C. R. Acad. Sci. Paris, Ser. I 357 (2019) 221–230

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Mathematical problems in mechanics

A confinement problem for a linearly elastic Koiter's shell

Un problème de confinement pour une coque de Koiter linéairement élastique

Philippe G. Ciarlet, Paolo Piersanti

Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong

A R T I C L E I N F O

Article history:
Received 18 January 2019
Accepted 18 January 2019
Available online 31 January 2019

Presented by Philippe G. Ciarlet

A B S T R A C T

In this Note, we propose a natural two-dimensional model of "Koiter's type" for a general linearly elastic shell confined in a half space. This model is governed by a set of variational inequalities posed over a non-empty closed and convex subset of the function space used for modeling the corresponding "unconstrained" Koiter's model. To study the limit behavior of the proposed model as the thickness of the shell, regarded as a small parameter, approaches zero, we perform a rigorous asymptotic analysis, distinguishing the cases where the shell is either an elliptic membrane shell, a generalized membrane shell of the first kind, or a flexural shell. Moreover, in the case where the shell is an elliptic membrane shell, we show that the limit model obtained via the asymptotic analysis of our proposed two-dimensional Koiter's model coincides with the limit model obtained via a rigorous asymptotic analysis of the corresponding three-dimensional "constrained" model.

R É S U M É

Dans cette Note, on propose un modèle naturel « de type de Koiter » pour une coque linéairement élastique générale confinée dans un demi-espace. Ce modèle est gouverné par un système d'inégalités variationnelles posées sur un sous-ensemble non vide convexe et fermé de l'espace fonctionnel utilisé dans la modélisation du modèle correspondant de Koiter « sans contrainte ». Afin d'étudier le comportement limite du modèle proposé lorsque l'épaisseur de la coque, considérée comme un petit paramètre, tend vers zéro, nous effectuons une analyse asymptotique rigoureuse, en distinguant les cas où la coque est elliptique membranaire, ou membranaire généralisée du premier type, ou en flexion. De plus, dans le cas où la coque est elliptique membranaire, nous montrons que le modèle limite obtenu par l'analyse asymptotique du modèle bidimensionnel de Koiter que nous proposons coïncide avec le modèle limite obtenu par une analyse asymptotique rigoureuse du modèle correspondant tri-dimensionnel « avec contrainte ».

E-mail addresses: mapgc@cityu.edu.hk (P.G. Ciarlet), ppiersan-c@my.cityu.edu.hk (P. Piersanti).

https://doi.org/10.1016/j.crma.2019.01.004
1631-073X/

1. Preliminaries

For definitions, notations and other preliminaries, we refer the reader to [11]. The complete proofs of the results presented in this Note will be found in [12].

2. A natural Koiter's model for a general shell subjected to a confinement condition

Let ω be a domain in $\mathbb{R}^2$ with boundary γ and let γ_0 be a non-empty relatively open subset of γ. For each $\varepsilon > 0$, we define the sets

$$\Omega^\varepsilon := \omega \times]-\varepsilon, \varepsilon[\text{ and } \Gamma_0^\varepsilon := \gamma_0 \times [-\varepsilon, \varepsilon];$$

we let $x^\varepsilon = (x_i^\varepsilon)$ designate a generic point in the set $\overline{\Omega^\varepsilon}$, and we let $\partial_i^\varepsilon := \partial/\partial x_i^\varepsilon$.

In all that follows, we are given an immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ and $\varepsilon > 0$, and we consider a *shell* with *middle surface* $\boldsymbol{\theta}(\overline{\omega})$ and with *constant thickness* 2ε. The *reference configuration* of the shell is the set $\boldsymbol{\Theta}(\overline{\Omega^\varepsilon})$, where the mapping $\boldsymbol{\Theta} : \overline{\Omega^\varepsilon} \to \mathbb{E}^3$ is defined by

$$\boldsymbol{\Theta}(x^\varepsilon) := \boldsymbol{\theta}(y) + x_3^\varepsilon \boldsymbol{a}^3(y) \text{ at each point } x^\varepsilon = (y, x_3^\varepsilon) \in \overline{\Omega^\varepsilon}.$$

One can then show (cf. Theorem 3.1-1 of [4] or Theorem 4.1-1 of [5]) that, if the thickness $\varepsilon > 0$ is small enough, such a mapping $\boldsymbol{\Theta} \in \mathcal{C}^2(\overline{\Omega^\varepsilon}; \mathbb{E}^3)$ is a $\mathcal{C}^2$-diffeomorphism from $\overline{\Omega^\varepsilon}$ onto $\boldsymbol{\Theta}(\overline{\Omega^\varepsilon})$. The three vectors

$$\boldsymbol{g}_i^\varepsilon(x^\varepsilon) := \partial_i^\varepsilon \boldsymbol{\Theta}(x^\varepsilon)$$

are linearly independent at each point $x^\varepsilon \in \overline{\Omega^\varepsilon}$; these vectors then constitute the *covariant basis* at the point $\boldsymbol{\Theta}(x^\varepsilon)$, while the three vectors $\boldsymbol{g}^{j,\varepsilon}(x^\varepsilon)$ defined by the relations

$$\boldsymbol{g}^{j,\varepsilon}(x^\varepsilon) \cdot \boldsymbol{g}_i^\varepsilon(x^\varepsilon) = \delta_i^j,$$

constitute the *contravariant basis* at the same point.

It will be implicitly assumed in what follows that the immersion $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ is *injective* and that $\varepsilon > 0$ *is small enough* so that $\boldsymbol{\Theta} : \overline{\Omega^\varepsilon} \to \mathbb{E}^3$ is a $\mathcal{C}^2$-diffeomorphism onto its image.

We also assume that the shell is made of a *homogeneous* and *isotropic linearly elastic material* characterized by its two *Lamé constants* $\lambda \geq 0$ and $\mu > 0$, that its reference configuration is a *natural state* and that it is subjected to *applied body forces* whose density per unit volume is defined by means of its *contravariant* components $f^{i,\varepsilon} \in L^2(\Omega^\varepsilon)$ and to a *homogeneous boundary condition of place* along the portion Γ_0^ε of its lateral face.

A commonly used *two-dimensional* set of equations for modeling such a shell ("two-dimensional" in the sense that the equations are posed over $\overline{\omega}$ instead of $\overline{\Omega^\varepsilon}$) was proposed in 1970 by W.T. Koiter in the seminal paper [14]. We now describe the modern formulation of this model.

Define the space

$$\boldsymbol{V}_K(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\},$$

where the symbol ∂_ν denotes the outer unit normal derivative operator along γ, and define the norm $\|\cdot\|_{\boldsymbol{V}_K(\omega)}$ by

$$\|\boldsymbol{\eta}\|_{\boldsymbol{V}_K(\omega)} := \left\{ \sum_\alpha \|\eta_\alpha\|_{1,\omega}^2 + \|\eta_3\|_{2,\omega}^2 \right\}^{1/2} \quad \text{for each } \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_K(\omega).$$

We then define the bilinear forms $B_M(\cdot,\cdot)$ and $B_F(\cdot,\cdot)$ by

$$B_M(\boldsymbol{\xi}, \boldsymbol{\eta}) := \int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\xi}) \gamma_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\, \mathrm{d}y,$$

$$B_F(\boldsymbol{\xi}, \boldsymbol{\eta}) := \frac{1}{3} \int_\omega a^{\alpha\beta\sigma\tau} \rho_{\sigma\tau}(\boldsymbol{\xi}) \rho_{\alpha\beta}(\boldsymbol{\eta}) \sqrt{a}\, \mathrm{d}y,$$

for each $\boldsymbol{\xi} \in \boldsymbol{V}_K(\omega)$ and each $\boldsymbol{\eta} \in \boldsymbol{V}_K(\omega)$, where

$$a^{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\sigma\tau} + 2\mu \left(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma} \right)$$

denote the contravariant components of the *two-dimensional elasticity tensor* of the shell, which is *uniformly positive-definite*,

$$\gamma_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}(\boldsymbol{a}_\alpha \cdot \partial_\beta \tilde{\boldsymbol{\eta}} + \partial_\alpha \tilde{\boldsymbol{\eta}} \cdot \boldsymbol{a}_\beta),$$
$$\rho_{\alpha\beta}(\boldsymbol{\eta}) := (\partial_{\alpha\beta} \tilde{\boldsymbol{\eta}} - \Gamma^\sigma_{\alpha\beta} \partial_\sigma \tilde{\boldsymbol{\eta}}) \cdot \boldsymbol{a}_3,$$

respectively denote for each $\boldsymbol{\eta} = (\eta_i)$ the *linearized change of metric*, and *linearized change of curvature*, *tensors* associated with an admissible displacement field $\tilde{\boldsymbol{\eta}} = \eta_i \boldsymbol{a}^i$ of the middle surface $\boldsymbol{\theta}(\overline{\omega})$ of the shell. We also define the linear form ℓ^ε by

$$\ell^\varepsilon(\boldsymbol{\eta}) := \int_\omega p^{i,\varepsilon} \eta_i \sqrt{a}\, \mathrm{d}y, \text{ for each } \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_K(\omega),$$

where $p^{i,\varepsilon}(y) := \int_{-\varepsilon}^{\varepsilon} f^{i,\varepsilon}(y, x_3^\varepsilon)\, \mathrm{d}x_3^\varepsilon$ at each $y \in \omega$.

Then the *total energy* of the shell is the *quadratic functional* $J : \boldsymbol{V}_K(\omega) \to \mathbb{R}$ defined by

$$J(\boldsymbol{\eta}) := \frac{\varepsilon}{2} B_M(\boldsymbol{\eta}, \boldsymbol{\eta}) + \frac{\varepsilon^3}{2} B_F(\boldsymbol{\eta}, \boldsymbol{\eta}) - \ell^\varepsilon(\boldsymbol{\eta}) \quad \text{for each } \boldsymbol{\eta} \in \boldsymbol{V}_K(\omega).$$

We assume that the shell is subjected to the following *confinement condition*: the unknown displacement field $\zeta^\varepsilon_{i,K} \boldsymbol{a}^i$ of the middle surface of the shell must be such that the *corresponding "deformed" middle surface remains in a given half-space*, *of the form*

$$\mathbb{H} := \{x \in \mathbb{E}^3; \boldsymbol{ox} \cdot \boldsymbol{p} \geq 0\},$$

where $\boldsymbol{p} \in \mathbb{E}^3$ *is a given non-zero vector*. It will be of course assumed that the "undeformed" middle surface satisfies this confinement condition, i.e. that $\boldsymbol{\theta}(\overline{\omega}) \subset \mathbb{H}$.

These assumptions lead to the following definition of a problem, denoted $\mathcal{P}^\varepsilon_K(\omega)$ in the next theorem, which constitutes our proposed *Koiter's model for a shell subjected to a confinement condition*.

Theorem 2.1. *The minimization problem: find*

$$\boldsymbol{\zeta}^\varepsilon_K \in \boldsymbol{U}_K(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_K(\omega); (\boldsymbol{\theta}(y) + \eta_i(y)\boldsymbol{a}^i(y)) \cdot \boldsymbol{p} \geq 0 \text{ for a.a. } y \in \omega\}$$

such that

$$J(\boldsymbol{\zeta}^\varepsilon_K) = \inf_{\boldsymbol{\eta} \in \boldsymbol{U}_K(\omega)} J(\boldsymbol{\eta})$$

has one and only one solution.

Besides, this solution is also the unique solution to problem $\mathcal{P}^\varepsilon_K(\omega)$*: find* $\boldsymbol{\zeta}^\varepsilon_K \in \boldsymbol{U}_K(\omega)$ *that satisfies the following variational inequalities:*

$$\varepsilon B_M(\boldsymbol{\zeta}^\varepsilon_K, \boldsymbol{\eta} - \boldsymbol{\zeta}^\varepsilon_K) + \varepsilon^3 B_F(\boldsymbol{\zeta}^\varepsilon_K, \boldsymbol{\eta} - \boldsymbol{\zeta}^\varepsilon_K) \geq \ell^\varepsilon(\boldsymbol{\eta} - \boldsymbol{\zeta}^\varepsilon_K) \quad \text{for all } \boldsymbol{\eta} \in \boldsymbol{U}_K(\omega).$$

Proof. The bilinear forms $B_M : \boldsymbol{V}_K(\omega) \times \boldsymbol{V}_K(\omega) \to \mathbb{R}$ and $B_F : \boldsymbol{V}_K(\omega) \times \boldsymbol{V}_K(\omega) \to \mathbb{R}$ and the linear form $\ell^\varepsilon : \boldsymbol{V}_K(\omega) \to \mathbb{R}$ are clearly continuous. Since the two-dimensional elasticity tensor $(a^{\alpha\beta\sigma\tau})$ is uniformly positive-definite, the bilinear form

$$(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \boldsymbol{V}_K(\omega) \times \boldsymbol{V}_K(\omega) \to \varepsilon B_M(\boldsymbol{\xi}, \boldsymbol{\eta}) + \varepsilon^3 B_F(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{R}$$

is $\boldsymbol{V}_K(\omega)$*-elliptic*, thanks to the *Korn's inequality on a general surface* recalled in Theorem 5.1 below (note that this inequality holds in particular because $d\gamma$-meas $\gamma_0 > 0$), on the one hand.

On the other hand, it is easily seen that $\boldsymbol{U}_K(\omega)$ is a *non-empty*, *closed*, and *convex* subset of the space $\boldsymbol{V}_K(\omega)$.

Hence, the above minimization problem, or equivalently problem $\mathcal{P}^\varepsilon_K(\omega)$, has one and only one solution. □

Depending on the type of shell under consideration and on the assumptions on the data, we study in what follows the behavior of the solution to problem $\mathcal{P}^\varepsilon_K(\omega)$ as $\varepsilon \to 0$.

3. Koiter's model for an elliptic membrane shell subjected to a confinement condition

Consider a *linearly elastic elliptic membrane shell* in the sense of Section 4.1 of [4]. The space $\boldsymbol{V}_K(\omega)$ introduced in Section 2 now reduces to

$$\boldsymbol{V}_K(\omega) = H^1_0(\omega) \times H^1_0(\omega) \times H^2_0(\omega).$$

To begin with, we recall a *Korn inequality on an elliptic surface*, which is due to [7] and [13] (see also [4, Theorem 2.7-3]).

Theorem 3.1. *Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ be an immersion such that $\boldsymbol{\theta}(\overline{\omega})$ is an elliptic surface. Define the space*

$$\boldsymbol{V}_M(\omega) := H_0^1(\omega) \times H_0^1(\omega) \times L^2(\omega),$$

and the norm $\|\cdot\|_{\boldsymbol{V}_M(\omega)}$ by

$$\|\boldsymbol{\eta}\|_{\boldsymbol{V}_M(\omega)} := \left\{ \sum_\alpha \|\eta_\alpha\|_{1,\omega}^2 + \|\eta_3\|_{0,\omega}^2 \right\}^{1/2} \quad \text{for each } \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_M(\omega).$$

Then there exists a constant $c > 0$ such that

$$\|\boldsymbol{\eta}\|_{\boldsymbol{V}_M(\omega)} \leq c \left\{ \sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|_{0,\omega}^2 \right\}^{1/2}$$

for all $\boldsymbol{\eta} \in \boldsymbol{V}_M(\omega)$.

The forthcoming analysis resorts to an argument similar to that in Theorem 7.2-1 of [4] (itself based on various ideas found in [6], [15], [3] and, especially, on Theorem 2.1 in [8]), and constitutes the *first convergence result* of this Note. The set $\boldsymbol{U}_K(\omega)$ appearing in the next theorem is defined in Theorem 2.1.

Theorem 3.2. *Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ be an immersion. Consider a family of elliptic membrane shells with thickness 2ε approaching zero and with each having the same middle surface $\boldsymbol{\theta}(\overline{\omega})$, and assume that there exist functions $f^i \in L^2(\Omega)$ independent of ε such that*

$$f^{i,\varepsilon}(x^\varepsilon) = f^i(x) \text{ at each } x^\varepsilon \in \Omega^\varepsilon \text{ for each } \varepsilon > 0.$$

Define the set

$$\boldsymbol{U}_M(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_M(\omega);\ \Big(\boldsymbol{\theta}(y) + \eta_i(y)\boldsymbol{a}^i(y)\Big) \cdot \boldsymbol{p} \geq 0 \text{ for a.a. } y \in \omega\},$$

and assume that the following "density property" holds:

$$\boldsymbol{U}_K(\omega) \text{ is dense in } \boldsymbol{U}_M(\omega) \text{ with respect to the norm } \|\cdot\|_{\boldsymbol{V}_M(\omega)}.$$

For each $\varepsilon > 0$, let $\boldsymbol{\zeta}_K^\varepsilon$ denote the solution to problem $\mathcal{P}_K^\varepsilon(\omega)$ (Theorem 2.1)*. Then the following convergences hold:*

$$\zeta_{\alpha,K}^\varepsilon \boldsymbol{a}^\alpha \to \zeta_\alpha \boldsymbol{a}^\alpha \text{ in } \boldsymbol{H}^1(\omega) \text{ as } \varepsilon \to 0,$$
$$\zeta_{3,K}^\varepsilon \boldsymbol{a}^3 \to \zeta_3 \boldsymbol{a}^3 \text{ in } \boldsymbol{L}^2(\omega) \text{ as } \varepsilon \to 0,$$

where $\boldsymbol{\zeta}$ is the unique solution to the following problem $\mathcal{P}(\omega)$: find

$$\boldsymbol{\zeta} \in \boldsymbol{U}_M(\omega) = \{\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_M(\omega);\ \Big(\boldsymbol{\theta}(y) + \eta_i(y)\boldsymbol{a}^i(y)\Big) \cdot \boldsymbol{p} \geq 0 \text{ for a.a. } y \in \omega\}$$

that satisfies the following variational inequalities:

$$\int_\omega a^{\alpha\beta\sigma\tau} \gamma_{\sigma\tau}(\boldsymbol{\zeta}) \gamma_{\alpha\beta}(\boldsymbol{\eta} - \boldsymbol{\zeta}) \sqrt{a}\, \mathrm{d}y \geq \int_\omega p^i (\eta_i - \zeta_i) \sqrt{a}\, \mathrm{d}y \quad \text{for all } \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{U}_M(\omega),$$

where

$$p^i := \int_{-1}^{1} f^i \, \mathrm{d}x_3 \in L^2(\omega).$$

Sketch of proof. In what follows, strong and weak convergences are respectively denoted $\to$ and $\rightharpoonup$.

Combining the $\boldsymbol{V}_M(\omega)$-ellipticity of the continuous bilinear form $B_M(\cdot,\cdot)$, the Korn's inequality of Theorem 3.1, the observation that $\boldsymbol{U}_M(\omega)$ is a non-empty closed and convex subset of $\boldsymbol{V}_M(\omega)$, and the continuity of the linear form ℓ defined by

$$\ell(\boldsymbol{\eta}) := \int_\omega p^i \eta_i \sqrt{a}\, \mathrm{d}y \quad \text{for all } \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_M(\omega),$$

we infer that problem $\mathcal{P}_M(\omega)$ has one and only one solution.

Thanks to the uniform positive-definiteness of the tensor $(a^{\alpha\beta\sigma\tau})$ and to Theorem 3.1, there exists a constant $C_1 > 0$ such that

$$\|\boldsymbol{\zeta}_K^\varepsilon\|^2_{\boldsymbol{V}_M(\omega)} \leq C_1 B_M(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\zeta}_K^\varepsilon).$$

Besides, the continuity of the bilinear forms $B_M(\cdot,\cdot)$ and $B_F(\cdot,\cdot)$ and the continuity of the linear form ℓ imply that there exists a constant $C_2 > 0$ such that

$$\begin{aligned}
&B_M(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\eta}) + \varepsilon^2 B_F(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\eta}) - \ell(\boldsymbol{\eta} - \boldsymbol{\zeta}_K^\varepsilon) \\
&\leq C_2(\|\boldsymbol{\zeta}_K^\varepsilon\|_{\boldsymbol{V}_M(\omega)}\|\boldsymbol{\eta}\|_{\boldsymbol{V}_M(\omega)} + \varepsilon^2\|\boldsymbol{\zeta}_K^\varepsilon\|_{\boldsymbol{V}_M(\omega)}\|\boldsymbol{\eta}\|_{\boldsymbol{V}_M(\omega)} + \|\boldsymbol{\zeta}_K^\varepsilon\|_{\boldsymbol{V}_M(\omega)} + \|\boldsymbol{\eta}\|_{\boldsymbol{V}_M(\omega)}),
\end{aligned}$$

for all $\boldsymbol{\eta} \in \boldsymbol{U}_K(\omega)$. Letting $\boldsymbol{\eta} = \boldsymbol{0}$ thus gives

$$\|\boldsymbol{\zeta}_K^\varepsilon\|_{\boldsymbol{V}_M(\omega)} \leq C_1 C_2 \quad \text{for all } \varepsilon > 0.$$

Therefore, there exists a subsequence, still denoted $(\boldsymbol{\zeta}_K^\varepsilon)_{\varepsilon>0}$, a vector field $\boldsymbol{\zeta}^* \in \boldsymbol{V}_M(\omega)$, and functions $\rho_{\alpha\beta}^{-1} \in L^2(\omega)$, such that:

$$\begin{aligned}
\boldsymbol{\zeta}_K^\varepsilon &\rightharpoonup \boldsymbol{\zeta}^* \quad \text{in } \boldsymbol{V}_M(\omega) \text{ as } \varepsilon \to 0, \\
\varepsilon\rho_{\alpha\beta}(\boldsymbol{\zeta}_K^\varepsilon) &\rightharpoonup \rho_{\alpha\beta}^{-1} \quad \text{in } L^2(\omega) \text{ as } \varepsilon \to 0,
\end{aligned}$$

the second convergence being a consequence of the uniform positive-definiteness of the tensor $(a^{\alpha\beta\sigma\tau})$. Then $\boldsymbol{\zeta}^* \in \boldsymbol{U}_M(\omega)$, since the set $\boldsymbol{U}_M(\omega)$ is non-empty closed and convex.

Simple computations yield

$$B_M(\boldsymbol{\zeta}^*, \boldsymbol{\eta} - \boldsymbol{\zeta}^*) \geq \ell(\boldsymbol{\eta} - \boldsymbol{\zeta}^*) \quad \text{for all } \boldsymbol{\eta} \in \boldsymbol{U}_K(\omega).$$

Furthermore, the assumed "density property" gives

$$B_M(\boldsymbol{\zeta}^*, \boldsymbol{\eta} - \boldsymbol{\zeta}^*) \geq \ell(\boldsymbol{\eta} - \boldsymbol{\zeta}^*) \text{ for all } \boldsymbol{\eta} \in \boldsymbol{U}_M(\omega),$$

which implies that $\boldsymbol{\zeta} = \boldsymbol{\zeta}^*$ and that the *whole* family $(\boldsymbol{\zeta}_K^\varepsilon)_{\varepsilon>0}$ weakly converges to $\boldsymbol{\zeta}$ in $\boldsymbol{V}_M(\omega)$ as $\varepsilon \to 0$.

The $\boldsymbol{V}_K(\omega)$-ellipticity of the bilinear form $B_M(\cdot,\cdot)$ and the assumed "density property" together give

$$0 \leq \frac{1}{C_1}\|\boldsymbol{\zeta}_K^\varepsilon - \boldsymbol{\zeta}\|^2_{\boldsymbol{V}_M(\omega)} \leq B_M(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\eta}) + \varepsilon^2 B_F(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\eta}) - \ell(\boldsymbol{\eta} - \boldsymbol{\zeta}_K^\varepsilon) - 2B_M(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\zeta}) + B_M(\boldsymbol{\zeta}, \boldsymbol{\zeta}),$$

for all $\boldsymbol{\eta} \in \boldsymbol{U}_M(\omega)$. Hence, letting $\boldsymbol{\eta} = \boldsymbol{\zeta}$ and letting $\varepsilon \to 0$ gives

$$\boldsymbol{\zeta}_K^\varepsilon \to \boldsymbol{\zeta} \quad \text{in } \boldsymbol{V}_M(\omega) \text{ as } \varepsilon \to 0. \quad \square$$

Note that realistic sufficient conditions insuring that the "density property" holds are given in [10] (see also [11]).

4. Koiter's model for a generalized membrane of the "first kind" subjected to a confinement condition

Consider a *linearly elastic generalized membrane shell "of the first kind"* subjected to *"admissible" applied forces*, in the sense of Section 5.1 of [4]. The forthcoming analysis resorts to an argument similar to that in Theorem 7.2-2 of [4] (itself based on [3] and, especially, Theorems 6.1 and 6.2 of [9]) and constitutes the *second convergence result* of this Note.

Theorem 4.1. *Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ be an immersion. Consider a family of generalized membrane shells "of the first kind" with thickness 2ε approaching zero and with each having the same middle surface $\boldsymbol{\theta}(\overline{\omega})$, and assume that each shell is subject to a boundary condition of place along a portion of its lateral face, whose middle curve is the set $\boldsymbol{\theta}(\gamma_0)$. Define the spaces*

$$\begin{aligned}
\boldsymbol{V}(\omega) &:= \{\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{H}^1(\omega);\ \boldsymbol{\eta} = \boldsymbol{0} \text{ on } \gamma_0\}, \\
\boldsymbol{V}_M^\sharp(\omega) &:= \text{completion of } \boldsymbol{V}(\omega) \text{ with respect to } |\cdot|_\omega^M.
\end{aligned}$$

For each $\varepsilon > 0$, let $\boldsymbol{\zeta}_K^\varepsilon$ denote the solution to problem $\mathcal{P}_K^\varepsilon(\omega)$ (Theorem 2.1)*. Then the following convergence holds:*

$$\boldsymbol{\zeta}_K^\varepsilon \to \boldsymbol{\zeta} \quad \text{in } \boldsymbol{V}_M^\sharp(\omega) \text{ as } \varepsilon \to 0,$$

where $\boldsymbol{\zeta}$ denotes the unique solution to problem $\mathcal{P}_M^\sharp(\omega)$: find

$$\boldsymbol{\zeta} \in \boldsymbol{U}^\sharp(\omega) := \text{closure of } \boldsymbol{U}_K(\omega) \text{ with respect to } |\cdot|_\omega^M$$

that satisfies the following variational inequalities:

$$B_M^\sharp(\boldsymbol{\zeta}, \boldsymbol{\eta} - \boldsymbol{\zeta}) \geq L_M^\sharp(\boldsymbol{\eta} - \boldsymbol{\zeta}) \quad \text{for all } \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{U}^\sharp(\omega),$$

where $B_M^\sharp(\cdot,\cdot)$ *and* $L_M^\sharp$ *designate the unique continuous linear extensions from* $\boldsymbol{V}(\omega)$ *to* $\boldsymbol{V}_M^\sharp(\omega)$ *of the bilinear form* $B_M(\cdot,\cdot)$*, and of the linear form* L_M *defined by*

$$L_M(\boldsymbol{\eta}) := \int_\omega \varphi^{\alpha\beta}\gamma_{\alpha\beta}(\boldsymbol{\eta})\sqrt{a}\,\mathrm{d}y \quad \text{for all } \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}(\omega),$$

where $\varphi^{\alpha\beta} = \varphi^{\beta\alpha} \in L^2(\omega)$ *are the functions entering the definition of admissible applied forces.*

Sketch of proof. Following [3] (see also Theorem 7.2-2 of [4]), we first observe that the space $\boldsymbol{V}_M^\sharp(\omega)$ is also the completion of the space $\boldsymbol{V}_K(\omega)$ with respect to the norm $|\cdot|_\omega^M$. Clearly, problem $\mathcal{P}_M^\sharp(\omega)$ admits a unique solution, since the bilinear form $B_M^\sharp(\cdot,\cdot)$ is continuous and $\boldsymbol{V}_M^\sharp(\omega)$-elliptic (recall that the tensor $(a^{\alpha\beta\sigma\tau})$ is uniformly positive-definite), the set $\boldsymbol{U}^\sharp(\omega)$ is non-empty, closed with respect to $|\cdot|_\omega^M$, and convex, and the linear form $L_M^\sharp$ is continuous.

Because the applied body forces are admissible, the variational inequalities appearing in problem $\mathcal{P}_K^\varepsilon(\omega)$ read

$$B_M(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\eta} - \boldsymbol{\zeta}_K^\varepsilon) + \varepsilon^2 B_F(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\eta} - \boldsymbol{\zeta}_K^\varepsilon) \geq L_M(\boldsymbol{\eta} - \boldsymbol{\zeta}_K^\varepsilon) \quad \text{for all } \boldsymbol{\eta} \in \boldsymbol{U}_K(\omega).$$

By virtue of the continuity of the linear form L_M with respect to the norm $|\cdot|_\omega^M$, there exists a constant $C_1 > 0$ such that

$$B_M(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\zeta}_K^\varepsilon) + \varepsilon^2 B_F(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\zeta}_K^\varepsilon) \leq B_M(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\eta}) + \varepsilon^2 B_F(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\eta}) + C_1|\boldsymbol{\eta} - \boldsymbol{\zeta}_K^\varepsilon|_\omega^M \quad \text{for all } \boldsymbol{\eta} \in \boldsymbol{U}_K(\omega).$$

Thanks to the uniform positive-definiteness of the tensor $(a^{\alpha\beta\sigma\tau})$, there exists a constant $C_2 > 0$ such that

$$(|\boldsymbol{\zeta}_K^\varepsilon|_\omega^M)^2 \leq C_2 B_M(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\zeta}_K^\varepsilon) \quad \text{and} \quad \sum_{\alpha,\beta} |\varepsilon\rho_{\alpha\beta}(\boldsymbol{\zeta}_K^\varepsilon)|_{0,\omega}^2 \leq C_2\varepsilon^2 B_F(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\zeta}_K^\varepsilon).$$

Hence, letting $\boldsymbol{\eta} = \boldsymbol{0}$ gives

$$(|\boldsymbol{\zeta}_K^\varepsilon|_\omega^M)^2 + \frac{1}{3}\sum_{\alpha,\beta} |\varepsilon\rho_{\alpha\beta}(\boldsymbol{\zeta}_K^\varepsilon)|_{0,\omega}^2 \leq C_1 C_2 |\boldsymbol{\zeta}_K^\varepsilon|_\omega^M.$$

Therefore, there exist a subsequence, still denoted $(\boldsymbol{\zeta}_K^\varepsilon)_{\varepsilon>0}$, a vector field $\boldsymbol{\zeta}^\sharp \in \boldsymbol{V}_M^\sharp(\omega)$, and functions $\rho_{\alpha\beta}^{-1} \in L^2(\omega)$, such that

$$\begin{aligned} \boldsymbol{\zeta}_K^\varepsilon &\rightharpoonup \boldsymbol{\zeta}^\sharp \quad \text{in } \boldsymbol{V}_M^\sharp(\omega) \text{ as } \varepsilon \to 0, \\ \varepsilon\rho_{\alpha\beta}(\boldsymbol{\zeta}_K^\varepsilon) &\rightharpoonup \rho_{\alpha\beta}^{-1} \quad \text{in } L^2(\omega) \text{ as } \varepsilon \to 0. \end{aligned}$$

Clearly, the vector field $\boldsymbol{\zeta}^\sharp$ belongs to the set $\boldsymbol{U}^\sharp(\omega)$. Simple computations yield

$$B_M^\sharp(\boldsymbol{\zeta}^\sharp, \boldsymbol{\eta} - \boldsymbol{\zeta}^\sharp) \geq L_M^\sharp(\boldsymbol{\eta} - \boldsymbol{\zeta}^\sharp) \quad \text{for all } \boldsymbol{\eta} \in \boldsymbol{U}^\sharp(\omega),$$

which implies that $\boldsymbol{\zeta}^\sharp = \boldsymbol{\zeta}$ and that the *whole* family $(\boldsymbol{\zeta}_K^\varepsilon)_{\varepsilon>0}$ weakly converges to $\boldsymbol{\zeta}$ in $\boldsymbol{V}_M^\sharp(\omega)$ as $\varepsilon \to 0$.

The positive definiteness of the two-dimensional fourth-order elasticity tensor of the shell together with the definition of the norm $|\cdot|_\omega^M$ and of the bilinear form $B_M(\cdot,\cdot)$ and its extension $B_M^\sharp(\cdot,\cdot)$ show that establishing the announced strong convergence is thus equivalent to establishing the convergence

$$B_M^\sharp(\boldsymbol{\zeta}_K^\varepsilon - \boldsymbol{\zeta}, \boldsymbol{\zeta}_K^\varepsilon - \boldsymbol{\zeta}) \to 0 \quad \text{as } \varepsilon \to 0.$$

Noting that the weak convergence $\boldsymbol{\zeta}_K^\varepsilon \rightharpoonup \boldsymbol{\zeta}$ in $\boldsymbol{V}_M^\sharp(\omega)$ as $\varepsilon \to 0$ gives

$$\limsup_{\varepsilon\to 0} B_M^\sharp(\boldsymbol{\zeta}_K^\varepsilon - \boldsymbol{\zeta}, \boldsymbol{\zeta}_K^\varepsilon - \boldsymbol{\zeta}) = 0$$

we infer that the strong convergence

$$\boldsymbol{\zeta}_K^\varepsilon \to \boldsymbol{\zeta} \quad \text{in } \boldsymbol{V}_M^\sharp(\omega) \text{ as } \varepsilon \to 0$$

holds. □

5. Koiter's model for a flexural shell subjected to a confinement condition

Consider a *linearly elastic flexural shell* in the sense of Section 6.1 of [4]. To begin with, we recall an example of a *Korn's inequality on a general surface*, which is due to [1] and was later on improved by [2] (see also [4, Theorem 2.6-4]).

Theorem 5.1. *Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ be an immersion. Let γ_0 be a non-empty relatively open subset of γ. Define the space*

$$\boldsymbol{V}_K(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega); \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\}.$$

Then there exists a constant $c > 0$ such that

$$\left\{\sum_\alpha \|\eta_\alpha\|^2_{1,\omega} + \|\eta_3\|^2_{2,\omega}\right\}^{1/2} \leq c \left\{\sum_{\alpha,\beta} \|\gamma_{\alpha\beta}(\boldsymbol{\eta})\|^2_{0,\omega} + \sum_{\alpha,\beta} \|\rho_{\alpha\beta}(\boldsymbol{\eta})\|^2_{0,\omega}\right\}^{1/2}$$

for all $\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_K(\omega)$.

The forthcoming analysis resorts to an argument similar to that used in Theorem 7.2-3 of [4] (itself based on [15] and, especially, on Theorem 2.2 of [8]), and constitutes the *third convergence result* of this Note.

Theorem 5.2. *Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ be an immersion. Consider a family of flexural shells with thickness 2ε approaching zero and with each having the same middle surface $\boldsymbol{\theta}(\overline{\omega})$. Let γ_0 be a non-empty relatively open subset of γ, and assume that each shell is subject to a boundary condition of place along a portion of its lateral face, whose middle curve is the set $\boldsymbol{\theta}(\gamma_0)$. Finally, assume that there exist functions $f^i \in L^2(\Omega)$ independent of ε such that:*

$$f^{i,\varepsilon}(x^\varepsilon) = \varepsilon^2 f^i(x) \text{ at each } x^\varepsilon \in \Omega^\varepsilon \text{ for each } \varepsilon > 0.$$

For each $\varepsilon > 0$, let $\boldsymbol{\zeta}^\varepsilon_K$ denote the solution to problem $\mathcal{P}^\varepsilon_K(\omega)$ (Theorem 2.1)*. Then the following convergence holds:*

$$\boldsymbol{\zeta}^\varepsilon_K \to \boldsymbol{\zeta} \quad \text{in } \boldsymbol{V}_K(\omega) \text{ as } \varepsilon \to 0,$$

where $\boldsymbol{\zeta}$ denotes the solution to problem $\mathcal{P}_F(\omega)$: find

$$\boldsymbol{\zeta} \in \boldsymbol{U}_F(\omega) := \{\boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_F(\omega); \Big(\boldsymbol{\theta}(y) + \eta_i(y)\boldsymbol{a}^i(y)\Big) \cdot \boldsymbol{p} \geq 0 \text{ for a.a. } y \in \omega\}$$

that satisfies the following variational inequalities:

$$\frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta} - \boldsymbol{\zeta})\sqrt{a}\,\mathrm{d}y \geq \int_\omega p^i(\eta_i - \zeta_i)\sqrt{a}\,\mathrm{d}y \quad \text{for all } \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{U}_F(\omega),$$

where

$$p^i := \int_{-1}^{1} f^i \,\mathrm{d}x_3 \in L^2(\omega).$$

Sketch of proof. Since the bilinear form $B_F(\cdot,\cdot)$ is continuous and $\boldsymbol{V}_F(\omega)$-elliptic, the set $\boldsymbol{U}_F(\omega)$ is a non-empty, closed, and convex, subset of $\boldsymbol{V}_F(\omega)$, and the linear form defined by

$$\ell(\boldsymbol{\eta}) := \int_\omega p^i \eta_i \sqrt{a}\,\mathrm{d}y \quad \text{for all } \boldsymbol{\eta} = (\eta_i) \in \boldsymbol{V}_F(\omega)$$

is continuous over $\boldsymbol{V}_F(\omega)$, problem $\mathcal{P}_F(\omega)$ admits a unique solution.

By virtue of the assumption on the applied body forces, the variational inequalities in problem $\mathcal{P}^\varepsilon_K(\omega)$ read

$$\frac{1}{\varepsilon^2}B_M(\boldsymbol{\zeta}^\varepsilon_K, \boldsymbol{\eta} - \boldsymbol{\zeta}^\varepsilon_K) + B_F(\boldsymbol{\zeta}^\varepsilon_K, \boldsymbol{\eta} - \boldsymbol{\zeta}^\varepsilon_K) \geq \ell(\boldsymbol{\eta} - \boldsymbol{\zeta}^\varepsilon_K) \quad \text{for all } \boldsymbol{\eta} \in \boldsymbol{U}_K(\omega).$$

By the continuity of the linear form ℓ, there exists a constant $C_1 > 0$ such that

$$\frac{1}{\varepsilon^2}B_M(\boldsymbol{\zeta}^\varepsilon_K, \boldsymbol{\zeta}^\varepsilon_K) + B_F(\boldsymbol{\zeta}^\varepsilon_K, \boldsymbol{\zeta}^\varepsilon_K) \leq \frac{1}{\varepsilon^2}B_M(\boldsymbol{\zeta}^\varepsilon_K, \boldsymbol{\eta}) + B_F(\boldsymbol{\zeta}^\varepsilon_K, \boldsymbol{\eta}) + C_1\|\boldsymbol{\eta} - \boldsymbol{\zeta}^\varepsilon_K\|_{\boldsymbol{V}_K(\omega)} \quad \text{for all } \boldsymbol{\eta} \in \boldsymbol{U}_K(\omega).$$

Thanks to the uniform positive-definiteness of the tensor $(a^{\alpha\beta\sigma\tau})$ and the Korn's inequality on a general surface (Theorem 5.1), there exists a constant $C_2 > 0$ such that

$$\|\boldsymbol{\zeta}_K^\varepsilon\|^2_{\boldsymbol{V}_K(\omega)} \le C_2 \left(\frac{1}{\varepsilon^2} B_M(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\zeta}_K^\varepsilon) + B_F(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\zeta}_K^\varepsilon) \right) \quad \text{for all } 0 < \varepsilon \le 1.$$

Hence, combining the two inequalities above and letting $\boldsymbol{\eta} = \boldsymbol{0}$ gives

$$\|\boldsymbol{\zeta}_K^\varepsilon\|_{\boldsymbol{V}_K(\omega)} \le C_1 C_2.$$

Therefore, there exist a subfamily, still denoted $(\boldsymbol{\zeta}_K^\varepsilon)_{\varepsilon>0}$, a vector field $\boldsymbol{\zeta}^* \in \boldsymbol{V}_K(\omega)$, and functions $\gamma^{-1}_{\alpha\beta} \in L^2(\omega)$, such that

$$\begin{aligned} \boldsymbol{\zeta}_K^\varepsilon &\rightharpoonup \boldsymbol{\zeta}^* \quad \text{in } \boldsymbol{V}_K(\omega) \text{ as } \varepsilon \to 0, \\ \frac{1}{\varepsilon}\gamma_{\alpha\beta}(\boldsymbol{\zeta}_K^\varepsilon) &\rightharpoonup \gamma^{-1}_{\alpha\beta} \quad \text{in } L^2(\omega) \text{ as } \varepsilon \to 0, \end{aligned}$$

the second weak convergence being a consequence of the uniform positive-definiteness of the tensor $(a^{\alpha\beta\sigma\tau})$. The same weak convergence in turn implies that

$$\gamma_{\alpha\beta}(\boldsymbol{\zeta}_K^\varepsilon) \to 0 \text{ in } L^2(\omega) \text{ as } \varepsilon \to 0$$

on the one hand. On the other hand, the weak convergence $\boldsymbol{\zeta}_K^\varepsilon \rightharpoonup \boldsymbol{\zeta}^*$ in $\boldsymbol{V}_K(\omega)$ as $\varepsilon \to 0$ clearly implies that

$$\gamma_{\alpha\beta}(\boldsymbol{\zeta}_K^\varepsilon) \rightharpoonup \gamma_{\alpha\beta}(\boldsymbol{\zeta}^*) \text{ as } \varepsilon \to 0.$$

Hence, the uniqueness of the weak limit shows that

$$\gamma_{\alpha\beta}(\boldsymbol{\zeta}^*) = 0.$$

In conclusion, $\boldsymbol{\zeta}^*$ belongs to $\boldsymbol{U}_F(\omega)$. Simple computations give

$$B_F(\boldsymbol{\zeta}^*, \boldsymbol{\eta} - \boldsymbol{\zeta}^*) \ge \ell(\boldsymbol{\eta} - \boldsymbol{\zeta}^*) \quad \text{for all } \boldsymbol{\eta} \in \boldsymbol{U}_F(\omega).$$

Hence we conclude that $\boldsymbol{\zeta} = \boldsymbol{\zeta}^*$ and that the *whole* family $(\boldsymbol{\zeta}_K^\varepsilon)_{\varepsilon>0}$ weakly converges to $\boldsymbol{\zeta}$ in $\boldsymbol{V}_K(\omega)$ as $\varepsilon \to 0$.

Combining the Korn's inequality on a general surface (Theorem 5.1), the strong convergence $\gamma_{\alpha\beta}(\boldsymbol{\zeta}_K^\varepsilon) \to \gamma_{\alpha\beta}(\boldsymbol{\zeta}) = 0$ in $L^2(\omega)$, and the uniform positive-definiteness of the tensor $(a^{\alpha\beta\sigma\tau})$, establishing the strong convergence $\boldsymbol{\zeta}_K^\varepsilon \to \boldsymbol{\zeta}$ in $\boldsymbol{V}_K(\omega)$ amounts to establishing that $B_F(\boldsymbol{\zeta}_K^\varepsilon - \boldsymbol{\zeta}, \boldsymbol{\zeta}_K^\varepsilon - \boldsymbol{\zeta}) \to 0$ as $\varepsilon \to 0$.

Noting that we have

$$\begin{aligned} 0 &\le B_F(\boldsymbol{\zeta}_K^\varepsilon - \boldsymbol{\zeta}, \boldsymbol{\zeta}_K^\varepsilon - \boldsymbol{\zeta}) = B_F(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\zeta}_K^\varepsilon) - 2B_F(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\zeta}) + B_F(\boldsymbol{\zeta}, \boldsymbol{\zeta}) \\ &\le B_F(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\zeta}) - \ell(\boldsymbol{\zeta} - \boldsymbol{\zeta}_K^\varepsilon) - 2B_F(\boldsymbol{\zeta}_K^\varepsilon, \boldsymbol{\zeta}) + B_F(\boldsymbol{\zeta}, \boldsymbol{\zeta}), \end{aligned}$$

since $\boldsymbol{\zeta} \in \boldsymbol{U}_F(\omega)$, we thus conclude that

$$\boldsymbol{\zeta}_K^\varepsilon \to \boldsymbol{\zeta} \quad \text{in } \boldsymbol{V}_K(\omega) \text{ as } \varepsilon \to 0. \quad \square$$

6. Justification of the proposed model for elliptic membrane shells

Consider the following *obstacle problem* for a "general" shell whose reference configuration is the set $\boldsymbol{\Theta}(\overline{\Omega^\varepsilon})$; cf. Section 2. We *assume* that the shell is subjected to a *confinement condition*, expressing that any *admissible displacement vector field* $v_i^\varepsilon \boldsymbol{g}^{i,\varepsilon}$ must be such that the corresponding deformed configuration remains in the same *half-space* as in Section 2, i.e.

$$\mathbb{H} := \{x \in \mathbb{E}^3;\ \boldsymbol{ox} \cdot \boldsymbol{p} \ge 0\}.$$

In other words, the covariant components v_i^ε of the admissible three-dimensional displacement field $v_i^\varepsilon \boldsymbol{g}^{i,\varepsilon}$ of the reference configuration must satisfy the following "constraint":

$$\Big(\boldsymbol{\Theta}(x^\varepsilon) + v_i^\varepsilon(x^\varepsilon)\boldsymbol{g}^{i,\varepsilon}(x^\varepsilon)\Big) \cdot \boldsymbol{p} \ge 0 \quad \text{for all } x^\varepsilon \in \overline{\Omega^\varepsilon},$$

or possibly only almost everywhere in Ω^ε.

We of course assume that the reference configuration satisfies the confinement condition, i.e. that

$$\boldsymbol{\Theta}(\overline{\Omega^\varepsilon}) \subset \mathbb{H}.$$

The unknown of the considered problem, which is the vector field $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon)$, where the functions u_i^ε are the covariant components of the unknown displacement field $u_i^\varepsilon \boldsymbol{g}^{i,\varepsilon}$ of the reference configuration, should minimize the *energy* $J^\varepsilon : \boldsymbol{H}^1(\Omega^\varepsilon) \to \mathbb{R}$ defined by

$$J^\varepsilon(\boldsymbol{v}^\varepsilon) := \frac{1}{2}\int_{\Omega^\varepsilon} A^{ijk\ell,\varepsilon} e^\varepsilon_{k\|\ell}(\boldsymbol{v}^\varepsilon) e^\varepsilon_{i\|j}(\boldsymbol{v}^\varepsilon)\sqrt{g^\varepsilon}\,\mathrm{d}x^\varepsilon - \int_{\Omega^\varepsilon} f^{i,\varepsilon} v_i^\varepsilon \sqrt{g^\varepsilon}\,\mathrm{d}x^\varepsilon,$$

when $\boldsymbol{v}^\varepsilon = (v_i^\varepsilon)$ varies over the following set:

$$\boldsymbol{U}(\Omega^\varepsilon) := \{\boldsymbol{v}^\varepsilon = (v_i^\varepsilon) \in \boldsymbol{H}^1(\Omega^\varepsilon);\ \boldsymbol{v}^\varepsilon = \boldsymbol{0} \text{ on } \Gamma_0^\varepsilon, \Big(\boldsymbol{\Theta}(x^\varepsilon) + v_i^\varepsilon(x^\varepsilon)\boldsymbol{g}^{i,\varepsilon}(x^\varepsilon)\Big)\cdot \boldsymbol{p} \geq 0 \text{ for a.a. } x^\varepsilon \in \Omega^\varepsilon\}.$$

The solution to this *minimization problem* exists and is unique, and it can be also characterized as the unique solution to a set of appropriate *variational inequalities* as shown in the next theorem.

Theorem 6.1. *The quadratic minimization problem: Find a vector field $\boldsymbol{u}^\varepsilon \in \boldsymbol{U}(\Omega^\varepsilon)$ such that*

$$J^\varepsilon(\boldsymbol{u}^\varepsilon) = \inf_{\boldsymbol{v}^\varepsilon \in \boldsymbol{U}(\Omega^\varepsilon)} J^\varepsilon(\boldsymbol{v}^\varepsilon)$$

has one and only one solution, which is also the unique solution to the variational problem $\mathcal{P}(\Omega^\varepsilon)$: find

$$\boldsymbol{u}^\varepsilon \in \boldsymbol{U}(\Omega^\varepsilon)$$

which satisfies the following variational inequalities:

$$\int_{\Omega^\varepsilon} A^{ijk\ell,\varepsilon} e^\varepsilon_{k\|\ell}(\boldsymbol{u}^\varepsilon)\Big(e^\varepsilon_{i\|j}(\boldsymbol{v}^\varepsilon) - e^\varepsilon_{i\|j}(\boldsymbol{u}^\varepsilon)\Big)\sqrt{g^\varepsilon}\,\mathrm{d}x^\varepsilon \geq \int_{\Omega^\varepsilon} f^{i,\varepsilon}(v_i^\varepsilon - u_i^\varepsilon)\sqrt{g^\varepsilon}\,\mathrm{d}x^\varepsilon$$

for all $\boldsymbol{v}^\varepsilon = (v_i^\varepsilon) \in \boldsymbol{U}(\Omega^\varepsilon)$.

Proof. Define the space

$$\boldsymbol{V}(\Omega^\varepsilon) := \{\boldsymbol{v}^\varepsilon = (v_i^\varepsilon) \in \boldsymbol{H}^1(\Omega^\varepsilon);\ \boldsymbol{v}^\varepsilon = \boldsymbol{0} \text{ on } \Gamma_0^\varepsilon\}.$$

Then, thanks to the uniform positive-definiteness of the elasticity tensor $(A^{ijk\ell,\varepsilon})$, to the *three-dimensional Korn inequality in curvilinear coordinates* (cf. Theorem 3.8-3 in [5]), and to the boundary condition of place satisfied on $\Gamma_0^\varepsilon = \gamma \times [-\varepsilon, \varepsilon]$, the continuous and symmetric bilinear form

$$(\boldsymbol{v}^\varepsilon, \boldsymbol{w}^\varepsilon) \in \boldsymbol{H}^1(\Omega^\varepsilon) \times \boldsymbol{H}^1(\Omega^\varepsilon) \mapsto \int_{\Omega^\varepsilon} A^{ijk\ell,\varepsilon} e^\varepsilon_{k\|\ell}(\boldsymbol{v}^\varepsilon) e^\varepsilon_{i\|j}(\boldsymbol{w}^\varepsilon)\sqrt{g^\varepsilon}\,\mathrm{d}x^\varepsilon$$

is $\boldsymbol{V}(\Omega^\varepsilon)$-elliptic; besides, the linear form

$$\boldsymbol{v}^\varepsilon = (v_i^\varepsilon) \in \boldsymbol{H}^1(\Omega^\varepsilon) \mapsto \int_{\Omega^\varepsilon} f^{i,\varepsilon} v_i^\varepsilon \sqrt{g^\varepsilon}\,\mathrm{d}x^\varepsilon$$

is clearly continuous. It is easily seen that the set $\boldsymbol{U}(\Omega^\varepsilon)$ is a non-empty, closed, and convex subset of the space $\boldsymbol{V}(\Omega^\varepsilon)$. The existence and uniqueness of the solution to the minimization problem and its characterization by means of variational inequalities is then classical. □

Then one can establish the following result; cf. [10, Theorem 4.2].

Theorem 6.2. *Let ω be a domain in $\mathbb{R}^2$ and let $\boldsymbol{\theta} \in \mathcal{C}^3(\overline{\omega}; \mathbb{E}^3)$ be an immersion. Consider a family of elliptic membrane shells with thickness 2ε approaching zero and with each having the same middle surface $S = \boldsymbol{\theta}(\overline{\omega})$, and assume that there exist functions $f^i \in L^2(\Omega)$ independent of ε such that the following assumption on the applied body force density holds:*

$$f^{i,\varepsilon}(x^\varepsilon) = f^i(x) \text{ at each } x^\varepsilon \in \Omega^\varepsilon \text{ for each } \varepsilon > 0.$$

Finally, assume that the following "density property" holds (the same as in Theorem 3.2)*:*

$$\boldsymbol{U}_K(\omega) \text{ is dense in } \boldsymbol{U}_M(\omega) \text{ with respect to the norm } \|\cdot\|_{H^1(\omega)\times H^1(\omega)\times L^2(\omega)}.$$

Let $\boldsymbol{\zeta} \in \boldsymbol{U}_M(\omega)$ denote the solution to problem $\mathcal{P}_M(\omega)$ (Theorem 3.2) *and for each $\varepsilon > 0$, let $\boldsymbol{u}^\varepsilon = (u_i^\varepsilon)$ denote the solution to problem $\mathcal{P}(\Omega^\varepsilon)$* (Theorem 2.1)*. Then the following convergences hold:*

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u^{\varepsilon}_{\alpha}\boldsymbol{g}^{\alpha,\varepsilon}\,dx^{\varepsilon}_3 \to \zeta_\alpha \boldsymbol{a}^{\alpha} \text{ in } \boldsymbol{H}^1(\omega) \text{ as } \varepsilon \to 0,$$

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} u^{\varepsilon}_{3}\boldsymbol{g}^{3,\varepsilon}\,dx^{\varepsilon}_3 \to \zeta_3 \boldsymbol{a}^{3} \text{ in } \boldsymbol{L}^2(\omega) \text{ as } \varepsilon \to 0. \quad \square$$

Comparing the convergences found in Theorem 6.2 with the convergences

$$\zeta^{\varepsilon}_{\alpha,K}\boldsymbol{a}^{\alpha} \to \zeta_\alpha \boldsymbol{a}^{\alpha} \text{ in } \boldsymbol{H}^1(\omega) \text{ as } \varepsilon \to 0,$$
$$\zeta^{\varepsilon}_{3,K}\boldsymbol{a}^{3} \to \zeta_3 \boldsymbol{a}^{3} \text{ in } \boldsymbol{L}^2(\omega) \text{ as } \varepsilon \to 0,$$

established in Theorem 3.2 thus shows that, if the shell under consideration is an elliptic membrane shell satisfying the "density property" in Theorem 3.2, *the solution to the three-dimensional obstacle problem* $\mathcal{P}(\Omega^{\varepsilon})$ *and to the two-dimensional obstacle problem* $\mathcal{P}^{\varepsilon}_K(\omega)$ *exhibit the same limit behavior as* $\varepsilon \to 0$. This observation then fully justifies the formulation of our proposed Koiter's model for an *elliptic membrane shell subjected to a confinement condition*.

Acknowledgements

The work described in this paper was substantially supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region [Project No. 9042536, CityU 11300317].

References

[1] M. Bernadou, P.G. Ciarlet, Sur l'ellipticité du modèle linéaire de coques de W. T. Koiter, in: Computing Methods in Applied Sciences and Engineering, Part 1, in: Lecture Notes in Economics and Mathematical Systems, vol. 134, Springer, Heidelberg, Germany, 1996, pp. 89–136.
[2] M. Bernadou, P.G. Ciarlet, B. Miara, Existence theorems for two-dimensional linear shell theories, J. Elast. 34 (1994) 111–138.
[3] D. Caillerie, É. Sanchez-Palencia, A new kind of singular stiff problems and application to thin elastic shells, Math. Models Methods Appl. Sci. 5 (1995) 47–66.
[4] P.G. Ciarlet, Mathematical Elasticity. Vol. III. Theory of Shells, North-Holland, Amsterdam, 2000.
[5] P.G. Ciarlet, An Introduction to Differential Geometry with Applications to Elasticity, Springer, Dordrecht, The Netherlands, 2005.
[6] P. Destuynder, A classification of thin shell theories, Acta Appl. Math. 4 (1985) 15–63.
[7] P.G. Ciarlet, V. Lods, On the ellipticity of linear membrane shell equations, J. Math. Pures Appl. 75 (1996) 107–124.
[8] P.G. Ciarlet, V. Lods, Asymptotic analysis of linearly elastic shells. III. Justification of Koiter's shell equations, Arch. Ration. Mech. Anal. 136 (1996) 191–200.
[9] P.G. Ciarlet, V. Lods, Asymptotic analysis of linearly elastic shells: "generalized membrane shells", J. Elast. 43 (1996) 147–188.
[10] P.G. Ciarlet, C. Mardare, P. Piersanti, An obstacle problem for elliptic membrane shells, Math. Mech. Solids (2019), https://doi.org/10.1177/1081286518800164.
[11] P.G. Ciarlet, C. Mardare, P. Piersanti, Un problème de confinement pour une coque membranaire linéairement élastique de type elliptique, C. R. Acad. Sci. Paris, Ser. I 356 (2018) 1040–1051.
[12] P.G. Ciarlet, P. Piersanti, An obstacle problem for Koiter's shells, Math. Mech. Solids (2019), https://doi.org/10.1177/1081286519825979.
[13] P.G. Ciarlet, E. Sanchez-Palencia, An existence and uniqueness theorem for the two-dimensional linear membrane shell equations, J. Math. Pures Appl. 75 (1996) 51–67.
[14] W.T. Koiter, On the foundations of the linear theory of thin elastic shells. I, II, Ned. Akad. Wet. Proc. Ser. B 73 (1970) 169–182, Ned. Akad. Wet. Proc. Ser. B 73 (1970) 183–195.
[15] É. Sanchez-Palencia, Statique et dynamique des coques minces. I. Cas de flexion pure non inhibée, C. R. Acad. Sci. Paris, Ser. I 309 (1989) 411–417.