrm{i}|^2}.$$

Hence,

$$1 - |\varphi_1^{[n]}(z)|^2 = 1 - |(\mathcal{C}^{-1} \circ \Phi^{[n]} \circ \mathcal{C})_1(z)|^2 = \frac{4\,\mathrm{Im}[(\Phi^{[n]} \circ \mathcal{C})_1(z)]}{|\mathrm{i} + (\Phi^{[n]} \circ \mathcal{C})_1(z)|^2}$$
$$= \frac{4\,\mathrm{Im}\{w_1 + 2\mathrm{i}n\langle w', \gamma\rangle + \mathrm{i}n(n-1)\|\gamma\|^2 + nb\}}{|\mathrm{i} + w_1 + 2\mathrm{i}n\langle w', \gamma\rangle + \mathrm{i}n(n-1)\|\gamma\|^2 + nb|^2}.$$

When $N = 1$, we have $\gamma = 0$, and the argument is the same as in [7]. When $N > 1$, if K is any compact subset of $\mathbb{B}_N$, then for sufficiently large n, there exists some constant $M > 0$ such that

$$(1 - |\varphi^{[n]}(z)|^2)^\alpha \leq (1 - |\varphi_1^{[n]}(z)|^2)^\alpha < \frac{M}{(n(n-1))^\alpha}.$$

Therefore, $\sum_{n=1}^{\infty} (1 - \|\varphi^{[n]}(z)\|^2)^\alpha$ converges locally uniformly on $\mathbb{B}_N$ whenever $\alpha > 1/2$. □

Theorem 3.4. *Let φ be a holomorphic linear fractional self-map of $\mathbb{B}_N$ with Denjoy–Wolff point $\tau \in \partial\mathbb{B}_N$, and let $u \in Lip_\theta(\tau)$, and $u(\tau) \neq 0$. Then $u(\tau)$ is an eigenvalue for uC_φ whenever φ is hyperbolic and $\theta > 0$ or φ is a parabolic automorphism and $\theta > 1\, forN = 1$, $\theta > 2\, forN > 1$. Moreover, if u never vanishes on $\mathbb{B}_N$, then the eigenfunction also never vanishes.*

Proof. The Julia–Carathédory Theorem provides a constant $c > 0$ such that

$$|1 - \langle\varphi^{[n]}(z), \tau\rangle|^2 \leq c(1 - \|\varphi^{[n]}(z)\|^2).$$

When $N = 1$, we have

$$|\tau - \varphi^{[n]}(z)| = |\langle\tau - \varphi^{[n]}(z), \tau\rangle| = |1 - \langle\varphi^{[n]}(z), \tau\rangle| \leq c(1 - \|\varphi^{[n]}(z)\|^2)^{1/2},$$

the theorem follows by an argument as in [7]. When $N > 1$, note that $\|z - \tau\|^2 = 2\mathrm{Re}(1 - \langle z, \tau\rangle)$, then $\|z - \tau\|^2 \leq 2|1 - \langle z, \tau\rangle|$. Our hypothesis shows that there exists some constant M and $\delta > 0$, such that $|u(z) - u(\tau)| < M\|z - \tau\|^\theta$ for every z with $\|z - \tau\| < \delta$. Now fix a compact subset K of $\mathbb{B}_N$. Therefore,

$$|u(\varphi^{[n]}(z)) - u(\tau)| < M\|\varphi^{[n]}(z) - \tau\|^\theta < M2^{\theta/2}|1 - \langle\varphi^{[n]}(z), \tau\rangle|^{\theta/2}$$
$$< M2^{\theta/2}c^{\theta/4}(1 - \|\varphi^{[n]}(z)\|^2)^{\theta/4}$$

for every n large enough. By Proposition 3.3, the series

$$\sum_{n=0}^{\infty} \left|1 - \frac{1}{u(\tau)} C_\varphi^n(u)\right|$$

converges locally uniformly on $\mathbb{B}_N$. Thus $g := \prod_{n=0}^{\infty} \frac{C_\varphi^n(u)}{u(\tau)} \in H(\mathbb{B}_N)$ is nonzero and if u never vanishes on $\mathbb{B}_N$, then g does so. Moreover, $uC_\varphi g = u(\tau)g$. □

Remark 3.5. It is well known that the adjoint of a hypercyclic operator has no eigenvector. If we add the assumption that $\|u\|_\infty = u(\tau)$ in Theorem 3.4, then we have $g \in H^\infty(\mathbb{B}_N)$. When considering the operator uC_φ acting on the Hilbert space, which contains the constant functions and $H^\infty(\mathbb{B}_N)$, the operator $(uC_\varphi)^*$ is not hypercyclic.

Remark 3.6. From Theorem 3.4, we actually see that

$$uC_\varphi \circ M_g = M_g \circ (u(\tau)C_\varphi),$$

which implies that uC_φ is quasi-conjugate to $u(\tau)C_\varphi$ if M_g is continuous with dense range, and some properties, such as mixing, frequent hypercyclicity, are preserved under quasiconjugacy. Thus, it is enough to deal with λC_φ.

For $N > 1$, our proof of Theorem 3.4 does not work for the case $1 < \theta \le 2$. This rises up the following questions.

- **Question 1:** When $N > 1$, φ is a parabolic automorphism and $\theta > 1$, does Theorem 3.4 still hold?
- **Question 2:** For $N > 1$, when the weighted composition operator has a hypercyclic subspace?

Acknowledgements

The authors would like to thank the editor and the referees for useful suggestions that improved the presentation of this paper.

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