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Ordinary differential equations

Phase portraits of integrable quadratic systems with an invariant parabola and an invariant straight line

Portraits de phase de systèmes quadratiques intégrables avec une parabole et une ligne droite invariantes

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A R T I C L E I N F O

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A B S T R A C T

We classify the phase portraits of the quadratic polynomial differential systems having an invariant parabola, an invariant straight line, and a Darboux first integral produced by these two invariant curves.

R É S U M É

Nous classifions les portraits de phase des systèmes différentiels polynomiaux quadratiques ayant une parabole invariante, une ligne droite invariante et une intégrale première de Darboux produite par ces deux invariants.

1. Introduction and statement of the main results

Consider planar polynomial differential systems of the form

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

where P and Q are real polynomials defined in $\mathbb{R}^2$. The dot denotes derivative with respect to the independent variable t. The *degree* of system (1) is the maximum of the degrees of the polynomials P and Q. When a polynomial differential system (1) has degree two, we call it simply a *quadratic system.*

Let U be a dense and open subset of $\mathbb{R}^2$. A *first integral* of a differential system (1) is a non-locally constant $\mathcal{C}^1$ function $H : U \to \mathbb{R}$ that is constant on the orbits of system (1) contained in U, i.e. H satisfies

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$$\frac{\partial H}{\partial x}(x,y)P(x,y)+\frac{\partial H}{\partial y}(x,y)Q(x,y)=0 \tag{2}$$

in the points $(x,y)\in U$. We say that a quadratic system is *integrable* if it has a first integral $H:U\to\mathbb{R}$.

After the linear differential systems in the plane, i.e. the polynomial differential systems of degree one, the quadratic systems are the easier ones and they have been studied intensively, and more than one thousand papers have been published on those systems, see the references quoted in the books of Ye [25], [24] and Reyn [19]. But the classification of all the integrable quadratic system is an open problem.

The *phase portrait* of a differential system is the decomposition of its domain of definition as the union of all its oriented orbits. The phase portraits of a polynomial differential system is drawn in the Poincaré disc, which, roughly speaking, is the closed disc centered at the origin of coordinates with radius one, the interior of this disc is diffeomorphic to $\mathbb{R}^2$, and its boundary $\mathbb{S}^1$ corresponds to the infinity of $\mathbb{R}^2$, each point of $\mathbb{S}^1$ provides a direction for going or coming from infinity. For more details, see section 3 or Chapter 5 of [6].

Many classes of integrable quadratic systems have been studied, and for these classes the topological phase portraits of their quadratic systems have been classified in the Poincaré disc. A relatively easy class of integrable quadratic systems is formed by the homogeneous quadratic systems studied by Lyagina [15], Markus [16], Korol [10], Sibirskii and Vulpe [21], Newton [17], Date [4] and Vdovina [22]... An important class of integrable quadratic systems are the ones with centers studied by many authors; see, for instance, Dulac [5], Kapteyn [8] and [9], Bautin [3], Lunkevich and Sibirskii [14], Schlomiuk [20], Vulpe [23], Żołądek [27], Ye and Ye [26], Artés, Llibre and Vulpe [2]... In general, all polynomial differential systems having a center with purely imaginary eigenvalues are integrable, see for instance Poincaré [18] and Liapunov [11], and one example in [12]. Another interesting class is the one formed by the Hamiltonian quadratic systems, see Artés and Llibre [1], Kalin and Vulpe [7] and Artés, Llibre and Vulpe [2].

We say that the algebraic curve $f(x,y)=0$ is *invariant* by the polynomial system (1) if

$$\frac{\partial f}{\partial x}P(x,y)+\frac{\partial f}{\partial y}Q(x,y)=K(x,y)f(x,y), \tag{3}$$

for some polynomial $K(x,y)$, this polynomial is called the *cofactor* of the invariant algebraic curve $f(x,y)=0$. Note that an invariant algebraic curve of system (1) is formed by orbits of that system.

Recently, the topological phase portraits in the Poincaré disc of a new class of integrable quadratic systems having an invariant ellipse and an invariant straight line have been classified in [13].

The objective of this paper is to classify the topological phase portraits in the Poincaré disc of a class of integrable quadratic systems having an invariant parabola and an invariant straight line. After an affine change of variables, we may suppose without loss of generality that the equation of the invariant parabola and the one of the invariant straight line are $y=x^2$ and $ax+by+c=0$, respectively, where a,b, and c are constants such that $a^2+b^2\neq 0$.

More precisely we shall classify the topological phase portraits in the Poincaré disc of the class of quadratic systems

$$\begin{aligned}\dot{x}&=b\lambda_1(y-x^2)-\lambda_2(ax+by+c),\\ \dot{y}&=-a\lambda_1(y-x^2)-2\lambda_2 x(ax+by+c),\end{aligned} \tag{4}$$

where a,b,c,λ_1 and λ_2 are real parameters such that $a^2+b^2\neq 0$ and $\lambda_1^2+\lambda_2^2\neq 0$; otherwise, we will have the null differential system. Note that system (4) depends on five parameters.

It is easy to check, using the definition of invariant algebraic curve (3), that the quadratic systems (4) have the invariant parabola $y-x^2=0$ and the invariant straight line $ax+by+c=0$. At the end of section 2, we will show that the quadratic systems (4) is a subclass of all quadratic systems having an invariant parabola and an invariant straight line.

Using the Darboux theory (see Theorem 8.7 of [6]), it is easy to obtain that

$$H(x,y)=(y-x^2)^{-\lambda_2}(ax+by+c)^{\lambda_1} \tag{5}$$

is a first integral of system (4) in the open and dense subset of $\mathbb{R}^2$ where it is defined. In fact, it is immediate to check that it is a first integral using the definition (2). The existence of this first integral shows that the quadratic systems have no limit cycles.

Our main result is the following theorem.

Theorem 1.1. *The phase portraits in the Poincaré disc of systems* (4) *with* $(\lambda_1^2+\lambda_2^2)(a^2+b^2)\neq 0$ *are topologically equivalent to one of the* 26 *phase portraits shown in Fig. 1. The correspondence between systems* (4) *and the phase portraits of Fig. 1 are provided in Table 1.*

This work is organized as follows. In section 2, we mainly present the definitions concerning the classification of singularities of planar vector fields. In section 3, we introduce the Poincaré compactification. We perform the study of the singularities of system (4) in sections 4 and 5. Finally, in section 6, we prove Theorem 1.1.

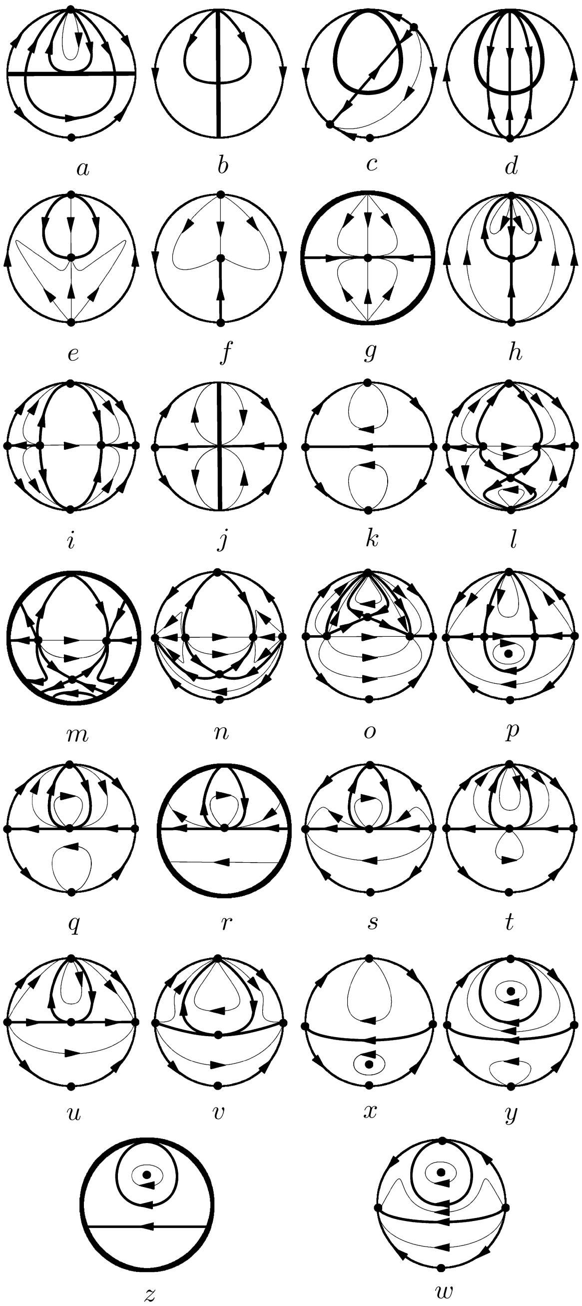

Fig. 1. Phase portraits of systems (4) in the Poincaré disc.

Fig. 1. Portraits de phase des systèmes (4) en le disque de Poincaré.

2. Preliminary definitions and results

Consider a vector field $X : U \to \mathbb{R}^2$ defined on the open subset U of $\mathbb{R}^2$ and denote the Jacobian matrix of X at $(x, y) \in U$ by $DX(x, y)$. Let $(x_0, y_0) \in U$ be a singularity of X. We say that (x_0, y_0) is *elementary* if one of the following three conditions holds: either the real part of both eigenvalues of $DX(x_0, y_0)$ are different from zero; in this case (x_0, y_0) is a *hyperbolic singular point*; or the eigenvalues of $DX(x_0, y_0)$ are purely imaginary and (x_0, y_0) is a focus or a center; or only one of the eigenvalues of $DX(x_0, y_0)$ is different from zero; in this case (x_0, y_0) is a *semi-hyperbolic singular point.* If both the eigenvalues of $DX(x_0, y_0)$ are zero then (x_0, y_0) is a *non-elementary singular point of* X.

The topological classification of the flow near a hyperbolic or semi-hyperbolic singular point of X is given, for instance in Theorems 2.15 and 2.19 of [6], respectively.

Table 1
Classification of the phase portraits in the Poincaré disc of the quadratic systems (4).

Tableau 1
Classification des portraits de phase dans le disque de Poincaré des systèmes quadratiques (4).

Global phase portrait of Fig. 1	Parameters of system (4)
a	$\lambda_1 = 0$ and $b\lambda_2 \neq 0$.
b	$\lambda_1 = 0, b = 0$ and $a\lambda_2 \neq 0$.
c	$\lambda_2 = 0$ and $b\lambda_1 \neq 0$.
d	$\lambda_2 = 0, b = 0$ and $a\lambda_1 \neq 0$.
e	$a\lambda_1 > 0, a\lambda_2 > 0, b = 0, a(\lambda_1 - 2\lambda_2) > 0$ or $a\lambda_1 < 0, a\lambda_2 < 0, b = 0, a(\lambda_1 - 2\lambda_2) < 0$.
f	$a\lambda_1 > 0, a\lambda_2 > 0, b = 0, a(\lambda_1 - 2\lambda_2) < 0$ or $a\lambda_1 < 0, a\lambda_2 < 0, b = 0, a(\lambda_1 - 2\lambda_2) > 0$.
g	$a\lambda_1 > 0, a\lambda_2 > 0, b = 0, \lambda_1 = 2\lambda_2$ or $a\lambda_1 < 0, a\lambda_2 < 0, b = 0, \lambda_1 = 2\lambda_2$.
h	$a\lambda_1 > 0, a\lambda_2 < 0, b = 0, a(\lambda_1 - 2\lambda_2) > 0$ or $a\lambda_1 < 0, a\lambda_2 > 0, b = 0, a(\lambda_1 - 2\lambda_2) < 0$.
i	$\lambda_1 = \lambda_2 \neq 0, b \neq 0, a^2 - 4bc > 0$.
j	$\lambda_1 = \lambda_2 \neq 0, b \neq 0, a^2 - 4bc = 0$.
k	$\lambda_1 = \lambda_2 \neq 0, b \neq 0, a^2 - 4bc < 0$.
l	$0 < \lambda_2 < \lambda_1 < 2\lambda_2, b \neq 0, a^2 - 4bc > 0$ or $2\lambda_2 < \lambda_1 < \lambda_2 < 0, b \neq 0, a^2 - 4bc > 0$.
m	$0 < 2\lambda_2 = \lambda_1, b \neq 0, a^2 - 4bc > 0$ or $\lambda_1 = 2\lambda_2 < 0, b \neq 0, a^2 - 4bc > 0$.
n	$0 < 2\lambda_2 < \lambda_1, b \neq 0, a^2 - 4bc > 0$ or $\lambda_1 < 2\lambda_2 < 0, b \neq 0, a^2 - 4bc > 0$.
o	$0 < \lambda_1 < \lambda_2, b \neq 0, a^2 - 4bc > 0$ or $\lambda_2 < \lambda_1 < 0, b \neq 0, a^2 - 4bc > 0$.
p	$\lambda_1\lambda_2 < 0, b \neq 0, a^2 - 4bc > 0$.
q	$0 < \lambda_2 < \lambda_1 < 2\lambda_2, b \neq 0, a^2 - 4bc = 0$ or $2\lambda_2 < \lambda_1 < \lambda_2 < 0, b \neq 0, a^2 - 4bc = 0$.
r	$0 < 2\lambda_2 = \lambda_1, b \neq 0, a^2 - 4bc = 0$ or $\lambda_1 = 2\lambda_2 < 0, b \neq 0, a^2 - 4bc = 0$.
s	$0 < 2\lambda_2 < \lambda_1, b \neq 0, a^2 - 4bc = 0$ or $\lambda_1 < 2\lambda_2 < 0, b \neq 0, a^2 - 4bc = 0$.
t	$0 < \lambda_1 < \lambda_2, b \neq 0, a^2 - 4bc = 0$ or $\lambda_2 < \lambda_1 < 0, b \neq 0, a^2 - 4bc = 0$.
u	$\lambda_1\lambda_2 < 0, b \neq 0, a^2 - 4bc = 0$.
v	$\lambda_1\lambda_2 < 0, b \neq 0, a^2 - 4bc < 0$.
x	$0 < \lambda_1 < \lambda_2, b \neq 0, a^2 - 4bc < 0$ or $\lambda_2 < \lambda_1 < 0, b \neq 0, a^2 - 4bc < 0$.
y	$0 < \lambda_2 < \lambda_1 < 2\lambda_2, b \neq 0, a^2 - 4bc < 0$ or $2\lambda_2 < \lambda_1 < \lambda_2 < 0, b \neq 0, a^2 - 4bc < 0$.
z	$0 < 2\lambda_2 = \lambda_1, b \neq 0, a^2 - 4bc < 0$ or $\lambda_1 = 2\lambda_2 < 0, b \neq 0, a^2 - 4bc < 0$.
w	$0 < 2\lambda_2 < \lambda_1, b \neq 0, a^2 - 4bc < 0$ or $\lambda_1 < 2\lambda_2 < 0, b \neq 0, a^2 - 4bc < 0$.

Suppose that (x_0, y_0) is a non-elementary singular point of X. If the matrix $DX(x_0, y_0)$ is not zero, then (x_0, y_0) is a *nilpotent singularity*. If $DX(x_0, y_0)$ is the zero matrix, then (x_0, y_0) is a *linearly zero singularity*. The topological classification of a nilpotent singular point is given, for instance in Theorem 3.5 of [6]. The main tool used for studying the local phase portraits of the linearly zero singularities are the changes of variables called blow-ups.

In general a *quasi-homogeneous blow-up* is a change of variables of the form

$$
\begin{aligned}
&(x, y) \mapsto (x^\alpha, x^\beta y), \text{ positive } x\text{-direction},\\
&(x, y) \mapsto (-x^\alpha, x^\beta y), \text{ negative } x\text{-direction},\\
&(x, y) \mapsto (xy^\alpha, y^\beta), \text{ positive } y\text{-direction},\\
&(x, y) \mapsto (xy^\alpha, -y^\beta), \text{ negative } y\text{-direction},
\end{aligned}
$$

where α, β are positive integers. For quasi-homogeneous blow-ups in the x-direction (respectively y-direction), when α (respectively β) is odd, the information obtained in the positive x-direction (respectively y-direction) is useful to the negative x-direction (respectively y-direction). A suitable choice of α and β avoids successive homogeneous blow-ups (see p. 104 of [6]).

In this work, all isolated singularities are hyperbolic, nilpotent, or linearly zero. The classification of nilpotent singularities could be performed using Theorem 3.5 of [6]. But, in order to obtain information on the localization of the separatrices of the nilpotent singularities, we will use quasi-homogeneous blow-ups (see Lemmas 5.3 and 5.8).

For showing that the class of quadratic systems (4) having the invariant parabola $y = x^2$, the invariant straight line $ax + by + c = 0$, and the first integral (5) are not all the quadratic systems having the invariant parabola $y = x^2$ and the invariant straight line $ax + by + c = 0$, it is sufficient to provide an example. Consider the invariant straight line

$$L = ax + by + c = 8x + 16y + 1 = 0,$$

and the subclass of quadratic systems (4) having this invariant straight line are

$$
\begin{aligned}
\dot{x} &= 16\lambda_1(y - x^2) - \lambda_2(8x + 16y + 1),\\
\dot{y} &= -8\lambda_1(y - x^2) - 2\lambda_2 x(8x + 16y + 1).
\end{aligned}
\tag{6}
$$

But the quadratic system

$$
\begin{aligned}
\dot{x} &= 1 + 3x + 33y - 36x^2 + 4xy,\\
\dot{y} &= 2(x - 4y + 7x^2 - 3xy + 4y^2)
\end{aligned}
$$

also has the invariant parabola $y = x^2$ with cofactor $k_P = -8(1+9x-y)$ and the invariant straight line L with cofactor $k_L = 8(1-x+y)$, it is not contained in the subclass and it has no first integral of the form (5), because, according to statement (i) of Theorem 8.7 of [6], such a first integral only exists if there are real numbers μ_1 and μ_2 not all zero such that

$$\mu_1 k_p + \mu_2 k_L = -8\mu_1(1+9x-y) + 8\mu_2(1-x+y) = 0,$$

but these μ_1 and μ_2 do not exist.

3. The Poincaré compactification

The Poincaré compactification is a helpful technique to study the behavior near the infinity of a polynomial vector field defined in $\mathbb{R}^2$. In this section, X is the planar polynomial vector field defined by the polynomial differential system (1) of degree d.

We shall use the notation $\mathbb{S}^2 = \{(z_1, z_2, z_3) \in \mathbb{R}^3; z_1^2 + z_2^2 + z_3^2 = 1\}$ and $\mathbb{S}^1 = \{(z_1, z_2, z_3) \in \mathbb{S}^2; z_3 = 0\}$ for the two-dimensional sphere and its equator, respectively. We identify $\mathbb{R}^2$ with the tangent plane π of $\mathbb{S}^2$ at the point $(0, 0, 1)$. Considering the central projection from π into $\mathbb{S}^2$, we obtain a vector field X' on $\mathbb{S}^2 \setminus \mathbb{S}^1$ such that the infinity points of π are projected in $\mathbb{S}^1$.

Observe that the vector field X' is symmetric with respect to the center of $\mathbb{S}^2$. X' is an unbounded vector field near $\mathbb{S}^1$, but after a multiplication by an appropriate factor, the resultant vector field $p(X)$, called the *Poincaré compactification* of X, is analytic and defined in the whole sphere $\mathbb{S}^2$. The symmetry means that it is sufficient to consider $p(X)$ defined only in the closed northern hemisphere H of $\mathbb{S}^2$. We project the vector field $p(X)$ on H using the orthogonal projection into a vector field on the disc $\{(z_1, z_2, z_3) \in \mathbb{R}^3; z_1^2 + z_2^2 \le 1, z_3 = 0\}$, called the *Poincaré disc*. It is on this disc that we draw the phase portraits of the polynomial differential systems; its interior is diffeomorphic to $\mathbb{R}^2$ and its boundary $\mathbb{S}^1$ corresponds to the infinity of $\mathbb{R}^2$.

To obtain an explicit expression of the Poincaré compactification $p(X)$, the northern and the southern hemispheres of $\mathbb{S}^2$ are denoted by $H^+ = \{(z_1, z_2, z_3) \in \mathbb{S}^2; z_3 > 0\}$ and $H^- = \{(z_1, z_2, z_3) \in \mathbb{S}^2; z_3 < 0\}$, respectively. We consider the central projections of $\pi := \{(z_1, z_2, z_3) \in \mathbb{R}^3; z_3 = 1\}$ in H^+ and H^-, which are defined by

$$\begin{array}{rcl} f^+ : \pi & \to & H^+ \\ z & \mapsto & \dfrac{1}{\Delta(z)}(z_1, z_2, 1) \end{array} \quad \text{and} \quad \begin{array}{rcl} f^- : \pi & \to & H^- \\ z & \mapsto & \dfrac{1}{\Delta(z)}(-z_1, -z_2, -1), \end{array}$$

respectively. Here $\Delta(z) = \sqrt{z_1^2 + z_2^2 + 1}, z \in \pi$. Define the vector field X' on $\mathbb{S}^2 \setminus \mathbb{S}^1$ by

$$X'(w) = \begin{cases} Df^+(z)X(z) & \text{if } w = f^+(z), \\ Df^-(z)X(z) & \text{if } w = f^-(z). \end{cases}$$

The vector field X' is unbounded in a neighborhood of $\mathbb{S}^1$, but the vector field $w_3^d X'(w)$ is an analytical extension of X' to the whole $\mathbb{S}^2$. This extension is *the Poincaré compactification of* X that we have denoted by $p(X)$.

In general, the vector field $p(X)$ is C^ω-equivalent, but not C^ω-conjugated, to X in each hemisphere of $\mathbb{S}^2$. Then we study the trajectories of X near a singularity using the correspondent singularities of $p(X)$. Since $p(X)$ may have singularities in $\mathbb{S}^1$, a singular point of $p(X)$ that belongs to $\mathbb{S}^2 \setminus \mathbb{S}^1$ or to $\mathbb{S}^1$ is called *finite* or *infinite* singular point of X, respectively.

We can prove that $\mathbb{S}^1$ is invariant under the flow of $p(X)$. Using the expressions of f^+ and f^-, we obtain that $p(X)$ is symmetric with respect to the origin, then it is sufficient to study the trajectories of $p(X)$ in $H^+ \cup \mathbb{S}^1$.

We will obtain the expressions of $p(X)$ in the local charts of $\mathbb{S}^2$. For $j = 1, 2, 3$ consider $U_j = \{(z_1, z_2, z_3) \in \mathbb{S}^2; z_j > 0\}$, $V_j = \{(z_1, z_2, z_3) \in \mathbb{S}^2; z_j < 0\}$ and $\varphi_j : U_j \to \mathbb{R}^2$, $\psi_j : V_j \to \mathbb{R}^2$ defined by

$$\varphi_1(z) = -\psi_1(z) = \frac{(z_2, z_3)}{z_1}, \; \varphi_2(z) = -\psi_2(z) = \frac{(z_1, z_3)}{z_2}, \; \varphi_3(z) = \frac{(z_1, z_2)}{z_3}.$$

We denote by (u, v) the value of φ_j or ψ_j at the point z. The expression of $p(X)$ in the chart (U_1, φ_1) is

$$\dot{u} = v^d \left[-uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right) \right], \;\; \dot{v} = -v^{d+1} P\left(\frac{1}{v}, \frac{u}{v}\right).$$

The expression of $p(X)$ in the chart (U_2, φ_2) is

$$\dot{u} = v^d \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right], \;\; \dot{v} = -v^{d+1} Q\left(\frac{u}{v}, \frac{1}{v}\right),$$

and the expression of $p(X)$ in the chart (U_3, φ_3) is

$$\dot{u} = P(u, v), \quad \dot{v} = Q(u, v).$$

The expression of $p(X)$ in the chart (V_j, ψ_j) is the expression of $p(X)$ in the chart (U_j, φ_j) multiplied by $(-1)^{d-1}$, for $j = 1, 2$ and 3.

Notice that $(u, v) \in U_j$ is an infinite singular point of X if and only if the expression of $p(X)$ in the chart (U_j, φ_j) vanishes at (u, v) and $v = 0$.

Due to the symmetry of $p(X)$ with respect to the origin of the coordinates, if z is an infinite singular point of X, then $-z$ is also an infinite singular point of X. Then we only need to study $p(X)$ in the charts (U_j, φ_j), $j = 1, 2, 3$.

If z is an infinite singular point of X with $z \in U_2$ and $z \neq (0, 0, 0)$, then $z \in U_1 \cup V_1$. Then, to study all the infinite singular points of $p(X)$, it is sufficient to study the singularities of $p(X)$ in U_1 and the origin of U_2.

From the definition of the Poincaré compactification follows Remarks 1–3.

Remark 1. Suppose that $k : \mathbb{R}^2 \to \mathbb{R}$ is a polynomial function of degree d_1 and $\widetilde{X}$ is a polynomial planar vector field with degree d_2. Define $X = k\widetilde{X}$. Let $p_j(X)$ and $p_j(\widetilde{X})$ be the expression of X and $\widetilde{X}$ in the chart (U_j, φ_j), respectively, for $j = 1$ and 2. Then

$$p_1(X) = v^{d_1} k\left(\frac{1}{v}, \frac{u}{v}\right) p_1(\widetilde{X}) \quad \text{and} \quad p_2(X) = v^{d_1} k\left(\frac{u}{v}, \frac{1}{v}\right) p_2(\widetilde{X}).$$

Remark 2. If $b \neq 0$, then the points of $\mathbb{R}^2$ of the straight line $ax + by + c = 0$ of $\mathbb{R}^2$ are mapped into the points of $\{a + bu + cv = 0\} \cap U_1$ with $v \neq 0$. Observe that the straight line $a + bu + cv = 0$ cuts the line of the infinity $v = 0$ at the point $(-a/b, 0)$.

Remark 3. If $b = 0$, then the points of $\mathbb{R}^2$ of the straight line $ax + by + c = 0$ of $\mathbb{R}^2$ are mapped into the points $\{au + cv = 0\} \cap U_2$ with $v \neq 0$. Observe that the straight line $au + cv = 0$ cuts the line of the infinity $v = 0$ at the point $(0, 0)$.

Remark 4. For each $k \in \mathbb{R}$ the points of $\mathbb{R}^2$ of the parabola $y = x^2 + k$ contained in the half-planes $y > 0$ or $y < 0$ are mapped into the points of the form $v = u^2 + kv^2$ of U_2, or into the points of the form $-v = u^2 + kv^2$ of V_2, respectively.

4. Phase portraits of system (4) for the case $\lambda_1\lambda_2 = 0$ and $\lambda_1^2 + \lambda_2^2 \neq 0$

In this section, we present the phase portrait in the Poincaré disc of system (4) for the case $\lambda_1\lambda_2 = 0$. If $\lambda_1 = \lambda_2 = 0$ and $\lambda_1^2 + \lambda_2^2 \neq 0$.

Lemma 4.1. *Suppose that $\lambda_1 = 0$ and $\lambda_2 \neq 0$. If $b \neq 0$ or $b = 0$, then the phase portrait in the Poincaré disc of system (4) is shown in Figs. 1a or 1b, respectively.*

Proof. After the rescaling given by $ds = \lambda_2(ax + by + c)\,dt$, system (4) becomes

$$x' = -1, \quad y' = -2x, \tag{7}$$

where the prime denotes the derivative with respect to the variable s. The phase portrait in $\mathbb{R}^2$ of system (7) is given in Fig. 2a.

The expression of the Poincaré compactification of system (7) in the charts (U_1, φ_1) and (U_2, φ_2) is

$$\dot{u} = uv - 2, \quad \dot{v} = v^2, \tag{8}$$

and

$$\dot{u} = 2u^2 - v, \quad \dot{v} = 2uv, \tag{9}$$

respectively. We present the phase portrait of system (8) in a sufficiently small neighborhood of the axis $v = 0$ in Fig. 2b.

Each orbit of system (7) is contained in a parabola given by $y = x^2 + k$, for an appropriated choice of $k \in \mathbb{R}$. From Remark 4, it follows that the phase portrait of system (9) is the one of Fig. 2c.

Combining the information in Fig. 2a–c, we obtain the phase portrait in the Poincaré disc of system (7) in Fig. 2d. Using Remark 1 with $k(x, y) = \lambda_2(ax + by + c)$ and $\widetilde{X} = (-1, -2x)$, we obtain the conclusions of Lemma 4.1. □

Lemma 4.2. *Suppose that $\lambda_1 \neq 0$ and $\lambda_2 = 0$. If $b \neq 0$ or $b = 0$, then the phase portrait in the Poincaré disc of system (4) is given in Figs. 1c or 1d, respectively.*

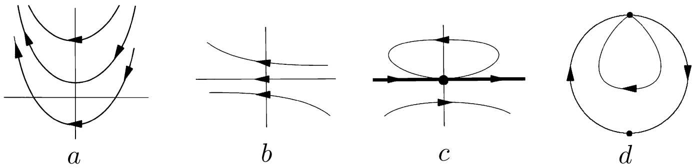

Fig. 2. (a) Phase portrait in $\mathbb{R}^2$ of system (7). (b) Phase portrait of system (8) in a neighborhood of the axis $v = 0$. (c) Phase portrait of system (9). (d) Phase portrait in the Poincaré disc of system (7).

Fig. 2. (a) Portrait de phase du système (7) dans $\mathbb{R}^2$. (b) Portrait de phase du système (8) dans un voisinage de l'axe $v = 0$. (c) Portrait de phase du système (9). (d) Portrait de phase dans le disque de Poincaré du système (7).

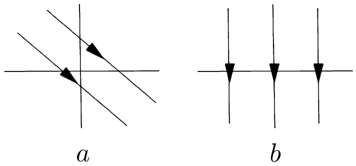

Fig. 3. Phase portrait of system (10). (a) $b > 0$, if $b < 0$ then the orientation of the orbits is reversed. (b) $b = 0$ and $a > 0$, if $b = 0$ and $a < 0$, then the orientation of the orbits is reversed.

Fig. 3. Portrait de phase du système (10). (a) $b > 0$; si $b < 0$, alors l'orientation des orbites doit être inversée. (b) $b = 0$ et $a > 0$; si $b = 0$ et $a < 0$, alors l'orientation des orbites doit être inversée.

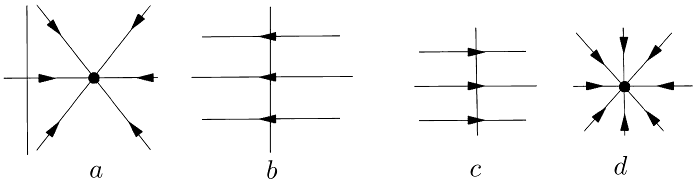

Fig. 4. (a) Phase portrait of system (11) with $b > 0$, if $b < 0$, then the orientation of the orbits is reversed. (b) Phase portrait of system (11) with $a > 0$ and $b = 0$, if $a < 0$ and $b = 0$, then the orientation of the orbits is reversed. (c) Phase portrait of system (12) with $b > 0$, if $b < 0$, then the orientation of the orbits is reversed. (d) Phase portrait of system (12) with $a > 0$ and $b = 0$, if $a < 0$ and $b = 0$ then the orientation of the orbits is reversed.

Fig. 4. (a) Portrait de phase du système (11) avec $b > 0$; si $b < 0$, alors l'orientation des orbites doit être inversée. (b) Portrait de phase du système (11) avec $a > 0$ et $b = 0$; si $a < 0$ et $b = 0$, alors l'orientation des orbites doit être inversée. (c) Portrait de phase du système (12) avec $b > 0$; si $b < 0$, alors l'orientation des orbites doit être inversée. (d) Portrait de phase du système (12) avec $a > 0$ et $b = 0$; si $a < 0$ et $b = 0$, alors l'orientation des orbites doit être inversée.

Proof. After the rescaling given by $ds = \lambda_1(y - x^2)\,dt$, system (4) becomes

$$x' = b, \quad y' = -a. \tag{10}$$

The phase portrait in $\mathbb{R}^2$ of system (10) is shown in Fig. 3.

The expression of the Poincaré compactification of system (10) in the charts (U_1, φ_1) and (U_2, φ_2) is

$$\dot{u} = -(a + bu), \quad \dot{v} = -bv, \tag{11}$$

and

$$\dot{u} = au + b, \quad \dot{v} = av, \tag{12}$$

respectively.

We present the phase portrait of systems (11) and (12) in Figs. 4a–d.

Combining the information of Figs. 3 and 4, we obtain the local and the phase portrait in the Poincaré disc of system (10) in Fig. 5. Using Remark 1 with $k(x, y) = \lambda_1(y - x^2)$ and $\widetilde{X} = (b, -a)$, we conclude the proof. □

5. Local phase portraits of system (4) for the case $\lambda_1\lambda_2 \neq 0$

In this section, we will study the local phase portraits in the Poincaré disc of system (4) when $\lambda_1\lambda_2 \neq 0$.

5.1. Finite singularities of system (4)

We denote the parabola $y = x^2$ and the straight line $ax + by + c = 0$ by p and l, respectively. In this subsection, we consider the cases $b = 0$; $b \neq 0$ and $\lambda_1 = \lambda_2$, $b \neq 0$ and $\lambda_1 \neq \lambda_2$ in Lemmas 5.1–5.3, respectively.

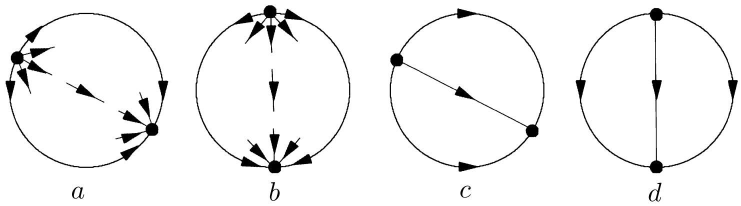

Fig. 5. (a) Local phase portrait in the Poincaré disc of system (10) with $b > 0$, if $b < 0$, then the orientation of the orbits is reversed. (b) Local phase portrait in the Poincaré disc of system (10) with $b = 0$ and $a > 0$, if $b = 0$ and $a < 0$, then the orientation of the orbits is reversed. (c) Phase portrait in the Poincaré disc of system (10) with $b > 0$, if $b < 0$, then the orientation of the orbits is reversed. (d) Phase portrait in the Poincaré disc of system (10) with $b = 0$ and $a > 0$, if $b = 0$ and $a < 0$, then the orientation of the orbits is reversed.

Fig. 5. (a) Portrait de phase dans le disque de Poincaré du système (10) avec $b > 0$ aux points singuliers ; si $b < 0$, alors l'orientation des orbites doit être inversée. (b) Portrait de phase dans le disque de Poincaré du système (10) avec $b = 0$ et $a > 0$ aux points singuliers ; si $b = 0$ et $a < 0$, alors l'orientation des orbites doit être inversée. (c) Portrait de phase dans le disque de Poincaré du système (10) avec $b > 0$; si $b < 0$, alors l'orientation des orbites doit être inversée. (d) Portrait de phase dans le disque de Poincaré du système (10) avec $b = 0$ et $a > 0$; si $b = 0$ et $a > 0$, alors l'orientation des orbites doit être inversée.

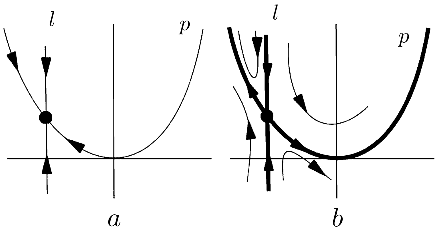

Fig. 6. Phase portrait at the singularity of system (4). (a) $b = 0, a\lambda_1 > 0$, and $a\lambda_2 > 0$. If $b = 0, a\lambda_1 < 0$, and $a\lambda_2 < 0$, then the orientation of the orbits is reversed. (b) $b = 0, a\lambda_1 > 0$ and $a\lambda_2 < 0$. If $b = 0, a\lambda_1 < 0$, and $a\lambda_2 > 0$, then the orientation of the orbits is reversed.

Fig. 6. Portrait de phase du système (4) au point singulier. (a) $b = 0, a\lambda_1 > 0$ et $a\lambda_2 > 0$. Si $b = 0, a\lambda_1 < 0$ et $a\lambda_2 < 0$, alors l'orientation des orbites doit être inversée. (b) $b = 0, a\lambda_1 > 0$ et $a\lambda_2 < 0$. Si $b = 0, a\lambda_1 < 0$ et $a\lambda_2 > 0$, alors l'orientation des orbites doit être inversée.

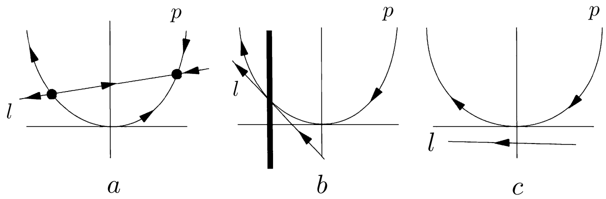

Fig. 7. Phase portrait at the singularities of system (4). (a) $\lambda_1 = \lambda_2, b\lambda_1 > 0$ and $a^2 - bc > 0$. If $\lambda_1 = \lambda_2, b\lambda_1 < 0$ and $a^2 - 4bc > 0$, then the orientation of the orbits is reversed. (b) $\lambda_1 = \lambda_2, b\lambda_1 > 0$ and $a^2 - 4bc = 0$. If $\lambda_1 = \lambda_2, b\lambda_1 < 0$ and $a^2 - 4bc = 0$, then the orientation of the orbits is reversed. (c) $\lambda_1 = \lambda_2$ and $b\lambda_1 > 0$. If $\lambda_1 = \lambda_2$ and $b\lambda_1 < 0$, then the orientation of the orbits is reversed.

Fig. 7. Portrait de phase du système (4) aux points singuliers. (a) $\lambda_1 = \lambda_2, b\lambda_1 > 0$ et $a^2 - bc > 0$. Si $\lambda_1 = \lambda_2, b\lambda_1 < 0$ et $a^2 - 4bc > 0$, alors l'orientation des orbites doit être inversée. (b) $\lambda_1 = \lambda_2, b\lambda_1 > 0$ et $a^2 - 4bc = 0$. Si $\lambda_1 = \lambda_2, b\lambda_1 < 0$ et $a^2 - 4bc = 0$, alors l'orientation des orbites doit être inversée. (c) $\lambda_1 = \lambda_2$ et $b\lambda_1 > 0$. If $\lambda_1 = \lambda_2$ et $b\lambda_1 < 0$, alors l'orientation des orbites doit être inversée.

Lemma 5.1. *Suppose that $\lambda_1\lambda_2 \neq 0$ and $b = 0$. Then system (4) has a unique singularity (x_0, y_0) and the local phase portrait in $\mathbb{R}^2$ of system (4) at (x_0, y_0) is given in Fig. 6.*

Proof. From the hypothesis $a^2 + b^2 \neq 0$, it follows that $a \neq 0$. The unique singularity of system (4) is (x_0, y_0) where $x_0 = -c/a$ and $y_0 = c^2/a^2$. Observe that $(x_0, y_0) \in p \cap l$ and l is given by the equation $x = x_0$. Moreover $\dot{y}|_{x=x_0} = -a\lambda_1(y - y_0)$.

The eigenvalues of the Jacobian matrix of the vector field defined by system (4) evaluated at (x_0, y_0) are $-a\lambda_2$ and $-a\lambda_1$. Then (x_0, y_0) is a hyperbolic singular point. Since $(x_0, y_0) \in p \cap l$, it follows for the saddle point case that the separatrices are contained in $p \cup l$. We presented the local phase portrait of system (4) in Fig. 6. □

Lemma 5.2. *Suppose that $\lambda_1\lambda_2 \neq 0, b \neq 0$ and $\lambda_1 = \lambda_2$. The local phase portrait of system (4) is shown in Fig. 7.*

Proof. Taking the rescaling given by $\mathrm{d}s = \lambda_1 \,\mathrm{d}t$, system (4) becomes

$$x' = -bx^2 - ax - c, \quad y' = -(2bx + a)y - ax^2 - 2cx. \tag{13}$$

In system (13), we have $x' = 0$ if and only if $x = (a \pm \sqrt{a^2 - 4bc}/(2b))$. Then we consider three cases.

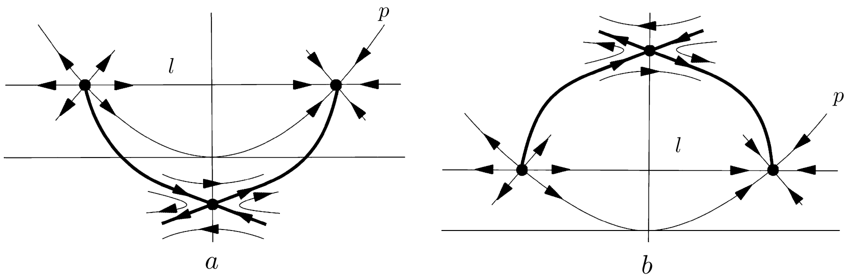

Fig. 8. Phase portrait at the singularities of system (4) with $\lambda_1\lambda_2 \neq 0, \lambda_1 \neq \lambda_2, b \neq 0$ and $a^2 - 4bc > 0$. (a) $\lambda_1 > \lambda_2 > 0$ and $b > 0$, or $\lambda_1 < \lambda_2 < 0$ and $b < 0$. If $\lambda_1 < \lambda_2 < 0$ and $b > 0$, or $0 < \lambda_2 < \lambda_1$ and $b < 0$, then the orientation of the orbits is reversed. (b) $0 < \lambda_1 < \lambda_2$ and $b > 0$, or $\lambda_2 < \lambda_1 < 0$ and $b < 0$. If $\lambda_2 < \lambda_1 < 0$ and $b > 0$, or $0 < \lambda_1 < \lambda_2$ and $b < 0$, then the orientation of the orbits is reversed.

Fig. 8. Portrait de phase du système (4) aux points singuliers avec $\lambda_1\lambda_2 \neq 0, \lambda_1 \neq \lambda_2, b \neq 0$ et $a^2 - 4bc > 0$. (a) $\lambda_1 > \lambda_2 > 0$ et $b > 0$, ou $\lambda_1 < \lambda_2 < 0$ et $b < 0$. Si $\lambda_1 < \lambda_2 < 0$ et $b > 0$, ou $0 < \lambda_2 < \lambda_1$ et $b < 0$, alors l'orientation des orbites doit être inversée. (b) $0 < \lambda_1 < \lambda_2$ et $b > 0$, ou $\lambda_2 < \lambda_1 < 0$ et $b < 0$. Si $\lambda_2 < \lambda_1 < 0$ et $b > 0$, ou $0 < \lambda_1 < \lambda_2$ et $b < 0$, alors l'orientation des orbites doit être inversée.

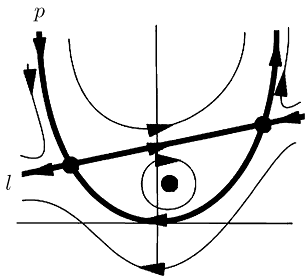

Fig. 9. Phase portrait at the singularities of system (4) with $\lambda_1\lambda_2 \neq 0, \lambda_1 \neq \lambda_2, b \neq 0$ and $a^2 - 4bc > 0$. The figure corresponds to the cases $\lambda_2 < 0 < \lambda_1$ and $b > 0$, or $\lambda_1 < 0 < \lambda_2$ and $b < 0$. If $\lambda_1 < 0 < \lambda_2$ and $b > 0$, or $\lambda_2 < 0 < \lambda_1$ and $b < 0$, then the orientation of the orbits is reversed.

Fig. 9. Portrait de phase du système (4) aux points singuliers avec $\lambda_1\lambda_2 \neq 0, \lambda_1 \neq \lambda_2, b \neq 0$ et $a^2 - 4bc > 0$. La figure correspond aux cas $\lambda_2 < 0 < \lambda_1$ et $b > 0$, ou $\lambda_1 < 0 < \lambda_2$ et $b < 0$. Si $\lambda_1 < 0 < \lambda_2$ et $b > 0$, ou $\lambda_2 < 0 < \lambda_1$ et $b < 0$, alors l'orientation des orbites doit être inversée.

Case 1: $a^2 - 4bc > 0$. In this case system, (13) has two singularities, (x_1, y_1) and (x_2, y_2), where $x_1 = -(\sqrt{a^2 - 4bc} + a)/(2b)$, $x_2 = (\sqrt{a^2 - 4bc} - a)/(2b)$ and $y_j = x_j^2$ for $j = 1, 2$.

Observe that, for $j = 1, 2$, we have $(x_j, y_j) \in p \cap l$. The eigenvalues of the Jacobian matrix of the vector field defined by system (13) evaluated at (x_j, y_j) are equal to $(-1)^{j+1}\sqrt{a^2 - 4bc}$. Then (x_j, y_j) is a hyperbolic singular point, $j = 1, 2$. We show the local phase portrait of system (4) at (x_1, y_1) and (x_2, y_2) in Fig. 7a.

Case 2: $a^2 - 4bc = 0$. In this case, we can write system (13) into the form

$$x' = -b(x - x_0)^2, \quad y' = -(2by + ax)(x - x_0), \tag{14}$$

with $x_0 = -a/(2b)$. After the rescaling given by $\mathrm{d}\tau = (x - x_0)\,\mathrm{d}t$, system (14) becomes

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = -b(x - x_0), \quad \frac{\mathrm{d}y}{\mathrm{d}\tau} = -(2by + ax). \tag{15}$$

The singularity of system (15) is (x_0, y_0), where $y_0 = x_0^2$. Observe that $(x_0, y_0) \in p \cap l$ and the tangent line to p at (x_0, y_0) is l.

The eigenvalues of the Jacobian matrix of the vector field defined by system (15) evaluated at (x_0, y_0) are $-b$ and $-2b$. Then (x_0, y_0) is a hyperbolic singular point of system (15). We give the local phase portrait of system (4) at (x_0, y_0) in Fig. 7b.

Case 3: $a^2 - 4bc < 0$. In this case, system (4) has no singularities. We present the phase portrait of system (4) in Fig. 7c. □

The α-limit and the ω-limit set of the orbit Γ are denoted by $\alpha(\Gamma)$ and $\omega(\Gamma)$, respectively. We say that the orbit Γ *connects* the singularities (x_1, y_1) and (x_2, y_2) when $\alpha(\Gamma) = \{(x_1, y_1)\}$ and $\omega(\Gamma) = \{(x_2, y_2)\}$, or $\alpha(\Gamma) = \{(x_2, y_2)\}$ and $\omega(\Gamma) = \{(x_1, y_1)\}$. From now on, for system (4), we denote by P, N, and Σ the subsets of $\mathbb{R}^2$ such that $\dot{x} > 0, \dot{x} < 0$ and $\dot{x} = 0$, respectively.

Lemma 5.3. *Suppose that $\lambda_1\lambda_2 \neq 0, b \neq 0$, and $\lambda_1 \neq \lambda_2$.*

(a) *The local phase portrait in $\mathbb{R}^2$ of system (4) is given in Figs. 8–11.*
(b) *For the case of Fig. 11a, if Γ is a separatrix of the saddle point and $\Gamma \cap P \neq \emptyset$ (respect. $\Gamma \cap N \neq \emptyset$), then $\Gamma \subset P$ (respect. $\Gamma \subset N$).*

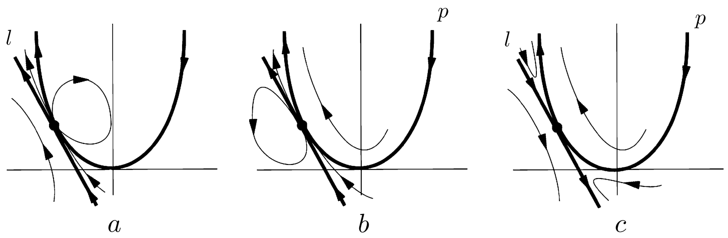

Fig. 10. Phase portrait of system (4) at (x_0, y_0) for the case $b\lambda_1\lambda_2(\lambda_1-\lambda_2)\neq 0$ and $4bc=a^2$. (a) $0<\lambda_2<\lambda_1$ and $b>0$, or $\lambda_1<\lambda_2<0$ and $b<0$. If $\lambda_1<\lambda_2<0$ and $b>0$, or $0<\lambda_2<\lambda_1$ and $b<0$, then the orientation of the orbits is reversed. The separatrices are contained in the straight line $ax+by+c=0$ and in the region $y\geq x^2$. (b) $0<\lambda_1<\lambda_2$ and $b>0$, or $\lambda_2<\lambda_1<0$ and $b<0$. If $\lambda_2<\lambda_1<0$ and $b>0$, or $0<\lambda_1<\lambda_2$ and $b<0$, then the orientation of the orbits is reversed. The separatrices are contained in the region $ax+by+c\leq 0$ and the parabola $y=x^2$. (c) $\lambda_1<0<\lambda_2$ and $b>0$, or $\lambda_2<0<\lambda_1$ and $b<0$. If $\lambda_2<0<\lambda_1$ and $b>0$, or $\lambda_1<0<\lambda_2$ and $b<0$, then the orientation of the orbits is reversed. The separatrices are contained in the straight line $ax+by+c=0$ and in the parabola $y=x^2$.

Fig. 10. Portrait de phase du système (4) au point singulier (x_0, y_0) pour le cas $b\lambda_1\lambda_2(\lambda_1-\lambda_2)\neq 0$ et $4bc=a^2$. (a) $0<\lambda_2<\lambda_1$ et $b>0$, ou $\lambda_1<\lambda_2<0$ et $b<0$. Si $\lambda_1<\lambda_2<0$ et $b>0$, ou $0<\lambda_2<\lambda_1$ et $b<0$, alors l'orientation des orbites doit être inversée. Les séparatrices sont contenues dans la ligne $ax+by+c=0$ et dans la région $y\geq x^2$. (b) $0<\lambda_1<\lambda_2$ et $b>0$, ou $\lambda_2<\lambda_1<0$ et $b<0$. Si $\lambda_2<\lambda_1<0$ et $b>0$, ou $0<\lambda_1<\lambda_2$ et $b<0$, alors l'orientation des orbites doit être inversée. Les séparatrices sont contenues dans la région $ax+by+c\leq 0$ et la parabole $y=x^2$. (c) $\lambda_1<0<\lambda_2$ et $b>0$, ou $\lambda_2<0<\lambda_1$ et $b<0$. Si $\lambda_2<0<\lambda_1$ et $b>0$, ou $\lambda_1<0<\lambda_2$ et $b<0$, alors l'orientation des orbites doit être inversée. Les séparatrices sont contenues dans la ligne $ax+by+c=0$ et dans la parabole $y=x^2$.

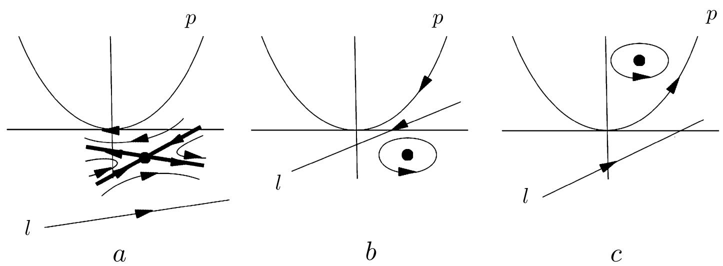

Fig. 11. Phase portrait of system (4) at (x_0, y_0). (a) $\lambda_1\lambda_2<0, b(\lambda_1-\lambda_2)<0$ and $a^2-4bc<0$. If $\lambda_1\lambda_2<0, b(\lambda_1-\lambda_2)>0$ and $a^2-4bc<0$, then the orientation of the orbits is reversed. (b) $\lambda_1\lambda_2>0, b(\lambda_1-\lambda_2)<0, \lambda_2(\lambda_1-\lambda_2)<0$ and $a^2-4bc<0$. If $\lambda_1\lambda_2>0, b(\lambda_1-\lambda_2)>0, \lambda_2(\lambda_1-\lambda_2)<0$ and $a^2-4bc<0$, then the orientation of the orbits is reversed. (c) $\lambda_1\lambda_2>0, b(\lambda_1-\lambda_2)<0, \lambda_2(\lambda_1-\lambda_2)>0$ and $a^2-4bc<0$. If $\lambda_1\lambda_2>0, b(\lambda_1-\lambda_2)>0, \lambda_2(\lambda_1-\lambda_2)>0$ and $a^2-4bc<0$, then the orientation of the orbits is reversed.

Fig. 11. Portrait de phase du système (4) au point singulier (x_0, y_0). (a) $\lambda_1\lambda_2<0, b(\lambda_1-\lambda_2)<0$ et $a^2-4bc<0$. Si $\lambda_1\lambda_2<0, b(\lambda_1-\lambda_2)>0$ et $a^2-4bc<0$, alors l'orientation des orbites doit être inversée. (b) $\lambda_1\lambda_2>0, b(\lambda_1-\lambda_2)<0, \lambda_2(\lambda_1-\lambda_2)<0$ et $a^2-4bc<0$. Si $\lambda_1\lambda_2>0, b(\lambda_1-\lambda_2)>0, \lambda_2(\lambda_1-\lambda_2)<0$ et $a^2-4bc<0$, alors l'orientation des orbites doit être inversée. (c) $\lambda_1\lambda_2>0, b(\lambda_1-\lambda_2)<0, \lambda_2(\lambda_1-\lambda_2)>0$ et $a^2-4bc<0$. Si $\lambda_1\lambda_2>0, b(\lambda_1-\lambda_2)>0, \lambda_2(\lambda_1-\lambda_2)>0$ et $a^2-4bc<0$, alors l'orientation des orbites doit être inversée.

Proof. (a) Solving the equation $\dot{x}=0$ for y and replacing the result in the equation $\dot{y}=0$, we obtain that a point (x, y) is a singularity of system (4) when x satisfies

$$2b^2\lambda_1\lambda_2x^3+3ab\lambda_1\lambda_2x^2+(2bc+a^2)\lambda_1\lambda_2x+ac\lambda_1\lambda_2=0 \tag{16}$$

and y is given by

$$y=-\frac{b\lambda_1x^2+a\lambda_2x+c\lambda_2}{b(\lambda_2-\lambda_1)}.$$

Dividing by $2b^2\lambda_1\lambda_2$ and performing the change of variables given by $x\mapsto x+a/(2b)$, Eq. (16) becomes

$$x^3+\frac{4bc-a^2}{4b^2}x=0. \tag{17}$$

Then we consider three cases.

Case 1: $a^2>4bc$. There exist three singularities (x_j, y_j), $j=0,1,2$, of system (4), where $x_0=-a/(2b), x_1=x_0-\sqrt{a^2/b^2-4c/b}, x_2=x_0+\sqrt{a^2/b^2-4c/b}$, $y_0=-(b\lambda_1x_0^2+a\lambda_2x_0+c\lambda_2)/(b\lambda_2-b\lambda_1)$ and $y_j=x_j^2$ for $j=1,2$. Notice that $(x_j, y_j)\in p\cap l$ for $j=1,2$ and $(x_0, y_0)\notin p\cup l$.

The eigenvalues of the Jacobian matrix of the vector field defined by system (4) evaluated at (x_0, y_0) are

$$\pm\frac{\sqrt{(a^2-4bc)\lambda_1\lambda_2}}{\sqrt{2}}. \tag{18}$$

The eigenvalues of the Jacobian matrix of the vector field defined by system (4) evaluated at (x_j, y_j) are

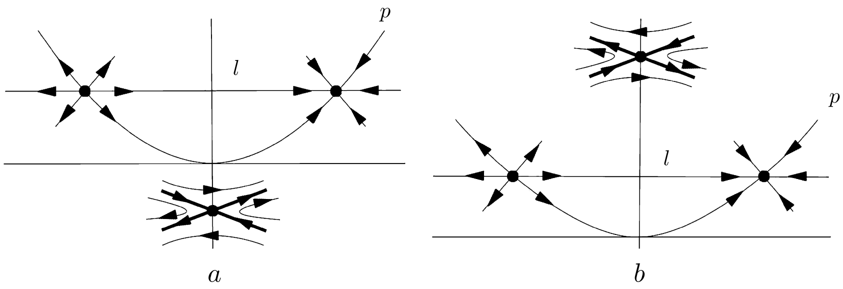

Fig. 12. Phase portrait at the singularities of system (4) with $\lambda_1\lambda_2 \neq 0, \lambda_1 \neq \lambda_2, b \neq 0$ and $a^2 - 4bc > 0$. (a) $\lambda_1 > \lambda_2 > 0$ and $b > 0$, or $\lambda_1 < \lambda_2 < 0$ and $b < 0$. If $\lambda_1 < \lambda_2 < 0$ and $b > 0$, or $0 < \lambda_2 < \lambda_1$ and $b < 0$, then the orientation of the orbits is reversed. (b) $0 < \lambda_1 < \lambda_2$ and $b > 0$, or $\lambda_2 < \lambda_1 < 0$ and $b < 0$. If $\lambda_2 < \lambda_1 < 0$ and $b > 0$, or $0 < \lambda_1 < \lambda_2$ and $b < 0$, then the orientation of the orbits is reversed.

Fig. 12. Portrait de phase du système (4) aux points singuliers avec $\lambda_1\lambda_2 \neq 0, \lambda_1 \neq \lambda_2, b \neq 0$ et $a^2 - 4bc > 0$. (a) $\lambda_1 > \lambda_2 > 0$ et $b > 0$, ou $\lambda_1 < \lambda_2 < 0$ et $b < 0$. Si $\lambda_1 < \lambda_2 < 0$ et $b > 0$, ou $0 < \lambda_2 < \lambda_1$ et $b < 0$, alors l'orientation des orbites doit être inversée. (b) $0 < \lambda_1 < \lambda_2$ et $b > 0$, ou $\lambda_2 < \lambda_1 < 0$ et $b < 0$. Si $\lambda_2 < \lambda_1 < 0$ et $b > 0$, ou $0 < \lambda_1 < \lambda_2$ et $b < 0$, alors l'orientation des orbites doit être inversée.

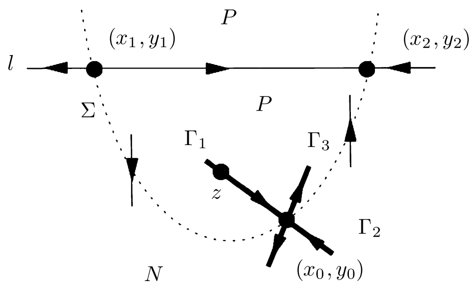

Fig. 13. Proof of $\alpha(\Gamma_1) = \{(x_1, y_1)\}$.

Fig. 13. Démonstration de $\alpha(\Gamma_1) = \{(x_1, y_1)\}$.

$$(-1)^{j+1}\frac{|b|\lambda_1\sqrt{a^2-4bc}}{b} \text{ and } (-1)^{j+1}\frac{|b|\lambda_2\sqrt{a^2-4bc}}{b}, \ j = 1, 2.$$

It follows that (x_j, y_j) is a hyperbolic singular point, for $j = 1, 2$. Moreover, if $\lambda_1\lambda_2 > 0$, then (x_0, y_0) is a hyperbolic singular point and, if $\lambda_1\lambda_2 < 0$, then (x_0, y_0) is a center. Since $(x_j, y_j) \in p \cap l$ for $j = 1, 2$, we obtain that p and l are separatrices of system (4). We present the local phase portrait of system (4) at the singularities (x_0, y_0), (x_1, y_1) and (x_2, y_2) in Figs. 9 and 12.

We will prove that the saddle point is connected with the attracting node and with the repelling node in the cases shown in Fig. 12. From this, we obtain Fig. 8.

We consider only the case given by Fig. 12a, because the case given by Fig. 12b follows in a similar way. The portion of Fig. 12a that corresponds to the region below to l is shown in Fig. 13. We will prove that $\alpha(\Gamma_1) = \{(x_1, y_1)\}$. It is sufficient to prove that $\Gamma_1 \subset P$. The proof of $\alpha(\Gamma_3) = \{(x_2, y_2)\}$ is analogous.

In Fig. 13, we suppose that $a/b \leq 0$. The case $a/b > 0$ is treated in a similar way.

Under the hypotheses of Lemma 6.1, we obtain that Σ is given by the points of the parabola $b(\lambda_1 - \lambda_2)y = b\lambda_1 x^2 + a\lambda_2 x + c\lambda_2$ and that it is represented as the dashed line in Fig. 13. Observe that $(x_j, y_j) \in \Sigma$, $j = 0, 1, 2$ and $x_1 < x_0 < x_2$.

The slope of the tangent to Σ at (x_0, y_0) is $-a/b$. The slope of the stable subspace of the linearized system at (x_0, y_0) is

$$-\frac{a}{b} - \frac{\sqrt{2}\sqrt{(a^2-4bc)\lambda_1\lambda_2}}{2b(\lambda_1-\lambda_2)},$$

which is smaller than $-a/b$, since $2b(\lambda_1 - \lambda_2) > 0$. From the Stable Manifold Theorem, it follows that $\Gamma_1 \cap P \neq \emptyset$ and $\Gamma_2 \cap N \neq \emptyset$.

Since $\dot{y}|_\Sigma$ is given by the polynomial function on the left side of Eq. (16) divided by $b(\lambda_2 - \lambda_1)$, we have

$$\dot{y}|_\Sigma > 0 \text{ if } x < x_1 \text{ or } x_0 < x < x_2 \tag{19}$$

and

$$\dot{y}|_\Sigma < 0 \text{ if } x_1 < x < x_0 \text{ or } x_2 < x. \tag{20}$$

Now we will prove that $\Gamma_1 \subset P$. Suppose that this statement is false. Let $\phi = (\phi_1, \phi_2)$ the flow of the vector field defined by system (4) and take $z \in \Gamma_1 \cap P$. Using continuity, it follows that there exists $t_0 \in \mathbb{R}$ such that $\phi(t_0, z) \in \Sigma$. Observe that $\phi(t_0, z) \neq (x_0, y_0)$ because $z \in \Gamma_1$. Then there exist two possibilities: $\phi_1(t_0, z) > x_0$ or $\phi_1(t_0, z) < x_0$.

Suppose that $\phi_1(t_0,z) > x_0$. Since $\dot{\phi}_1(t_0,z) = 0$, from (19) it follows that the solution $\phi(t_0,z)$ is transversal to Σ at $\phi(t_0,z)$. Using the Flow Box Theorem, there exists $t_2 \in \mathbb{R}$ such that $t_0 < t_2$ and

$$t_0 < t \le t_2 \Rightarrow \phi(t,z) \in P.$$

Define

$$\theta = \{t \ge t_2; \phi(s,z) \in P, \forall s \in [t_2,t]\}$$

and $t_2^* = \inf\theta$. From the continuity of ϕ, we obtain that $\phi(t_2^*,z) \in \Sigma$.

Using these arguments, we obtain that $\phi(t,z) \in P \cup \Sigma, \forall t \ge t_0$. Since $t \mapsto \phi_1(t,z)$ is a nondecreasing function when $t \ge t_0$, we have that $\phi_1(t,z) \ge \phi_1(t_0,z) > x_0, \forall t \ge t_0$. This is a contradiction with $\lim_{t\to+\infty}\phi_1(t,z) = x_0$.

If $\phi_1(t_0,z) < x_0$, using (20) instead (19), we obtain in a similar way a contradiction with $\lim_{t\to+\infty}\phi_1(t,z) = x_0$.

Case 2: $a^2 = 4bc$. The unique solution to Eq. (17) is $x = 0$; then the unique singularity (x_0, y_0) of system (4) is given by $x_0 = -a/(2b)$ and $y_0 = c/b$. Observe that $(x_0, y_0) \in p \cap l$. Moreover, the tangent line to p at (x_0, y_0) is l.

The Jacobian matrix of the vector field defined by system (4) evaluated at (x_0, y_0) is

$$(\lambda_1 - \lambda_2)\begin{pmatrix} a & b \\ -a^2/b & -a \end{pmatrix}.$$

Then (x_0, y_0) is a nilpotent singularity of system (4).

Doing the translation given by $x \mapsto x - x_0$, $y \mapsto y - y_0$, system (4) becomes

$$\begin{aligned} \dot{x} &= a(\lambda_1-\lambda_2)x + b(\lambda_1-\lambda_2)y - b\lambda_1 x^2, \\ \dot{y} &= -4c(\lambda_1-\lambda_2)x - a(\lambda_1-\lambda_2)y - a(2\lambda_2-\lambda_1)x^2 - 2b\lambda_2 xy. \end{aligned} \tag{21}$$

After the linear change of coordinates defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1/b & 0 \\ a/b^2 & 1/b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

system (21) becomes

$$\dot{x} = b(\lambda_1-\lambda_2)y - b^2\lambda_1 x^2, \quad y' = -2b^2\lambda_2 xy. \tag{22}$$

In order to obtain information on the localization of the separatrices, we take the quasi-homogeneous blow-up in the x-direction given by $x = u$, $y = u^2 v$. Then system (22) becomes

$$\dot{u} = b(\lambda_1-\lambda_2)u^2 v - b^2\lambda_1 u^2, \quad \dot{v} = -2b(\lambda_1-\lambda_2)uv^2 + 2b^2(\lambda_1-\lambda_2)uv. \tag{23}$$

Taking the rescaling given by $\mathrm{d}s = u\,\mathrm{d}t$, system (23) writes

$$u' = b(\lambda_1-\lambda_2)uv - b^2\lambda_1 u, \quad v' = -2b(\lambda_1-\lambda_2)v^2 + 2b^2(\lambda_1-\lambda_2)v. \tag{24}$$

The singularities (u_0, v_0) of system (24) with $u_0 = 0$ are $(0,0)$ and $(0,b)$. The eigenvalues of the Jacobian matrix of the vector field defined by system (24) evaluated at $(0,0)$ and $(0,b)$ are $-b^2\lambda_1, 2b^2(\lambda_1-\lambda_2)$ and $-b^2, -2b^2(\lambda_1-\lambda_2)$, respectively.

Changing y by $-y$ and v by $-v$, Fig. 14 corresponds also to the case $b < 0$ and $\lambda_1 < \lambda_2 < 0$. If $b < 0$ and $0 < \lambda_2 < \lambda_1$, then the orientation of the orbits is reversed.

Changing y by $-y$ and v by $-v$, Fig. 15 corresponds also to the case $b < 0$ and $\lambda_2 < \lambda_1 < 0$. If $b < 0$ and $0 < \lambda_1 < \lambda_2$, then the orientation of the orbits is reversed.

Changing y by $-y$ and v by $-v$, Fig. 16 corresponds also to the case $b < 0$ and $\lambda_1 < 0 < \lambda_2$. If $b < 0$ and $\lambda_2 < 0 < \lambda_1$, then the orientation of the orbits is reversed.

Observe that, for system (22), we have $\dot{x}|_{x=0} = b(\lambda_1-\lambda_2)y$. System (22) is invariant under the change of coordinates $(t,x,y) \mapsto (-t,-x,y)$, then system (22) is symmetric with respect to the y-axis. Then we obtain the following conclusions. If $\lambda_1\lambda_2 > 0$, then the sectorial decomposition of system (22) at $(0,0)$ is composed by one elliptic sector, one hyperbolic sector, one attracting sector, and one repelling sector. If $\lambda_1\lambda_2 < 0$, then $(0,0)$ is a saddle point of system (22).

Observe that from the blow-up we obtain that the straight line $v = b$ of system (24) corresponds to the parabola $y = bx^2$ of system (22). Then we have the following conclusions on the separatrices of system (22). For the cases of Fig. 14, the separatrices are contained in the axis $y = 0$ and in the region $y \ge bx^2$. For the cases of Fig. 15, the separatrices are contained in $y = bx^2$ and in the region $y \le 0$. For the cases of Fig. 16, the separatrices are contained in $y = bx^2$ and in $y = 0$.

We present the sectorial decomposition of system (4) at (x_0, y_0) in Fig. 10.

Case 3: $a^2 < 4bc$. In this case, the unique solution to Eq. (17) is $x = 0$, then the unique singularity (x_0, y_0) of system (4) is given by $x_0 = -a/(2b)$ and $y_0 = ((4bc - a^2)\lambda_2 + a^2(\lambda_1-\lambda_2))/(4b^2(\lambda_1-\lambda_2))$. Observe that $(x_0, y_0) \notin p \cup l$.

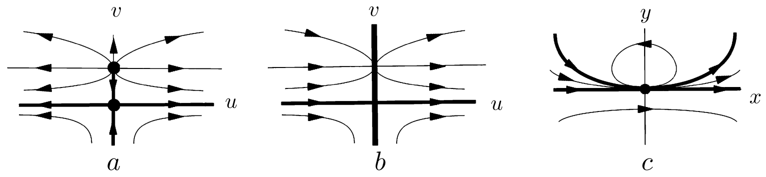

Fig. 14. Phase portrait of system (22) at (0, 0) for the case $b > 0$ and $\lambda_1 < \lambda_2 < 0$. If $b > 0$ and $0 < \lambda_2 < \lambda_1$, then the orientation of the orbits is reversed. (a) System (24). (b) System (23). (c) System (22).

Fig. 14. Portrait de phase du système (22) au point singulier (0, 0) avec $b > 0$ et $\lambda_1 < \lambda_2 < 0$. Si $b > 0$ et $0 < \lambda_2 < \lambda_1$ alors l'orientation des orbites doit être inversée. (a) Système (24). (b) Système (23). (c) Système (22).

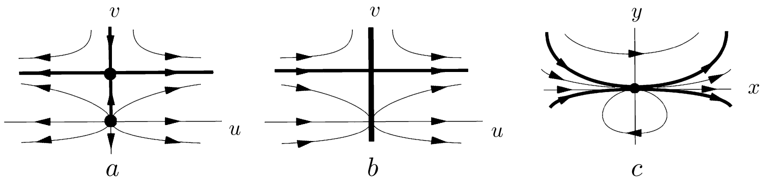

Fig. 15. Phase portrait of system (22) at (0, 0) for the case $b > 0$ and $\lambda_2 < \lambda_1 < 0$. If $b > 0$ and $\lambda_2 < \lambda_1 < 0$, then the orientation of the orbits is reversed. (a) System (24). (b) System (23). (c) System (22).

Fig. 15. Portrait de phase du système (22) au point singulier (0, 0) avec $b > 0$ et $\lambda_2 < \lambda_1 < 0$. Si $b > 0$ et $\lambda_2 < \lambda_1 < 0$, alors l'orientation des orbites doit être inversée. (a) Système (24). (b) Système (23). (c) Système (22).

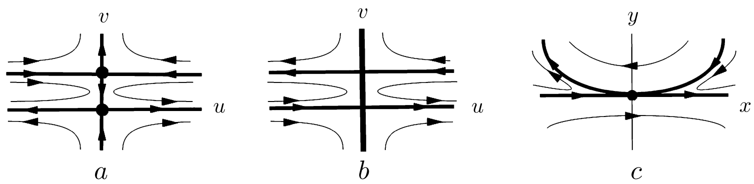

Fig. 16. Phase portrait of system (22) at (0, 0) for the case $b > 0$ and $\lambda_1 < 0 < \lambda_2$. If $b > 0$ and $\lambda_2 < 0 < \lambda_1$, then the orientation of the orbits is reversed. (a) System (24). (b) System (23). (c) System (22).

Fig. 16. Portrait de phase du système (22) au point singulier (0, 0) avec $b > 0$ et $\lambda_1 < 0 < \lambda_2$. Si $b > 0$ et $\lambda_2 < 0 < \lambda_1$, alors l'orientation des orbites doit être inversée. (a) Système (24). (b) Système (23). (c) Système (22).

The eigenvalues of the Jacobian matrix of the vector field defined by system (4) evaluated at (x_0, y_0) are given by (18). If $\lambda_1\lambda_2 < 0$, then (x_0, y_0) is a hyperbolic saddle point of system (4). If $\lambda_1\lambda_2 > 0$, then the eigenvalues are purely imaginary and (x_0, y_0) is a center of system (4). We show the sectorial decomposition of system (4) at (x_0, y_0) in Fig. 11. The proof of statement (a) is complete.

The proof of statement (b) is analogous to the proof of $\alpha(\Gamma_1) = \{(x_1, y_1)\}$ done in Case 2. □

5.2. *Infinite singularities of system (4)*

The expression of the Poincaré compactification of system (4) in the charts (U_1, φ_1) and (U_2, φ_2) is

$$\begin{aligned} \dot{u} &= \lambda_2(a + bu + cv)(uv - 2) + \lambda_1(1 - uv)(a + bu), \\ \dot{v} &= \lambda_2(a + bu + cv)v^2 + b\lambda_1(1 - uv)v, \end{aligned} \tag{25}$$

and

$$\begin{aligned} \dot{u} &= \lambda_2(au + b + cv)(2u^2 - v) + \lambda_1(v - u^2)(au + b), \\ \dot{v} &= 2\lambda_2(au + b + cv)uv + a\lambda_1(v - u^2)v, \end{aligned} \tag{26}$$

respectively.

5.2.1. *Infinite singularities in U_1*

In this section, we will determine the sectorial decomposition of the singularities of system (25), which are of the form $(u, 0)$, with $u \in \mathbb{R}$. We consider the cases $\lambda_1 - 2\lambda_2 \neq 0$ and $\lambda_1 - 2\lambda_2 = 0$ separately. The first one, studied in Lemma 5.4, follows from the Grobman–Hartman Theorem and the proof is omitted. In the second case, $v = 0$ is a line of singularities and we study the phase portrait of the corresponding system in Lemma 5.5.

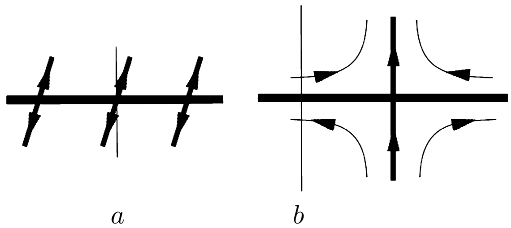

Fig. 17. Phase portrait of system (25). (a) $\lambda_1 = 2\lambda_2$ and $b\lambda_2 > 0$. If $\lambda_1 = 2\lambda_2$ and $b\lambda_2 < 0$, then the orientation of the orbits is reversed. (b) $\lambda_1 = 2\lambda_2, b = 0$ and $a\lambda_2 > 0$. If $\lambda_1 = 2\lambda_2, b = 0$ and $a\lambda_2 < 0$, then the orientation of the orbits is reversed.

Fig. 17. Portrait de phase du système (25). (a) $\lambda_1 = 2\lambda_2$ et $b\lambda_2 > 0$. Si $\lambda_1 = 2\lambda_2$ et $b\lambda_2 < 0$, alors l'orientation des orbites doit être inversée. (b) $\lambda_1 = 2\lambda_2, b = 0$ et $a\lambda_2 > 0$. Si $\lambda_1 = 2\lambda_2, b = 0$ et $a\lambda_2 < 0$, alors l'orientation des orbites doit être inversée.

Lemma 5.4. *Suppose that* $\lambda_1\lambda_2 \neq 0$ *and* $\lambda_1 - 2\lambda_2 \neq 0$.

(a) *If* $b = 0$, *then system* (25) *has no singularities in a neighborhood of the axis* $v = 0$.
(b) *Suppose* $b \neq 0$. *Then the unique singularity* (u_0, v_0) *of system* (25) *with* $v_0 = 0$ *is given by* $u_0 = -a/b$. *If* $b\lambda_1 > 0$ *and* $b(\lambda_1 - 2\lambda_2) > 0$, *then* $(u_0, 0)$ *is a hyperbolic repelling node of system* (25). *If* $b\lambda_1 < 0$ *and* $b(\lambda_1 - 2\lambda_2) < 0$, *then* $(u_0, 0)$ *is a hyperbolic attracting node of system* (25). *If* $\lambda_1(\lambda_1 - 2\lambda_2) < 0$, *then* $(u_0, 0)$ *is a hyperbolic saddle point of system* (25).

Lemma 5.5. *Suppose that* $\lambda_1\lambda_2 \neq 0$ *and* $\lambda_1 - 2\lambda_2 = 0$. *The phase portrait of system* (25) *in* $\mathbb{R}^2$ *is shown in Fig. 17.*

Proof. After the rescaling given by $ds = \lambda_2 v\, dt$ system (25) becomes

$$u' = cuv - bu^2 - au - 2c, \quad v' = cv^2 + (a - bu)v + 2b. \tag{27}$$

We consider the cases $b \neq 0$ and $b = 0$ separately.

Case 1: $b \neq 0$. In this case, system (27) has no singularities (u_0, v_0) with $v_0 = 0$ and we show the phase portrait of system (25) in a sufficiently small neighborhood of the axis $v = 0$ in Fig. 17a.

Case 2: $b = 0$. From the hypothesis $a^2 + b^2 \neq 0$, it follows that $a \neq 0$. In this case, the unique singularity (u_0, v_0) of system (27) with $v_0 = 0$ is $u_0 = -2c/a$.

The eigenvalues of the Jacobian matrix of the vector field defined by system (27) evaluated at $(u_0, 0)$ are $-a$ and $-2c^2/a$. Then $(u_0, 0)$ is a hyperbolic saddle point of system (27). We present the phase portrait of system (25) in a sufficiently small neighborhood of the axis $v = 0$ in Fig. 17b. □

5.2.2. The origin of U_2

We denote by p' the parabola given by $v = u^2$. From Remark 4, it follows that the points of $p \setminus \{(0, 0)\}$ are mapped into the points of p'.

The Jacobian matrix of the vector field defined by system (26) evaluated at $(0, 0)$ is

$$\begin{pmatrix} 0 & b(\lambda_1 - \lambda_2) \\ 0 & 0 \end{pmatrix}.$$

Then we consider the cases $b = 0$, $b \neq 0$ and $\lambda_1 = \lambda_2$, and $b(\lambda_1 - \lambda_2) \neq 0$ separately. In the last case, the origin is a nilpotent singularity of system (26). In order to obtain information on the localization of the separatrices, we will use blow-ups to study the singularity $(0, 0)$.

Lemma 5.6. *Suppose that* $\lambda_1\lambda_2 \neq 0$ *and* $b = 0$.

(a) *If* $\lambda_1 \neq 2\lambda_2$, *then the sectorial decomposition of system* (26) *at the origin is shown in Fig. 18.*
(b) *If* $\lambda_1 = 2\lambda_2$, *then the sectorial decomposition of system* (26) *at the origin is shown in Fig. 19.*

Proof. We consider the cases $\lambda_1 - 2\lambda_2 \neq 0$ and $\lambda_1 - 2\lambda_2 = 0$ separately.

Case 1: $\lambda_1 - 2\lambda_2 \neq 0$. We take the quasi-homogeneous blow-up in the positive v-direction given by $u = \overline{u}\ \overline{v}$, $v = \overline{v}^2$. Then system (26) becomes

$$\begin{aligned} \dot{\overline{u}} &= \left[2c\lambda_2(\overline{u}^2 - 1)\overline{v}^3 + a(2\lambda_2 - \lambda_1)(\overline{u}^2 - 1)\overline{u}\,\overline{v}^2\right]/2, \\ \dot{\overline{v}} &= \left[c\lambda_2\overline{u}\,\overline{v} + ((2\lambda_2 - \lambda_1)\overline{u}^2 + a\lambda_1)\right]\overline{v}^3/2. \end{aligned} \tag{28}$$

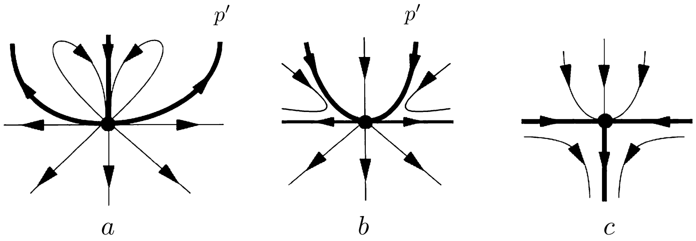

Fig. 18. Phase portrait of system (26) at $(0,0)$ for the case $b=0$. (a) $a\lambda_1<0, a\lambda_2>0$ and $a(\lambda_1-2\lambda_2)<0$. If $a\lambda_1>0, a\lambda_2<0$ and $a(\lambda_1-2\lambda_2)>0$, then the orientation of the orbits is reversed. (b) $a\lambda_1<0, a\lambda_2<0$ and $a(\lambda_1-2\lambda_2)<0$. If $a\lambda_1>0, a\lambda_2>0$ and $a(\lambda_1-2\lambda_2)>0$, then the orientation of the orbits is reversed. (c) $a\lambda_1<0, a\lambda_2<0$ and $a(\lambda_1-2\lambda_2)>0$. If $a\lambda_1>0, a\lambda_2>0$ and $a(\lambda_1-2\lambda_2)<0$, then the orientation of the orbits is reversed.

Fig. 18. Portrait de phase du système (26) au point singulier $(0,0)$ avec $b=0$. (a) $a\lambda_1<0, a\lambda_2>0$ et $a(\lambda_1-2\lambda_2)<0$. Si $a\lambda_1>0, a\lambda_2<0$ et $a(\lambda_1-2\lambda_2)>0$, alors l'orientation des orbites doit être inversée. (b) $a\lambda_1<0, a\lambda_2<0$ et $a(\lambda_1-2\lambda_2)<0$. Si $a\lambda_1>0, a\lambda_2>0$ et $a(\lambda_1-2\lambda_2)>0$, alors l'orientation des orbites doit être inversée. (c) $a\lambda_1<0, a\lambda_2<0$ et $a(\lambda_1-2\lambda_2)>0$. Si $a\lambda_1>0, a\lambda_2>0$ et $a(\lambda_1-2\lambda_2)<0$, alors l'orientation des orbites doit être inversée.

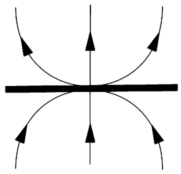

Fig. 19. Phase portrait of system (26) at $(0,0)$ for the case $\lambda_1\lambda_2\neq 0, b=0, \lambda_1=2\lambda_2$ and $a\lambda_2>0$. If $\lambda_1\lambda_2\neq 0, b=0, \lambda_1=2\lambda_2$ and $a\lambda_2<0$, then the orientation of the orbits is reversed.

Fig. 19. Portrait de phase du système (26) au point singulier $(0,0)$ avec $\lambda_1\lambda_2\neq 0, b=0, \lambda_1=2\lambda_2$ et $a\lambda_2>0$. Si $\lambda_1\lambda_2\neq 0, b=0, \lambda_1=2\lambda_2$ et $a\lambda_2<0$, alors l'orientation des orbites doit être inversée.

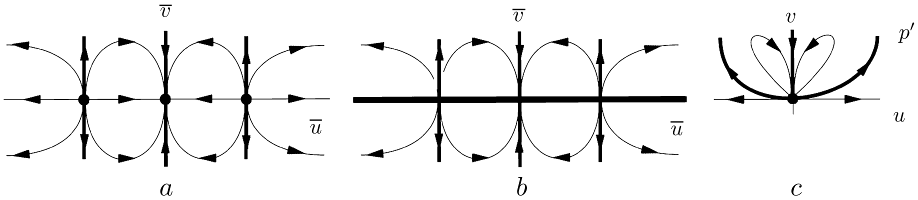

Fig. 20. Phase portrait of system (26) at $(0,0)$ with $v\geq 0$, for the case $b=0, a(2\lambda_2-\lambda_1)>0, a\lambda_2>0$ and $a\lambda_1<0$. If $b=0, a(2\lambda_2-\lambda_1)<0, a\lambda_2<0$ and $a\lambda_1>0$, then the orientation of the orbits is reversed. (a) System (29). (b) System (28). (c) Phase portrait of system (26) at $(0,0)$ with $v\geq 0$.

Fig. 20. Portrait de phase du système (26) au point singulier $(0,0)$ avec $v\geq 0$, pour le cas $b=0, a(2\lambda_2-\lambda_1)>0, a\lambda_2>0$ et $a\lambda_1<0$. Si $b=0, a(2\lambda_2-\lambda_1)<0, a\lambda_2<0$ et $a\lambda_1>0$, alors l'orientation des orbites doit être inversée. (a) Système (29). (b) Système (28). (c) Portrait de phase du système (26) à $(0,0)$ avec $v\geq 0$.

Taking the rescaling given by $ds=\overline{v}^2/2\,dt$, system (28) becomes

$$\begin{aligned}\overline{u}' &= c\lambda_2(\overline{u}^2-1)\overline{v}+a(2\lambda_2-\lambda_1)(\overline{u}^2-1)\overline{u},\\ \overline{v}' &= \left[c\lambda_2\overline{u}\,\overline{v}+(a(2\lambda_2-\lambda_1)\overline{u}^2+a\lambda_1)\right]\overline{v}.\end{aligned} \tag{29}$$

The singularities $(\overline{u}_0,\overline{v}_0)$ of system (29) with $\overline{v}_0=0$ are $(0,0)$ and $(\pm 1,0)$. The eigenvalues of the Jacobian matrix of the vector field defined by system (29) evaluated at $(0,0)$ are $-a(2\lambda_2-\lambda_1)$ and $a\lambda_1$. The eigenvalues of the Jacobian matrix of the vector field defined by system (29) evaluated at $(1,0)$ or $(-1,0)$ are $2a(2\lambda_2-\lambda_1)$ and $2a\lambda_2$. Then $(0,0)$ and $(\pm1,0)$ are hyperbolic singularities of system (29).

For system (26), we have that

$$\dot{u}|_{v=0}=a(2\lambda_2-\lambda_1)u^3 \text{ and } \dot{v}|_{u=0}=a\lambda_1 v^2. \tag{30}$$

Observe that the straight lines $\overline{u}=\pm1$ are invariant under the flow of system (29), then the parabola p' contain separatrices of system (26) for the cases present in Figs. 20 and 21. The quasi-homogeneous blow-up in the positive v-direction is summarized in Figs. 20–22.

We take the quasi-homogeneous blow-up in the negative v-direction given by $u=\overline{u}\;\overline{v},\;\; v=-\overline{v}^2$. Then system (26) becomes

$$\begin{aligned}\dot{\overline{u}} &= -\left[2c\lambda_2(\overline{u}^2+1)\overline{v}+a(\lambda_1-2\lambda_2)(\overline{u}^3+\overline{u})\right]\overline{v}^2/2,\\ \dot{\overline{v}} &= -\left[2c\lambda_2\overline{u}\,\overline{v}+a((\lambda_1-2\lambda_2)\overline{u}^2+\lambda_1)\right]\overline{v}^3/2.\end{aligned} \tag{31}$$

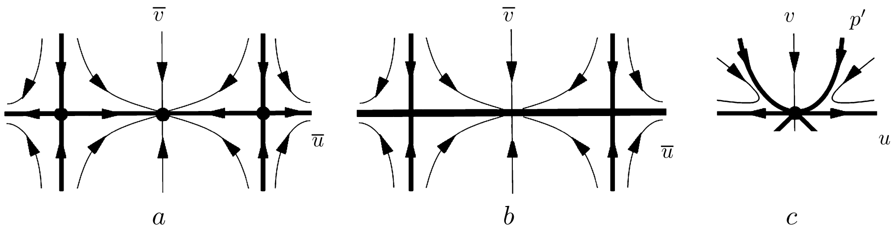

Fig. 21. Phase portrait of system (26) at $(0,0)$ with $v \geq 0$, for the case $b = 0, a(2\lambda_2 - \lambda_1) > 0, a\lambda_2 < 0$ and $a\lambda_1 < 0$. If $b = 0, a(2\lambda_2 - \lambda_1) < 0, a\lambda_2 > 0$ and $a\lambda_1 > 0$, then the orientation of the orbits is reversed. (a) System (29). (b) System (28). (c) Phase portrait of system (26) at $(0,0)$ with $v \geq 0$.

Fig. 21. Portrait de phase du système (26) au point singulier $(0,0)$ avec $v \geq 0$, pour le cas $b = 0, a(2\lambda_2 - \lambda_1) > 0, a\lambda_2 < 0$ et $a\lambda_1 < 0$. Si $b = 0, a(2\lambda_2 - \lambda_1) < 0, a\lambda_2 > 0$ et $a\lambda_1 > 0$, alors l'orientation des orbites doit être inversée. (a) Système (29). (b) Système (28). (c) Portrait de phase du système (26) à $(0,0)$ avec $v \geq 0$.

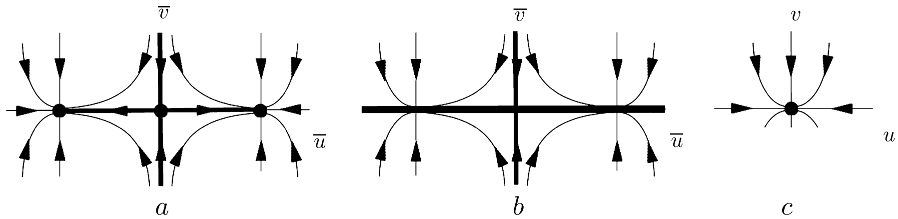

Fig. 22. Phase portrait of system (26) at $(0,0)$ with $v \geq 0$, for the case $b = 0, a(2\lambda_2 - \lambda_1) < 0, a\lambda_2 < 0$ and $a\lambda_1 < 0$. If $b = 0, a(2\lambda_2 - \lambda_1) > 0, a\lambda_2 > 0$ and $a\lambda_1 > 0$, then the orientation of the orbits is reversed. (a) System (29). (b) System (28). (c) Phase portrait of system (26) at $(0,0)$ with $v \geq 0$.

Fig. 22. Portrait de phase du système (26) au point singulier $(0,0)$ avec $v \geq 0$, pour le cas $b = 0, a(2\lambda_2 - \lambda_1) < 0, a\lambda_2 < 0$ et $a\lambda_1 < 0$. Si $b = 0, a(2\lambda_2 - \lambda_1) > 0, a\lambda_2 > 0$ et $a\lambda_1 > 0$, alors l'orientation des orbites doit être inversée. (a) Système (29). (b) Système (28). (c) Portrait de phase du système (26) au point singulier $(0,0)$ avec $v \geq 0$.

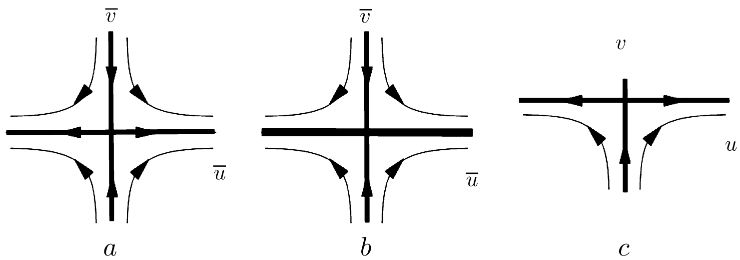

Fig. 23. Phase portrait of system (26) at $(0,0)$ with $v \leq 0$, for the case $\lambda_2 \neq 0, b = 0, a\lambda_1 > 0$ and $a(2\lambda_2 - \lambda_1) > 0$. If $\lambda_2 \neq 0, b = 0, a\lambda_1 < 0$ and $a(2\lambda_2 - \lambda_1) < 0$, then the orientation of the orbits is reversed. (a) System (32). (b) System (31). (c) Phase portrait of system (26) at $(0,0)$ with $v \leq 0$.

Fig. 23. Portrait de phase du système (26) au point singulier $(0,0)$ avec $v \leq 0$, pour le cas $\lambda_2 \neq 0, b = 0, a\lambda_1 > 0$ et $a(2\lambda_2 - \lambda_1) > 0$. Si $\lambda_2 \neq 0, b = 0, a\lambda_1 < 0$ et $a(2\lambda_2 - \lambda_1) < 0$, alors l'orientation des orbites doit être inversée. (a) Système (32). (b) Système (31). (c) Portrait de phase du système (26) au point singulier $(0,0)$ avec $v \leq 0$.

Observe that this change of coordinates maps points $(\overline{u}, \overline{v})$ with $\overline{v} > 0$ in points (u, v) with $v < 0$. Taking the rescaling given by $ds = \overline{v}^2/2\,dt$, system (31) becomes

$$\begin{aligned} \overline{u}' &= -\left[2c\lambda_2(\overline{u}^2+1)\overline{v} + a(\lambda_1 - 2\lambda_2)(\overline{u}^3 + \overline{u})\right], \\ \overline{v}' &= -\left[2c\lambda_2\overline{u}\,\overline{v} + a((\lambda_1 - 2\lambda_2)\overline{u}^2 + \lambda_1)\right]\overline{v}. \end{aligned} \tag{32}$$

The singularity $(\overline{u}_0, \overline{v}_0)$ of system (32) with $\overline{v}_0 = 0$ is $(0,0)$. The eigenvalues of the Jacobian matrix of the vector field defined by system (32) evaluated at $(0,0)$ are $a(2\lambda_2 - \lambda_1)$ and $-a\lambda_1$. Then $(0,0)$ is a hyperbolic singular point of system (32).

Using (30), we summarize the quasi-homogeneous blow-up in the negative v-direction in Figs. 23 and 24.

The sectorial decomposition of system (26) at $(0,0)$ shown in Fig. 18a follows from Fig. 20c and Fig. 24c (with reversed orientation). The sectorial decomposition of system (26) at $(0,0)$ given in Fig. 18b follows from Figs. 21c and 24c (with reversed orientation). The sectorial decomposition of system (26) at $(0,0)$ shown in Fig. 18c follows from Figs. 22c and 23c (with reversed orientation).

Case 2: $\lambda_1 = 2\lambda_2$. After the rescaling given by $ds = \lambda_2 v\,dt$, system (26) becomes

$$u' = -cv + 2cu^2 + au, \quad v' = 2(cu + a)v. \tag{33}$$

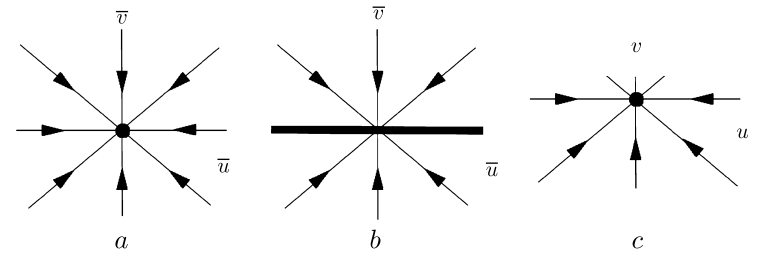

Fig. 24. Phase portrait of system (26) at (0, 0) with $v \le 0$, for the case $\lambda_2 \neq 0, b = 0, a\lambda_1 > 0$ and $a(2\lambda_2 - \lambda_1) < 0$. If $\lambda_2 \neq 0, b = 0, a\lambda_1 < 0$ and $a(2\lambda_2 - \lambda_1) > 0$, then the orientation of the orbits is reversed. (a) System (32). (b) System (31). (c) Phase portrait of system (26) at (0, 0) with $v \le 0$.

Fig. 24. Portrait de phase du système (26) au point singulier (0, 0) avec $v \le 0$, pour le cas $\lambda_2 \neq 0, b = 0, a\lambda_1 > 0$ et $a(2\lambda_2 - \lambda_1) < 0$. Si $\lambda_2 \neq 0, b = 0, a\lambda_1 < 0$ et $a(2\lambda_2 - \lambda_1) > 0$, alors l'orientation des orbites doit être inversée. (a) Système (32). (b) Système (31). (c) Portrait de phase du système (26) au point singulier (0, 0) avec $v \le 0$.

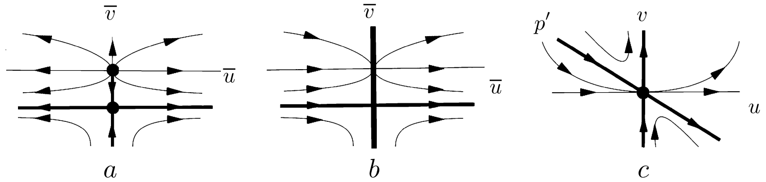

Fig. 25. Phase portrait of system (26) at (0, 0) for the case $\lambda_1 = \lambda_2 > 0, a > 0, b > 0$ and $c = 0$, or $\lambda_1 = \lambda_2 < 0, a < 0, b < 0$ and $c = 0$. If $\lambda_1 = \lambda_2 > 0, a < 0, b < 0$ and $c = 0$, or $\lambda_1 = \lambda_2 < 0, a > 0, b > 0$ and $c = 0$, then the orientation of the orbits is reversed. (a) System (36). (b) System (35). (c) Phase portrait of system (26) at (0, 0).

Fig. 25. Portrait de phase du système (26) au point singulier (0, 0) pour le cas $\lambda_1 = \lambda_2 > 0, a > 0, b > 0$ et $c = 0$, ou $\lambda_1 = \lambda_2 < 0, a < 0, b < 0$ et $c = 0$. Si $\lambda_1 = \lambda_2 > 0, a < 0, b < 0$ et $c = 0$, ou $\lambda_1 = \lambda_2 < 0, a > 0, b > 0$ et $c = 0$, alors l'orientation des orbites doit être inversée. (a) Système (36). (b) Système (35). (c) Portrait de phase du système (26) au point singulier (0, 0).

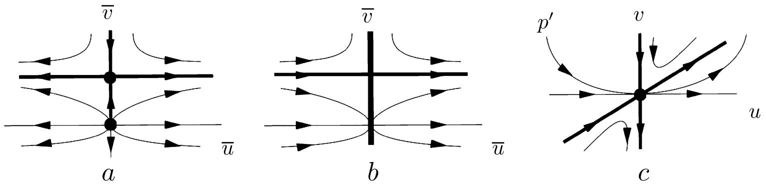

Fig. 26. Phase portrait of system (26) at (0, 0) for the case $\lambda_1 = \lambda_2 > 0, a < 0, b > 0$ and $c = 0$, or $\lambda_1 = \lambda_2 < 0, a > 0, b < 0$ and $c = 0$. If $\lambda_1 = \lambda_2 < 0, a < 0, b > 0$ and $c = 0$, or $\lambda_1 = \lambda_2 > 0, a > 0, b < 0$ and $c = 0$, then the orientation of the orbits is reversed. (a) System (36). (b) System (35). (c) Phase portrait of system (26) at (0, 0).

Fig. 26. Portrait de phase du système (26) au point singulier (0, 0) pour le cas $\lambda_1 = \lambda_2 > 0, a < 0, b > 0$ et $c = 0$, ou $\lambda_1 = \lambda_2 < 0, a > 0, b < 0$ et $c = 0$. If $\lambda_1 = \lambda_2 < 0, a < 0, b > 0$ et $c = 0$, ou $\lambda_1 = \lambda_2 > 0, a > 0, b < 0$ et $c = 0$, alors l'orientation des orbites doit être inversée. (a) Système (36). (b) Système (35). (c) Portrait de phase du système (26) au point singulier (0, 0).

The eigenvalues of the Jacobian matrix of the vector field defined by system (33) evaluated at (0, 0) are a and $2a$. Then (0, 0) is a hyperbolic singular point of system (33). We present the local phase portrait of system (26) at (0, 0) in Fig. 19. □

Observe that in Lemma 5.7-e, we will present a finite number of possibilities for the sectorial decomposition of system (26). But, in the proof of Theorem 1.1, we will prove that the sectorial decomposition of system (26) at the origin, for this case, is composed by two elliptic sectors.

Lemma 5.7. *Suppose that $\lambda_1\lambda_2 \neq 0, b \neq 0$ and $\lambda_1 = \lambda_2$.*

(a) *If $a = c = 0$, then the sectorial decomposition of system (26) at the origin is shown in Fig. 30.*
(b) *If $c = 0$ and $a \neq 0$, then the sectorial decomposition of system (26) at the origin is given in Figs. 25c and 26c.*
(c) *If $c \neq 0$ and $a^2 > 4bc$, then the sectorial decomposition of system (26) at the origin is shown in Figs. 27c, 28c, and 29c.*
(d) *If $c \neq 0$ and $a^2 = 4bc$, then the sectorial decomposition of system (26) at the origin is shown in Figs. 31d and 32d.*
(e) *If $c \neq 0$ and $a^2 < 4bc$, then the sectorial decomposition of system (26) at the origin is given in Fig. 33c.*

Proof. Taking the rescaling given by $ds = \lambda_1\, dt$, system (26) becomes

$$u' = 2cu^2v + au^3 - cv^2 + bu^2, \quad v' = 2cuv^2 + au^2v + av^2 + 2buv. \tag{34}$$

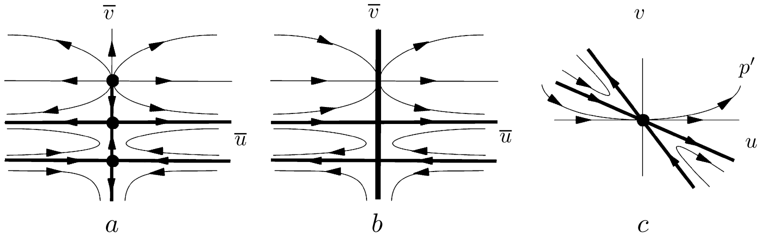

Fig. 27. Phase portrait of system (26) at (0, 0) for the case $\lambda_1 = \lambda_2 > 0, a > 0, b > 0, c > 0$ and $a^2 - 4bc > 0$, or $\lambda_1 = \lambda_2 < 0, a < 0, b < 0, c < 0$ and $a^2 - 4bc > 0$. If $\lambda_1 = \lambda_2 < 0, a > 0, b > 0, c > 0$ and $a^2 - 4bc > 0$, or $\lambda_1 = \lambda_2 > 0, a < 0, b < 0, c < 0$ and $a^2 - 4bc > 0$ then the orientation of the orbits is reversed. (a) System (36). (b) System (35). (c) Phase portrait of system (26) at (0, 0).

Fig. 27. Portrait de phase du système (26) au point singulier (0, 0) pour le cas $\lambda_1 = \lambda_2 > 0, a > 0, b > 0, c > 0$ et $a^2 - 4bc > 0$, ou $\lambda_1 = \lambda_2 < 0, a < 0, b < 0, c < 0$ et $a^2 - 4bc > 0$. Si $\lambda_1 = \lambda_2 < 0, a > 0, b > 0, c > 0$ et $a^2 - 4bc > 0$, ou $\lambda_1 = \lambda_2 > 0, a < 0, b < 0, c < 0$ et $a^2 - 4bc > 0$, alors l'orientation des orbites doit être inversée. (a) Système (36). (b) Système (35). (c) Portrait de phase du système (26) au point singulier (0, 0).

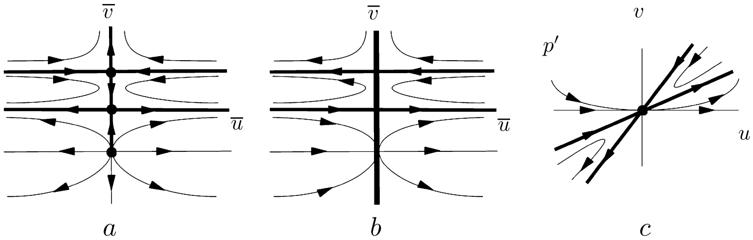

Fig. 28. Phase portrait of system (26) at (0, 0) for the case $\lambda_1 = \lambda_2 > 0, a < 0, b > 0, c > 0$ and $a^2 - 4bc > 0$, or $\lambda_1 = \lambda_2 < 0, a > 0, b < 0, c < 0$ and $a^2 - 4bc > 0$. If $\lambda_1 = \lambda_2 < 0, a < 0, b > 0, c > 0$ and $a^2 - 4bc > 0$, or $\lambda_1 = \lambda_2 > 0, a > 0, b < 0, c < 0$ and $a^2 - 4bc > 0$, then the orientation of the orbits is reversed. (a) System (36). (b) System (35). (c) Phase portrait of system (26) at (0, 0).

Fig. 28. Portrait de phase du système (26) au point singulier (0, 0) pour le cas $\lambda_1 = \lambda_2 > 0, a < 0, b > 0, c > 0$ et $a^2 - 4bc > 0$, ou $\lambda_1 = \lambda_2 < 0, a > 0, b < 0, c < 0$ et $a^2 - 4bc > 0$. Si $\lambda_1 = \lambda_2 < 0, a < 0, b > 0, c > 0$ et $a^2 - 4bc > 0$, ou $\lambda_1 = \lambda_2 > 0, a > 0, b < 0, c < 0$ and $a^2 - 4bc > 0$, alors l'orientation des orbites doit être inversée. (a) Système (36). (b) Système (35). (c) Portrait de phase du système (26) au point singulier (0, 0).

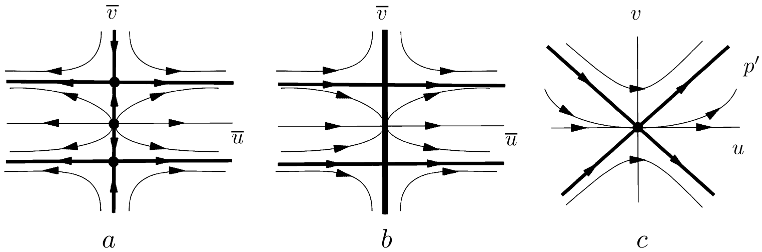

Fig. 29. Phase portrait of system (26) at (0, 0) for the case $\lambda_1 = \lambda_2 > 0, b > 0, c < 0$ and $a^2 - 4bc > 0$, or $\lambda_1 = \lambda_2 < 0, b < 0, c > 0$ and $a^2 - 4bc > 0$. If $\lambda_1 = \lambda_2 < 0, b > 0, c < 0$ and $a^2 - 4bc > 0$, or $\lambda_1 = \lambda_2 > 0, b < 0, c > 0$ and $a^2 - 4bc > 0$ then the orientation of the orbits is reversed. (a) System (36). (b) System (35). (c) Phase portrait of system (26) at (0, 0).

Fig. 29. Portrait de phase du système (26) au point singulier (0, 0) pour le cas $\lambda_1 = \lambda_2 > 0, b > 0, c < 0$ et $a^2 - 4bc > 0$, ou $\lambda_1 = \lambda_2 < 0, b < 0, c > 0$ et $a^2 - 4bc > 0$. Si $\lambda_1 = \lambda_2 < 0, b > 0, c < 0$ et $a^2 - 4bc > 0$, ou $\lambda_1 = \lambda_2 > 0, b < 0, c > 0$ et $a^2 - 4bc > 0$ alors l'orientation des orbites doit être inversée. (a) Système (36). (b) Système (35). (c) Portrait de phase du système (26) au point singulier (0, 0).

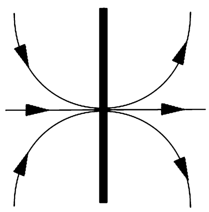

Fig. 30. Phase portrait of system (26) at (0, 0) for the case $\lambda_1 = \lambda_2 \neq 0, a = c = 0$ and $b\lambda_1 > 0$. If $\lambda_1 = \lambda_2 \neq 0, a = c = 0$, and $b\lambda_1 < 0$, then the orientation of the orbits is reversed.

Fig. 30. Portrait de phase du système (26) au point singulier (0, 0) pour le cas $\lambda_1 = \lambda_2 \neq 0, a = c = 0$ et $b\lambda_1 > 0$. Si $\lambda_1 = \lambda_2 \neq 0, a = c = 0$ et $b\lambda_1 < 0$, alors l'orientation des orbites doit être inversée.

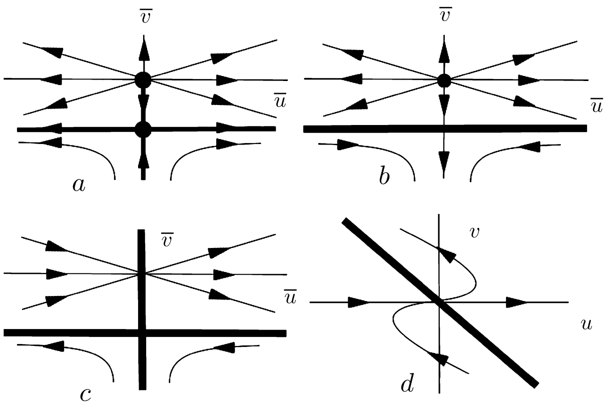

Fig. 31. Phase portrait of system (26) at $(0,0)$ for the case $\lambda_1 = \lambda_2 > 0, a > 0, b > 0, c > 0$ and $a^2 - 4bc = 0$, or $\lambda_1 = \lambda_2 < 0, a < 0, b < 0, c < 0$ and $a^2 - 4bc = 0$. If $\lambda_1 = \lambda_2 < 0, a > 0, b > 0, c > 0$ and $a^2 - 4bc = 0$, or $\lambda_1 = \lambda_2 > 0, a < 0, b < 0, c < 0$ and $a^2 - 4bc = 0$ then the orientation of the orbits is reversed. (a) System (37). (b) System (36). (c) System (35). (d) Phase portrait of system (26) at $(0,0)$.

Fig. 31. Portrait de phase du système (26) au point singulier $(0,0)$ pour le cas $\lambda_1 = \lambda_2 > 0, a > 0, b > 0, c > 0$ et $a^2 - 4bc = 0$, ou $\lambda_1 = \lambda_2 < 0, a < 0, b < 0, c < 0$ et $a^2 - 4bc = 0$. Si $\lambda_1 = \lambda_2 < 0, a > 0, b > 0, c > 0$ et $a^2 - 4bc = 0$, ou $\lambda_1 = \lambda_2 > 0, a < 0, b < 0, c < 0$ et $a^2 - 4bc = 0$, alors l'orientation des orbites doit être inversée. (a) Système (37). (b) Système (36). (c) Système (35). (d) Portrait de phase du système (26) au point singulier $(0,0)$.

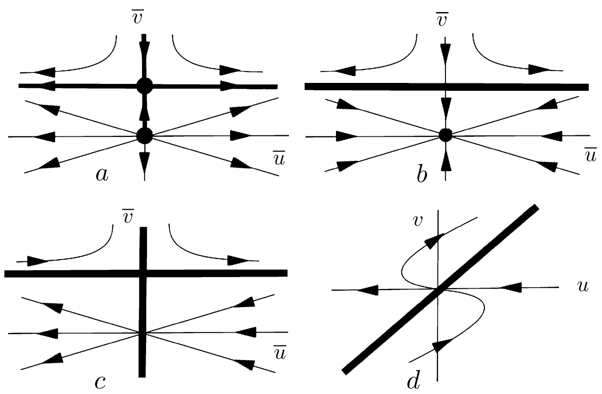

Fig. 32. Phase portrait of system (26) at $(0,0)$ for the case $\lambda_1 = \lambda_2 > 0, a > 0, b < 0, c < 0$ and $a^2 - 4bc = 0$, or $\lambda_1 = \lambda_2 < 0, a < 0, b > 0, c > 0$ and $a^2 - 4bc = 0$. If $\lambda_1 = \lambda_2 > 0, a < 0, b > 0, c > 0$ and $a^2 - 4bc = 0$, or $\lambda_1 = \lambda_2 < 0, a > 0, b < 0, c < 0$ and $a^2 - 4bc = 0$ then the orientation of the orbits is reversed. (a) System (37). (b) System (36). (c) System (35). (d) Phase portrait of system (26) at $(0,0)$.

Fig. 32. Portrait de phase du système (26) au point singulier $(0,0)$ pour le cas $\lambda_1 = \lambda_2 > 0, a > 0, b < 0, c < 0$ et $a^2 - 4bc = 0$, ou $\lambda_1 = \lambda_2 < 0, a < 0, b > 0, c > 0$ et $a^2 - 4bc = 0$. Si $\lambda_1 = \lambda_2 > 0, a < 0, b > 0, c > 0$ et $a^2 - 4bc = 0$, ou $\lambda_1 = \lambda_2 < 0, a > 0, b < 0, c < 0$ et $a^2 - 4bc = 0$, alors l'orientation des orbites doit être inversée. (a) Système (37). (b) Système (36). (c) Système (35). (d) Portrait de phase du système (26) au point singulier $(0,0)$.

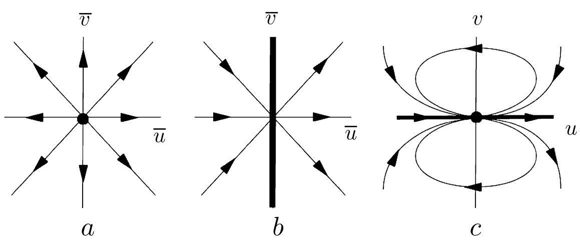

Fig. 33. Phase portrait of system (26) at $(0,0)$ for the case $\lambda_1 = \lambda_2 > 0, b > 0, c > 0$ and $a^2 - 4bc < 0$, or $\lambda_1 = \lambda_2 < 0, b < 0, c < 0$ and $a^2 - 4bc < 0$. If $\lambda_1 = \lambda_2 < 0, b > 0, c > 0$ and $a^2 - 4bc < 0$, or $\lambda_1 = \lambda_2 > 0, b < 0, c < 0$ and $a^2 - 4bc < 0$ then the orientation of the orbits is reversed. (a) System (36). (b) System (35). (c) Phase portrait of system (26) at $(0,0)$.

Fig. 33. Portrait de phase du système (26) au point singulier $(0,0)$ pour le cas $\lambda_1 = \lambda_2 > 0, b > 0, c > 0$ et $a^2 - 4bc < 0$, ou $\lambda_1 = \lambda_2 < 0, b < 0, c < 0$ et $a^2 - 4bc < 0$. Si $\lambda_1 = \lambda_2 < 0, b > 0, c > 0$ et $a^2 - 4bc < 0$, ou $\lambda_1 = \lambda_2 > 0, b < 0, c < 0$ et $a^2 - 4bc < 0$, alors l'orientation des orbites doit être inversée. (a) Système (36). (b) Système (35). (c) Portrait de phase du système (26) au point singulier $(0,0)$.

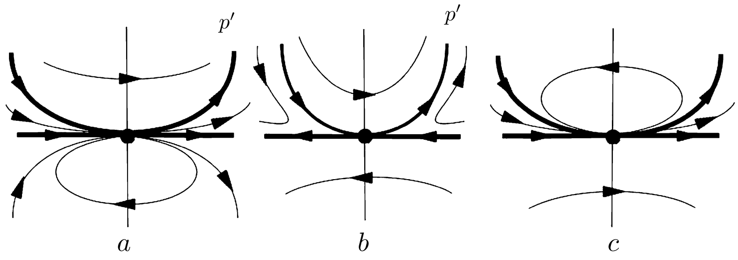

Fig. 34. Phase portrait of system (26) at the origin. (a) $b > 0$ and $0 < \lambda_2 < \lambda_1 < 2\lambda_2$, or $b < 0$ and $2\lambda_2 < \lambda_1 < \lambda_2 < 0$. If $b > 0$ and $2\lambda_2 < \lambda_1 < \lambda_2 < 0$, or $b < 0$ and $0 < \lambda_2 < \lambda_1 < 2\lambda_2$, then the orientation of the orbits is reversed. The separatrices are contained in $v = u^2$ and in the region $v \le 0$. (b) $b > 0$ and $0 < 2\lambda_2 < \lambda_1$, or $b < 0$ and $\lambda_1 < 2\lambda_2 < 0$. If $b > 0$ and $\lambda_1 < 2\lambda_2 < 0$, or $b < 0$ and $0 < 2\lambda_2 < \lambda_1$, then the orientation of the orbits is reversed. The separatrices are contained in $v = u^2$ and in $v = 0$. (c) $b > 0, \lambda_2 > 0$ and $\lambda_1 < \lambda_2$, or $b < 0, \lambda_2 < 0$ and $\lambda_1 > \lambda_2$. If $b > 0, \lambda_2 < 0$ and $\lambda_1 > \lambda_2$, or $b < 0, \lambda_2 > 0$ and $\lambda_1 < \lambda_2$, then the orientation of the orbits is reversed. The separatrices are contained in the region $v \ge u^2$ and in $v = 0$.

Fig. 34. Portrait de phase du système (26) au point singulier $(0, 0)$. (a) $b > 0$ et $0 < \lambda_2 < \lambda_1 < 2\lambda_2$, ou $b < 0$ et $2\lambda_2 < \lambda_1 < \lambda_2 < 0$. Si $b > 0$ et $2\lambda_2 < \lambda_1 < \lambda_2 < 0$, ou $b < 0$ et $0 < \lambda_2 < \lambda_1 < 2\lambda_2$, alors l'orientation des orbites doit être inversée. Les séparatrices sont contenues dans $v = u^2$ et $v \le 0$. (b) $b > 0$ et $0 < 2\lambda_2 < \lambda_1$, ou $b < 0$ et $\lambda_1 < 2\lambda_2 < 0$. Si $b > 0$ et $\lambda_1 < 2\lambda_2 < 0$, ou $b < 0$ et $0 < 2\lambda_2 < \lambda_1$, alors l'orientation des orbites doit être inversée. Les séparatrices sont contenues dans $v = u^2$ et $v = 0$. (c) $b > 0, \lambda_2 > 0$ et $\lambda_1 < \lambda_2$, ou $b < 0, \lambda_2 < 0$ et $\lambda_1 > \lambda_2$. Si $b > 0, \lambda_2 < 0$ et $\lambda_1 > \lambda_2$, ou $b < 0, \lambda_2 > 0$ et $\lambda_1 < \lambda_2$, alors l'orientation des orbites doit être inversée. Les séparatrices sont contenues dans $v \ge u^2$ et $v = 0$.

We take the blow-up in the u-direction given by $u = \overline{u},\ \ v = \overline{u}\ \overline{v}$, which maps points of the form $u < 0,\ \ v > 0$ into points of the form $\overline{u} < 0,\ \ \overline{v} < 0$, and points of the form $u < 0,\ \ v < 0$ into points of the form $\overline{u} < 0,\ \ \overline{v} > 0$. Then system (34) becomes

$$\dot{\overline{u}} = -c\overline{u}^2\overline{v}^2 + 2c\overline{u}^3\overline{v} + a\overline{u}^3 + b\overline{u}^2,\ \ \dot{\overline{v}} = c\,\overline{u}\,\overline{v}^3 + a\overline{u}\,\overline{v}^2 + b\,\overline{u}\,\overline{v}. \tag{35}$$

Taking the rescaling given by $\mathrm{d}s = \overline{u}\,\mathrm{d}t$, system (35) becomes

$$\overline{u}' = -c\,\overline{u}\,\overline{v}^2 + 2c\overline{u}^2\overline{v} + a\overline{u}^2 + b\overline{u},\ \ \overline{v}' = c\overline{v}^3 + a\overline{v}^2 + b\overline{v}. \tag{36}$$

The singularities $(\overline{u}_0, \overline{v}_0)$ of system (36) with $\overline{u}_0 = 0$ are $(0, 0)$ and $(0, \overline{v}_0)$, where $\overline{v}_0$ is a solution to the equation $c\overline{v}^2 + a\overline{v} + b = 0$. The eigenvalues of the Jacobian matrix of the vector field defined by system (36) evaluated at $(0, 0)$ are equal to b, then $(0, 0)$ is a hyperbolic singular point of system (36). In order to study the other singularities of system (36), we consider three cases.

Case 1: $a^2 - 4bc > 0$.

Case 1.1: $c = 0$. Then $a \ne 0$. The singularities $(\overline{u}_0, \overline{v}_0)$ of system (36) with $\overline{u} = 0$ are $(0, 0)$ and $(0, -b/a)$. The eigenvalues of the Jacobian matrix of the vector field defined by system (36) evaluated at $(0, -b/a)$ are b and $-b$. Observe that the straight line $\overline{v} = -b/a$ is invariant for system (36), then the straight line $v = -(b/a)u$ is invariant for system (26). Moreover, since $\lambda_1 = \lambda_2$ and $c = 0$, the axis $u = 0$ is invariant for system (26). We summarize the blow-up in Figs. 25 and 26.

Case 1.2: $c \ne 0$. The singularities $(\overline{u}_0, \overline{v}_0)$ of system (36) with $\overline{u} = 0$ are $(0, 0)$, $(0, \overline{v}_1)$ and $(0, \overline{v}_2)$, where $\overline{v}_1 = (-a + \sqrt{a^2 - 4bc})/(2c)$ and $\overline{v}_2 = -(a + \sqrt{a^2 - 4bc})/(2c)$. The eigenvalues of the Jacobian matrix of the vector field defined by system (36) evaluated at $(0, \overline{v}_j)$ are $\pm\overline{v}_j\sqrt{a^2 - 4bc}$, for $j = 1, 2$. Observe that the straight line $\overline{v} = \overline{v}_j$ is invariant for system (36), then the straight line $v = \overline{v}_j u$ is invariant for system (26), for $j = 1, 2$. The blow-up is summarized in Figs. 27–29.

Case 2: $a^2 = 4bc$.

Case 2.1: $a = c = 0$. In this case, the blow-up is superfluous. Taking the rescaling given by $\mathrm{d}s = u\,\mathrm{d}t$ in system (34), we obtain the sectorial decomposition of system (26) at $(0, 0)$ shown in Fig. 30.

Case 2.2: $ac \ne 0$. The singularities $(\overline{u}_0, \overline{v}_0)$ of system (36) with $\overline{u}_0 = 0$ are $(0, 0)$ and $(0, \overline{v}_1)$, where $\overline{v}_1 = -a/(2c)$. Taking the rescaling given by $\mathrm{d}\tau = (\overline{v} - \overline{v}_1)\,\mathrm{d}s$, system (36) becomes

$$\frac{\mathrm{d}\overline{u}}{\mathrm{d}\tau} = -c\overline{u}\,\overline{v} + 2c\overline{u}^2 + \frac{a\overline{u}}{2},\ \ \frac{\mathrm{d}\overline{v}}{\mathrm{d}\tau} = c\overline{v}(\overline{v} - \overline{v}_1). \tag{37}$$

The eigenvalues of the Jacobian matrix of the vector field defined by system (37) evaluated at $(0, 0)$ are equal to $a/2$. The eigenvalues of the Jacobian matrix of the vector field defined by system (37) evaluated at $(0, \overline{v}_1)$ are a and $-a/2$. Since $\lambda_1 = \lambda_2$, for system (26) we observe that $\dot{u}|_{u=0} = -\lambda_1 c v^2$, $\dot{u}|_{v=0} = \lambda_1 u^2(au + b)$ and $\dot{v}|_{u=0} = \lambda_1 a v^2$. The blow-up is summarized in Figs. 31 and 32.

Case 3. $a^2 < 4bc$. The unique singularity $(\overline{u}_0, \overline{v}_0)$ of system (36) with $\overline{u}_0 = 0$ is $(0, 0)$. The blow-up is summarized in Fig. 33. □

Lemma 5.8. *Suppose that* $\lambda_1\lambda_2 \ne 0$ *and* $b(\lambda_1 - \lambda_2) \ne 0$.

(a) *If* $\lambda_1 \ne 2\lambda_2$, *then the sectorial decomposition of system* (26) *at the origin is given in Fig. 34.*
(b) *If* $\lambda_1 = 2\lambda_2$, *then the sectorial decomposition of system* (26) *at the origin is shown in Fig. 35.*

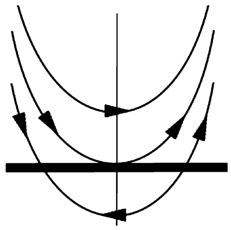

Fig. 35. Phase portrait of system (26) at $(0,0)$, for the case $\lambda_1\lambda_2 \neq 0, b(\lambda_1-\lambda_2) \neq 0, \lambda_1 = 2\lambda_2$ and $b\lambda_2 > 0$. If $\lambda_1\lambda_2 \neq 0, b(\lambda_1-\lambda_2) \neq 0, \lambda_1 = 2\lambda_2$ and $b\lambda_2 < 0$, then the orientation of the orbits is reversed.

Fig. 35. Portrait de phase du système (26) au point singulier $(0,0)$, pour le cas $\lambda_1\lambda_2 \neq 0, b(\lambda_1-\lambda_2) \neq 0, \lambda_1 = 2\lambda_2$ et $b\lambda_2 > 0$. Si $\lambda_1\lambda_2 \neq 0, b(\lambda_1-\lambda_2) \neq 0, \lambda_1 = 2\lambda_2$ et $b\lambda_2 < 0$, alors l'orientation des orbites doit être inversée.

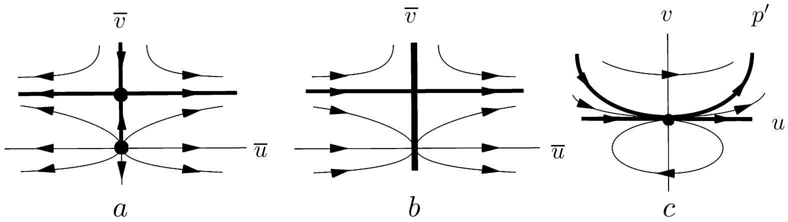

Fig. 36. Phase portrait of system (26) at $(0,0)$ for the case $b > 0$ and $0 < \lambda_2 < \lambda_1 < 2\lambda_2$, or $b < 0$ and $2\lambda_2 < \lambda_1 < \lambda_2 < 0$. If $b > 0$ and $2\lambda_2 < \lambda_1 < \lambda_2 < 0$, or $b < 0$ and $0 < \lambda_2 < \lambda_1 < 2\lambda_2$, then the orientation of the orbits is reversed. (a) System (39). (b) System (38). (c) System (26).

Fig. 36. Portrait de phase du système (26) au point singulier $(0,0)$ pour le cas $b > 0$ et $0 < \lambda_2 < \lambda_1 < 2\lambda_2$, ou $b < 0$ et $2\lambda_2 < \lambda_1 < \lambda_2 < 0$. Si $b > 0$ et $2\lambda_2 < \lambda_1 < \lambda_2 < 0$, ou $b < 0$ et $0 < \lambda_2 < \lambda_1 < 2\lambda_2$, alors l'orientation des orbites doit être inversée. (a) Système (39). (b) Système (38). (c) Système (26).

Proof. We consider the cases $\lambda_1 \neq 2\lambda_2$ and $\lambda_1 = 2\lambda_2$ separately.

Case 1: $\lambda_1 \neq 2\lambda_2$. In order to obtain information on the localization of the separatrices of the sectorial decomposition of system (26) at $(0,0)$, we will use blow-up. Moreover, if we use Theorem 3.5 of [6], then the case $\lambda_1 = 3\lambda_2$ should be treated separately.

We take the quasi-homogeneous blow-up in the u-direction given by $u = \overline{u}$, $v = \overline{u}^2\,\overline{v}$. Then system (26) becomes

$$\begin{aligned}\dot{\overline{u}} = {} & -c\lambda_2\overline{u}^4\overline{v}^2 + (2c\lambda_2\overline{u}^4 + a(\lambda_1-\lambda_2)\overline{u}^3 + b(\lambda_1-\lambda_2)\overline{u}^2)\overline{v} \\ & + a(2\lambda_2-\lambda_1)\overline{u}^3 + b(2\lambda_2-\lambda_1)\overline{u}^2, \\ \dot{\overline{v}} = {} & 2c\lambda_2\overline{u}^3\overline{v}^3 + (-2c\lambda_2\overline{u}^3 + a(2\lambda_2-\lambda_1)\overline{u}^2 + 2b(\lambda_2-\lambda_1)\overline{u})\overline{v}^2 \\ & + (a(\lambda_1-2\lambda_2)\overline{u}^2 + 2b(\lambda_1-\lambda_2)\overline{u})\overline{v}.\end{aligned} \tag{38}$$

After the rescaling given by $ds = \overline{u}\,dt$, system (38) becomes:

$$\begin{aligned}\overline{u}' = {} & -c\lambda_2\overline{u}^3\overline{v}^2 + (2c\lambda_2\overline{u}^3 + a(\lambda_1-\lambda_2)\overline{u}^2 + b(\lambda_1-\lambda_2)\overline{u})\overline{v} \\ & + a(2\lambda_2-\lambda_1)\overline{u}^2 + b(2\lambda_2-\lambda_1)\overline{u}, \\ \overline{v}' = {} & 2c\lambda_2\overline{u}^2\overline{v}^3 + (-2c\lambda_2\overline{u}^2 + a(2\lambda_2-\lambda_1)\overline{u} + 2b(\lambda_2-\lambda_1)\overline{u})\overline{v}^2 \\ & + (a(\lambda_1-2\lambda_2)\overline{u} + 2b(\lambda_1-\lambda_2))\overline{v}.\end{aligned} \tag{39}$$

The singularities $(\overline{u}_0, \overline{v}_0)$ of system (39) with $\overline{u}_0 = 0$ are $(0,0)$ and $(0,1)$. The eigenvalues of the Jacobian matrix of the vector field defined by system (39) evaluated at $(0,0)$ and $(0,1)$ are $b(2\lambda_2-\lambda_1), 2b(\lambda_1-\lambda_2)$ and $b\lambda_2, -2b(\lambda_1-\lambda_2)$, respectively.

Observe that the straight line $\overline{v} = 1$ is an invariant of system (39), then the parabola p' is an invariant of system (26). If $u = 0$ and $|v|$ is sufficiently small, then the sign of $\dot{u}$ in system (26) is the same than $b(\lambda_1-\lambda_2)v$. Then we will present only the figures of the blow-up for the case $b > 0$. We summarize the quasi-homogeneous blow-up in the u-direction in Figs. 36–38.

Case 3.2: $\lambda_1 = 2\lambda_2$. After the rescaling given by $ds = \lambda_2 v\,dt$, system (26) becomes

$$u' = -cv + 2cu^2 + au + b, \quad v' = 2(cu + a)v + 2bu.$$

Then $u' = b$ when $u = v = 0$ and $v' = 2bu$ when $v = 0$. We present the sectorial decomposition of system (26) at the origin in Fig. 35. □

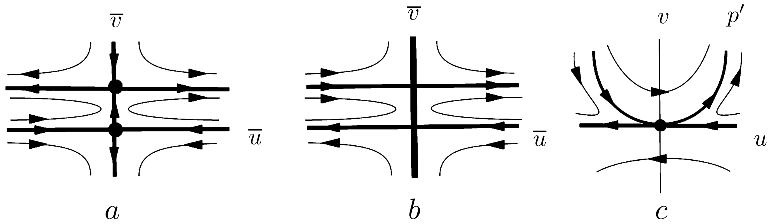

Fig. 37. Phase portrait of system (26) at $(0,0)$ for the case $b>0$ and $0<2\lambda_2<\lambda_1$, or $b<0$ and $\lambda_1<2\lambda_2<0$. If $b>0$ and $\lambda_1<2\lambda_2<0$, or $b<0$ and $0<2\lambda_2<\lambda_1$ then the orientation of the orbits is reversed. (a) System (39). (b) System (38). (c) System (26).

Fig. 37. Portrait de phase du système (26) au point singulier $(0,0)$ pour le cas $b>0$ et $0<2\lambda_2<\lambda_1$, ou $b<0$ et $\lambda_1<2\lambda_2<0$. Si $b>0$ et $\lambda_1<2\lambda_2<0$, ou $b<0$ et $0<2\lambda_2<\lambda_1$ alors l'orientation des orbites doit être inversée. (a) Système (39). (b) Système (38). (c) Système (26).

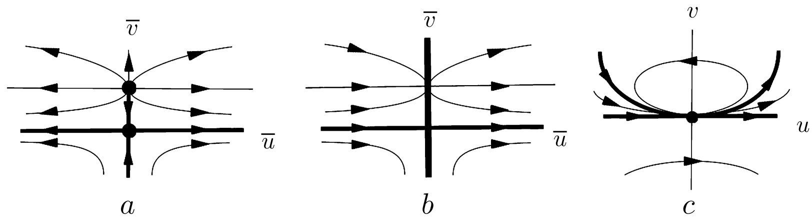

Fig. 38. Phase portrait of system (26) at $(0,0)$ for the case $b>0, \lambda_2>0$ and $\lambda_1<\lambda_2$, or $b<0, \lambda_2<0$ and $\lambda_2<\lambda_1$. If $b>0, \lambda_2<0$ and $\lambda_2<\lambda_1$, or $b<0, \lambda_2>0$ and $\lambda_1<\lambda_2$, then the orientation of the orbits is reversed. (a) System (39). (b) System (38). (c) System (26).

Fig. 38. Portrait de phase du système (26) au point singulier $(0,0)$ pour le cas $b>0, \lambda_2>0$ et $\lambda_1<\lambda_2$, ou $b<0, \lambda_2<0$ et $\lambda_2<\lambda_1$. Si $b>0, \lambda_2<0$ et $\lambda_2<\lambda_1$, ou $b<0, \lambda_2>0$ et $\lambda_1<\lambda_2$, alors l'orientation des orbites doit être inversée. (a) Système (39). (b) Système (38). (c) Système (26).

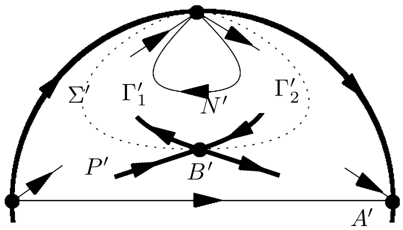

Fig. 39. Lemma 6.1.

6. Proof of Theorem 1.1

Combining the information of Lemmas 4.1–5.8, we obtain the local phase portrait of all the finite and infinite singularities in the Poincaré disc of systems (4) with $\lambda_1^2+\lambda_2^2\neq 0$, as is shown in Fig. 40. The correspondence between system (4) and the local phase portrait in the Poincaré disc shown in Fig. 40 is given in Table 1.

Using the Poincaré–Bendixson Theorem, Remarks 2–4, and the fact that we have the invariant parabola $y=x^2$ and the invariant straight line $ax+by+c=0$, we obtain that each local phase portrait in the Poincaré disc given in Fig. 40a–w determines a unique global phase portrait in the Poincaré disc of Fig. 1a–w respectively, except for the case given by Figs. 40v and 1v. The proof of the phase portrait in the Poincaré disc of Fig. 40v is given by Fig. 1v and follows from the next result.

Lemma 6.1. *Consider the local phase portrait given in Fig. 40v, whose portion corresponding to the region above the line l is shown in Fig. 39. Then $\omega(\Gamma_1')$ and $\alpha(\Gamma_2')$ are the origin of U_2.*

Proof. As before, for system (4), we denote by P, N and Σ the subsets of $\mathbb{R}^2$ such that $\dot{x}>0, \dot{x}<0$, and $\dot{x}=0$, respectively. The corresponding subsets in the Poincaré disc are denoted by P', N' and Σ'. In Fig. 39, we suppose that $a/b\leq 0$. The case $a/b>0$ is treated in a similar way.

Since Σ is defined by the equation $b(\lambda_1-\lambda_2)y=b\lambda_1x^2+a\lambda_2x+c\lambda_2$, the equation of Σ' in the charts U_1 and U_2 is given by $b(\lambda_1-\lambda_2)uv=b\lambda_1+a\lambda_2v+c\lambda_2v^2$ and $b(\lambda_1-\lambda_2)v=b\lambda_1u^2+a\lambda_2uv+c\lambda_2v^2$, respectively. Then $A\notin\Sigma'$ and the origin of U_2 is contained in Σ'.

If $\omega(\Gamma_1)$ is different than the origin of U_2, then $\omega(\Gamma_1)=\{A\}$. It follows that $\Gamma_1\cap N'\neq\emptyset$. This is a contradiction with Lemma 5.3(b). $\square$

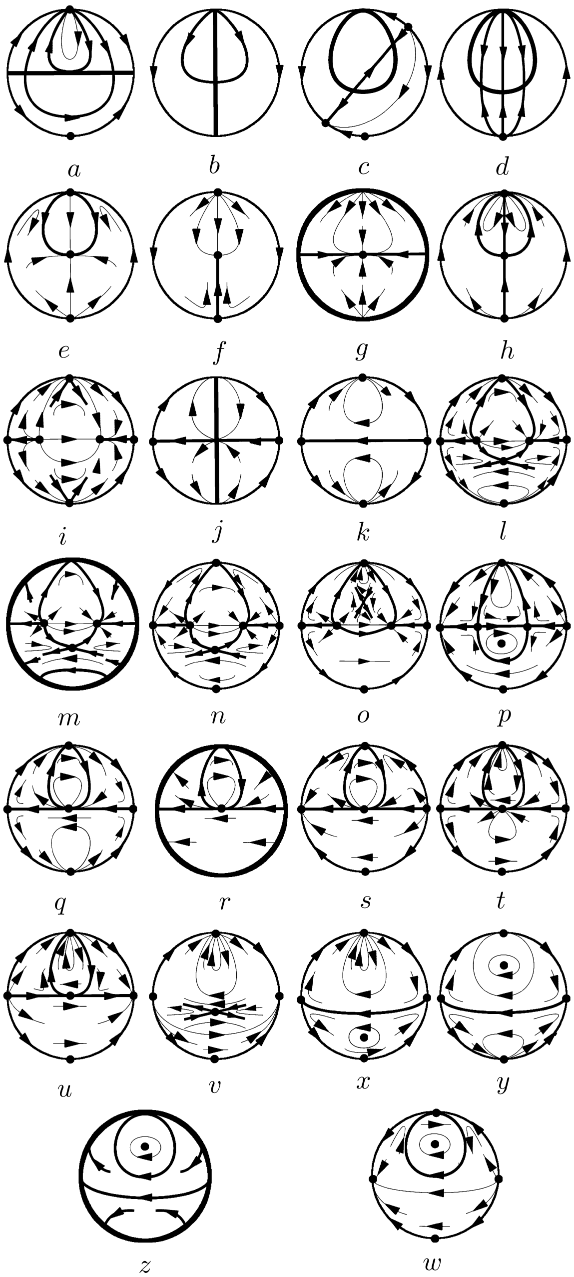

Fig. 40. Local phase portraits in the Poincaré disc of systems (4).

Fig. 40. Portraits de phase aux points singuliers dans le disque de Poincaré dès systemes (4).

References

[1] J.C. Artés, J. Llibre, Quadratic Hamiltonian vector fields, J. Differ. Equ. 107 (1994) 80–95.

[2] J.C. Artés, J. Llibre, N. Vulpe, Complete geometric invariant study of two classes of quadratic systems, Electron. J. Differ. Equ. 2012 (9) (2012) 1–35.

[3] N.N. Bautin, On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type, Mat. Sb. 30 (1952) 181–196, Amer. Math. Soc. Transl. 100 (1954) 1–19.

[4] T. Date, Classification and analysis of two-dimensional homogeneous quadratic differential equations systems, J. Differ. Equ. 32 (1979) 311–334.

[5] H. Dulac, Détermination et integration d'une certaine classe d'équations différentielle ayant par point singulier un centre, Bull. Sci. Math. Sér. (2) 32 (1908) 230–252.

[6] F. Dumortier, J. Llibre, J.C. Artés, Qualitative Theory of Planar Differential Systems, Universitext, Springer-Verlag, 2006.

[7] Y.F. Kalin, N.I. Vulpe, Affine-invariant conditions for the topological discrimination of quadratic Hamiltonian differential systems, Differ. Equ. 34 (3) (1998) 297–301.

[8] W. Kapteyn, On the midpoints of integral curves of differential equations of the first degree, Nederl. Akad. Wetensch. Verslag. Afd. Natuurk. Konikl. Nederland (1911) 1446–1457 (in Dutch).

[9] W. Kapteyn, New investigations on the midpoints of integrals of differential equations of the first degree, Nederl. Akad. Wetensch. Verslag Afd. Natuurk. 20 (1912) 1354–1365; 21 (1912) 27–33 (in Dutch).
[10] N.A. Korol, The integral curves of a certain differential equation, Minsk. Gos. Ped. Inst. Minsk (1973) 47–51 (in Russian).
[11] M.A. Liapunov, Problème général de la stabilité du mouvement, Ann. Math. Stud., vol. 17, Princeton University Press, Princeton, NJ, USA, 1947.
[12] J. Llibre, M.F. da Silva, Global phase portraits of Kukles differential systems with homogeneous polynomial nonlinearities of degree 6 having a center and their small limit cycles, Int. J. Bifurc. Chaos 26 (2016) 1650044, 25 p.
[13] J. Llibre, J. Yu, Phase portraits of quadratic systems with an ellipse and a straight line as invariant algebraic curves, Electron. J. Differ. Equ. 314 (2015), 14 p.
[14] V.A. Lunkevich, K.S. Sibirskii, Integrals of a general quadratic differential system in cases of a center, Differ. Equ. 18 (1982) 563–568.
[15] L.S. Lyagina, The integral curves of the equation $y' = (ax^2 + bxy + cy^2)/(dx^2 + exy + fy^2)$, Usp. Mat. Nauk 6-2 (42) (1951) 171–183 (in Russian).
[16] L. Markus, Quadratic Differential Equations and Non-Associative Algebras, Ann. Math. Stud., vol. 45, Princeton University Press, Princeton, NJ, USA, 1960, pp. 185–213.
[17] T.A. Newton, Two dimensional homogeneous quadratic differential systems, SIAM Rev. 20 (1978) 120–138.
[18] H. Poincaré, Mémoire sur les courbes définies par les équations différentielles, J. Math. 37 (1881) 375–422; Œuvres d'Henri Poincaré, vol. I, Gauthier-Villars, Paris, 1951, pp. 3–84.
[19] J.W. Reyn, Phase Portraits of Planar Quadratic Systems, Mathematics and Its Applications, vol. 583, Springer, New York, 2007.
[20] D. Schlomiuk, Algebraic particular integrals, integrability and the problem of the center, Trans. Amer. Math. Soc. 338 (1993) 799–841.
[21] K.S. Sibirskii, N.I. Vulpe, Geometric classification of quadratic differential systems, Differ. Equ. 13 (1977) 548–556.
[22] E.V. Vdovina, Classification of singular points of the equation $y' = (a_0x^2 + a_1xy + a_2y^2)/(b_0x^2 + b_1xy + b_2y^2)$ by Forster's method, Differ. Equ. 20 (1984) 1809–1813 (in Russian).
[23] N.I. Vulpe, Affine-invariant conditions for the topological discrimination of quadratic systems with a center, Differ. Equ. 1 (1983) 273–280.
[24] Y. Ye, Qualitative Theory of Polynomial Differential Systems, Shanghai Scientific & Technical Publishers, Shanghai, 1995 (in Chinese).
[25] Y. Ye, et al., Theory of Limit Cycles, Transl. Math. Monogr., vol. 66, Amer. Math. Soc., Providence, RI, USA, 1984.
[26] W.Y. Ye, Y. Ye, On the conditions of a center and general integrals of quadratic differential systems, Acta Math. Sin. (Engl. Ser.) 17 (2001) 229–236.
[27] H. Żołądek, Quadratic systems with center and their perturbations, J. Differ. Equ. 109 (1994) 223–273.