C. R. Acad. Sci. Paris, Ser. I 357 (2019) 296–298

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Topology/Differential topology

Geography of simply connected spin symplectic 4-manifolds, II

Géographie des variétés de spin symplectiques, simplement connexes, de dimension 4. II

Anar Akhmedov [a], B. Doug Park [b]

[a] *School of Mathematics, University of Minnesota, Minneapolis, MN, 55455, USA*
[b] *Department of Pure Mathematics, University of Waterloo, Waterloo, ON, N2L 3G1, Canada*

ARTICLE INFO

Article history:
Received 12 December 2018
Accepted 19 February 2019
Available online 5 March 2019

Presented by Claire Voisin

ABSTRACT

Building upon our early work, we construct infinitely many new smooth structures on closed simply connected spin 4-manifolds with nonnegative signature.

RÉSUMÉ

Dans la continuité de notre travail précédent, nous construisons une infinité de nouvelles structures lisses sur les variétés de spin simplement connexes, fermées, de dimension 4 et de signature positive ou nulle.

1. Introduction

This paper is a short sequel to [1] and addresses the geography problem for closed simply connected spin symplectic 4-manifolds. For some background and history, we refer the readers to the introductions found in [1] and [2]. First we need to recall the following definitions from [1].

Definition 1. We say that a smooth 4-manifold M has ∞^2*-property* if there exist infinitely many pairwise nondiffeomorphic irreducible symplectic 4-manifolds and infinitely many pairwise nondiffeomorphic irreducible nonsymplectic 4-manifolds, all of which are homeomorphic to M. We also say that a symmetric bilinear form has ∞^2*-property* if it is the intersection form of infinitely many pairwise nondiffeomorphic simply connected irreducible symplectic 4-manifolds and infinitely many pairwise nondiffeomorphic simply connected irreducible nonsymplectic 4-manifolds.

Definition 2. For an even integer $p \geq 0$, let Λ_p denote the smallest positive odd integer such that the symmetric bilinear form $pE_8 \oplus qH$ has ∞^2-property for every odd integer $q \geq \Lambda_p$.

E-mail addresses: akhmedov@math.umn.edu (A. Akhmedov), bdpark@uwaterloo.ca (B.D. Park).

https://doi.org/10.1016/j.crma.2019.02.002

Here, we have

$$E_8 = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \text{ and } H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

so that the rank and the signature of $pE_8 \oplus qH$ are $8p + 2q$ and $8p$, respectively. Recall from [5] that a closed simply connected smooth 4-manifold is spin if and only if its intersection form is $pE_8 \oplus qH$ for some integers p and q with p even. Also recall that if a closed simply connected smooth spin 4-manifold with the intersection form $pE_8 \oplus qH$ is symplectic, then $q \equiv 1 \pmod 2$.

The famous 11/8 Conjecture (Problem 4.92 in [8]), which remains unresolved, would imply an a priori lower bound $\Lambda_p \geq \frac{3}{2}p$. Accordingly, we made the following optimistic conjecture in [1].

Conjecture 3. *Λ_p is the smallest positive odd integer that is greater than or equal to $\frac{3}{2}p$.*

Unfortunately, Conjecture 3 seems out of our reach at the moment. The best known lower bound for Λ_p comes from a recent work [7], which gives $\Lambda_p \geq p + \epsilon_p$ when $p \geq 4$, where

$$\epsilon_p = \begin{cases} 2 & \text{if } p \equiv 1, 2, 5, 6 \pmod 8, \\ 3 & \text{if } p \equiv 3, 4, 7 \pmod 8, \\ 4 & \text{if } p \equiv 0 \pmod 8). \end{cases}$$

In [1], we also presented a recipe for checking ∞^2-property for $pE_8 \oplus qH$ starting from a suitable surface bundle over a surface (see Theorem 6 below). In this paper, we apply our recipe to a surface bundle found in [4] and its analogues, and obtain the following new upper bound for Λ_p, which will be proved in the next section.

Theorem 4. *Let $p \geq 0$ be an even integer. If m is any positive integer satisfying $p \leq 6m - 2$, then $\Lambda_p \leq 162m + 13 - 10p$.*

Let $S^2 \times S^2$ denote the Cartesian product of two 2-spheres with the intersection form H. Let $q(S^2 \times S^2)$ denote the connected sum of q copies of $S^2 \times S^2$. Let $\overline{K3}$ denote the complex $K3$ surface equipped with the noncomplex orientation and thus with the intersection form $2E_8 \oplus 3H$. When $p = 0$ and $m = 1$, Theorem 4 implies that $\Lambda_0 \leq 175$, i.e. $q(S^2 \times S^2)$ has ∞^2-property for every odd integer $q \geq 175$. This is an improvement over the upper bound $\Lambda_0 \leq 275$ in [1]. Similarly, when p is 2 or 4 and $m = 1$, Theorem 4 implies that the connected sum $\frac{p}{2}(\overline{K3})\#(q - \frac{23}{2}p)(S^2 \times S^2)$ has ∞^2-property for every odd integer $q \geq 175$. For many small values of p, Theorem 4 provides upper bounds for Λ_p that are lower (and hence better) than the upper bounds in [1] and [2].

Given a nonnegative even integer p, there is a positive integer m such that $6m - 6 \leq p \leq 6m - 2$. Thus Theorem 4 immediately implies the following simpler upper bound on Λ_p.

Corollary 5. *For any even $p \geq 0$, we have $\Lambda_p \leq 17p + 175$.*

The corollary states that $\Lambda_p \leq 17p + O(1)$ as $p \to \infty$. This should be compared to the asymptotic upper bound $\Lambda_p \leq 8p + O(p^{6/7})$ that was proved in [2].

2. Proof of Theorem 4

We will need the following theorem that was proved in [1].

Theorem 6. *Let X be a spin 4-manifold that is the total space of a genus-f surface bundle over a genus-b surface. Assume that the signature of X is $\sigma(X) = 16s$, and X has a section whose image is a genus-b surface of self-intersection $-2t$ for some integer t. Let r be a positive integer satisfying*

$$1 - t \leq r \leq \min\{s,\ f + b + 1 - t\}. \tag{1}$$

If p and q are nonnegative integers satisfying

$$p \equiv 0 \pmod 2, \quad 0 \le p \le 2(s-r),$$

$$q \equiv 1 \pmod 2, \quad q \ge 2fb + 12s - 1 - 10p,$$

then the symmetric bilinear form $pE_8 \oplus qH$ has ∞^2-property (cf. Definition 1) and

$$\Lambda_p \le 2fb + 12s - 1 - 10p.$$

We now apply Theorem 6 to the following example and its generalizations. We will let Σ_b denote a closed genus-b Riemann surface.

Example 7. Recall from Example 5.9 in [4] that there is a genus-7 surface bundle X whose total space is obtained as a certain 3-fold cyclic branched cover of $\Sigma_b \times \Sigma_2$ with branch locus D', which is a disjoint union of the graphs of 3 maps $\phi_i : \Sigma_b \to \Sigma_2$ $(i = 1, 2, 3)$. If $\pi : X \to \Sigma_b \times \Sigma_2$ is this branched covering map and $\mathrm{pr}_1 : \Sigma_b \times \Sigma_2 \to \Sigma_b$ is the projection map onto the first factor, then our surface bundle map is the composition $\Pi = \mathrm{pr}_1 \circ \pi$. In Example 6.5 of [9], it was shown that the base genus of this surface bundle X is $b = 10$ and $\sigma(X) = 48$.

We now proceed to construct infinitely many surface bundles that generalize Example 7. For any pair of positive integers b and m, there is an m-fold unbranched covering map $\rho_{b,m} : \Sigma_{m(b-1)+1} \to \Sigma_b$. Let $\Pi_m : X_m \to \Sigma_{9m+1}$ be the pullback of the surface bundle $\Pi : X \to \Sigma_{10}$ in Example 7 by the covering map $\rho_{10,m} : \Sigma_{9m+1} \to \Sigma_{10}$. Of course, we have $\Pi_1 = \Pi$ and $X_1 = X$. The total space X_m is the 3-fold cyclic branched cover of $\Sigma_{9m+1} \times \Sigma_2$ with branch locus D'_m, which is the disjoint union of the graphs of the compositions $\phi_i \circ \rho_{10,m} : \Sigma_{9m+1} \to \Sigma_2$ $(i = 1, 2, 3)$. (Note that the homology class $[D'_m] = (PD \circ (\rho_{10,m} \times \mathrm{id})^* \circ PD)[D']$ is divisible by 3, where PD denotes the Poincaré duality map and $\mathrm{id} : \Sigma_2 \to \Sigma_2$ is the identity map.)

By a formula of Brand in [3], the second Stiefel–Whitney class of X_m is

$$w_2(X_m) = \frac{2}{3}(\pi_m^*(PD[D'_m])) \equiv 0 \pmod 2,$$

where $\pi_m : X_m \to \Sigma_{9m+1} \times \Sigma_2$ is the 3-fold cyclic branched covering map, and thus X_m is spin. Since the induced map $X_m \to X$ between the total spaces is an unbranched covering map, we have $\sigma(X_m) = m\sigma(X) = 48m$. By Hirzebruch's signature formula in [6], we have

$$\sigma(X_m) = -\frac{8}{9}[D'_m]^2,$$

and thus $[D'_m]^2 = -54m$. Since the branching index is the same at each component of D'_m, the inclusion of each component of D'_m gives a section of the surface bundle Π_m whose image has self-intersection $-54m/3 = -18m$ inside X_m.

In conclusion, for each positive integer m, we get a surface bundle X_m with parameters $f = 7$, $b = 9m + 1$, $s = 3m$, and $t = 9m$. Plugging these numbers into (1), we get $1 - 9m \le r \le \min\{3m, 9\}$. By choosing $r = 1$, we obtain Theorem 4 from Theorem 6.

Acknowledgements

The first author was partially supported by Simons Research Fellowship and Collaboration Grant for Mathematicians from the Simons Foundation. The second author was partially supported by an NSERC discovery grant (261491). The authors thank F. Catanese and S. Rollenske for valuable e-mail exchanges regarding their work [4].

References

[1] A. Akhmedov, B.D. Park, Geography of simply connected spin symplectic 4-manifolds, Math. Res. Lett. 17 (2010) 483–492.
[2] A. Akhmedov, B.D. Park, G. Urzúa, Spin symplectic 4-manifolds near Bogomolov–Miyaoka–Yau line, J. Gökova Geom. Topol. GGT 4 (2010) 55–66.
[3] N. Brand, Necessary conditions for the existence of branched coverings, Invent. Math. 54 (1979) 1–10.
[4] F. Catanese, S. Rollenske, Double Kodaira fibrations, J. Reine Angew. Math. 628 (2009) 205–233.
[5] R.E. Gompf, A.I. Stipsicz, 4-manifolds and Kirby calculus, in: Grad. Stud. Math., vol. 20, American Mathematical Society, Providence, RI, USA, 1999.
[6] F. Hirzebruch, The signature of ramified coverings, in: Global Analysis (Papers in Honor of K. Kodaira), University of Tokyo Press, Tokyo, 1969, pp. 253–265.
[7] M.J. Hopkins, J. Lin, X.D. Shi, Z. Xu, Intersection forms of spin 4-manifolds and the Pin(2)-equivariant Mahowald invariant, arXiv:1812.04052.
[8] R. Kirby, Problems in low-dimensional topology, math.berkeley.edu/~kirby/problems.ps.gz.
[9] J. Lee, M. Lönne, S. Rollenske, Double Kodaira fibrations with small signature, arXiv:1711.01792.