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Algebra/Homological algebra

Homotopy G-algebra structure on the cochain complex of hom-type algebras

Structure de G-algèbre à homotopie près sur le complexe des co-chaînes des algèbres de type hom

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A R T I C L E I N F O

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A B S T R A C T

A hom-associative algebra is an algebra whose associativity is twisted by an algebra homomorphism. We show that the Hochschild type cochain complex of a hom-associative algebra carries a homotopy G-algebra structure. As a consequence, we get a Gerstenhaber algebra structure on the cohomology of a hom-associative algebra. We also find similar results for hom-dialgebras.

R É S U M É

Une algèbre hom-associative est une algèbre dont l'associativité est tordue par un homomorphisme d'algèbre. Nous montrons que le complexe des co-chaînes de type Hochschild d'une algèbre hom-associative porte une structure de G-algèbre à homotopie près. Comme conséquence, nous obtenons une structure d'algèbre de Gerstenhaber sur la cohomologie des algèbres hom-associatives. Nous arrivons également à des résultats similaires pour les hom-dialgèbres.

1. Introduction

In [4], Gerstenhaber showed that the Hochschild cohomology $H^\bullet(A, A)$ of an associative algebra A carries a certain algebraic structure. This algebraic structure is now known as Gerstenhaber algebra. A Gerstenhaber algebra is a graded commutative associative algebra together with a degree -1 graded Lie bracket that are compatible in the sense of a suitable Leibniz rule. An alternative proof of the same fact has been carried out by Gerstenhaber and Voronov [5]. More precisely, they prove a more general statement, that is, the Hochschild complex $C^\bullet(A, A)$ carries a homotopy G-algebra structure.

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A homotopy G-algebra is a brace algebra $(\mathcal{O} = \oplus\mathcal{O}(n), \{-\}\{-,\dots,-\})$ together with a differential graded associative algebra structure on $\mathcal{O}$ satisfying some compatibility conditions [5]. The brace algebra structure on $C^\bullet(A, A)$ is given by the classical braces introduced by Getzler–Jones [6], the differential graded associative algebra structure on $C^\bullet(A, A)$ is given by the usual cup product and the Hochschild coboundary (up to some signs). In [5], the authors showed that the existence of the homotopy G-algebra structure on $C^\bullet(A, A)$ is based on the non-symmetric endomorphism operad structure on $C^\bullet(A, A)$ together with a multiplication on that operad. The same idea has been used to define a homotopy G-algebra structure on the dialgebra complex $CY^\bullet(D, D)$ of a dialgebra D [9].

In this paper, we deal with certain types of algebras, called hom-type algebras. In these algebras, the identities defining the structures are twisted by homomorphisms. Recently, hom-type algebras have been studied by many authors. The notion of hom-Lie algebras was first introduced by Hartwig, Larsson, and Silvestrov [7]. Hom-Lie algebras appeared in examples of q-deformations of the Witt and Virasoro algebras. Another type of algebras (e.g., associative, Leibniz, Poisson, Hopf...) twisted by homomorphisms have also been studied. See [10,11] (and references there in) for more details. Our main objective in this paper is the notion of hom-associative algebra introduced by Makhlouf and Silvestrov [10]. A hom-associative algebra is an algebra (A, μ) whose associativity is twisted by an algebra homomorphism $\alpha : A \to A$ (cf. Definition 3.1). When α is the identity map, we recover the classical notion of associative algebras as a subclass.

In [1,11], the authors studied the formal one-parameter deformation of hom-associative algebras and introduced a Hochschild-type cohomology theory for hom-associative algebras. Given a hom-associative algebra (A, μ, α), its n-th cochain group $C^n_\alpha(A, A)$ consists of multilinear maps $f : A^{\otimes n} \to A$ that satisfy $\alpha \circ f = f \circ \alpha^{\otimes n}$, and the coboundary operator δ_α is similar to the Hochschild coboundary, but suitably twisted by α. In [1], the authors also introduce a degree -1 graded Lie bracket $[-,-]_\alpha$ on the cochain groups $C^\bullet_\alpha(A, A)$, which passes on to cohomology. In [2], the present author defines a cup product $\cup_\alpha$ on the cochain groups $C^\bullet_\alpha(A, A)$ and shows that it induces a graded commutative, associative product on the cohomology $H^\bullet_\alpha(A, A)$. Moreover, it was shown that the induced structures on the cohomology $H^\bullet_\alpha(A, A)$ makes it a Gerstenhaber algebra.

In this paper, we follow the method of Gerstenhaber and Voronov [5]. We show that the cochain complex $C^\bullet_\alpha(A, A)$ carries a non-symmetric operad structure. This operad structure is similar to the endomorphism operad on A, however, twisted by α. Moreover, the multiplication defining the hom-associative structure gives a multiplication in the above operad. Hence, by a result of [5], it follows that the cochain complex $C^\bullet_\alpha(A, A)$ carries a homotopy G-algebra structure. As a consequence, we get a Gerstenhaber algebra structure on cohomology. This gives an alternative approach to the same result proved by the author [2].

The notion of (diassociative) dialgebras was introduced by Loday as a generalization of associative algebras [8]. The hom-analogue of a dialgebra is known as a hom-dialgebra [13]. We discuss the above results for hom-dialgebras. Given a hom-dialgebra D, we show that the cochain complex $CY^\bullet_\alpha(D, D)$ defining the cohomology of a hom-dialgebra carries a non-symmetric operad structure. Moreover, the operations defining the hom-dialgebra structure induces a multiplication on the operad. Hence, we conclude that the cochain complex $CY^\bullet_\alpha(D, D)$ inherits a homotopy G-algebra structure and the corresponding cohomology $HY^\bullet_\alpha(D, D)$ carries a Gerstenhaber algebra structure.

In Section 2, we recall some basic preliminaries on operads, braces, and homotopy G-algebras. In Section 3, we first revise hom-associative algebras and prove our results for hom-associative algebras. Finally, in Section 4, we deal with hom-dialgebras.

2. Preliminaries

In this section, we recall some basic definitions. See [5,6] for more details.

2.1. Definition. A non-symmetric operad (non-$\sum$ operad in short) in the category of vector spaces is a collection of vector spaces $\{\mathcal{O}(k) | \; k \geq 1\}$ together with compositions

$$\gamma : \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \to \mathcal{O}(n_1 + \cdots + n_k)$$
$$f \otimes g_1 \otimes \cdots \otimes g_k \mapsto \gamma(f; g_1, \dots, g_k)$$

which is associative in the sense that

$$\gamma\big(\gamma(f; g_1, \dots, g_k); h_1, \dots, h_{n_1+\cdots+n_k}\big)$$
$$= \gamma\big(f;\ \gamma(g_1; h_1, \dots, h_{n_1}),\ \gamma(g_2; h_{n_1+1}, \dots, h_{n_1+n_2}), \dots,\ \gamma(g_k; h_{n_1+\cdots+n_{k-1}+1}, \dots, h_{n_1+\cdots+n_k})\big)$$

and there is an identity element $\mathrm{id} \in \mathcal{O}(1)$ such that

$$\gamma(f; \underbrace{\mathrm{id}, \dots, \mathrm{id}}_{k \text{ times}}) = f = \gamma(\mathrm{id}; f), \quad \text{for } f \in \mathcal{O}(k).$$

A non-$\sum$ operad can also be described by compositions (called partial compositions)

$$\circ_i : \mathcal{O}(m) \otimes \mathcal{O}(n) \to \mathcal{O}(m+n-1), \quad 1 \leq i \leq m$$

satisfying

$$\begin{cases} (f \circ_i g) \circ_{i+j-1} h = f \circ_i (g \circ_j h), & \text{for } 1 \leq i \leq m,\ 1 \leq j \leq n, \\ (f \circ_i g) \circ_{j+n-1} h = (f \circ_j h) \circ_i g, & \text{for } 1 \leq i < j \leq m, \end{cases}$$

for $f \in \mathcal{O}(m)$, $g \in \mathcal{O}(n)$, $h \in \mathcal{O}(p)$, and an identity element satisfying $f \circ_i \mathrm{id} = f = \mathrm{id} \circ_1 f$, for all $f \in \mathcal{O}(k)$ and $1 \leq i \leq m$. The two definitions of non-$\sum$ operad are related by

$$f \circ_i g = \gamma(f;\ \overbrace{\mathrm{id}, \ldots, \mathrm{id}, \underbrace{g}_{i\text{-th place}}, \mathrm{id}, \ldots, \mathrm{id}}^{m\text{-tuple}}), \qquad \text{for } f \in \mathcal{O}(m), \tag{1}$$

$$\gamma(f; g_1, \ldots, g_k) = (\cdots((f \circ_k g_k) \circ_{k-1} g_{k-1}) \cdots) \circ_1 g_1, \qquad \text{for } f \in \mathcal{O}(k). \tag{2}$$

A toy example of an operad is given by the endomorphisms of a vector space. Let A be a vector space and define $\mathcal{O}(k) = Hom(A^{\otimes k}, A)$, for $k \geq 1$. The compositions γ are substitutions of the values of k operations in a k-ary operation as inputs.

Next, consider the graded vector space $\mathcal{O} = \oplus_{k \geq 1} \mathcal{O}(k)$ of an operad. If $f \in \mathcal{O}(n)$, we define $\deg f = n$ and $|f| = n-1$. We use the same notation for any graded vector space as well. Consider the braces

$$\{f\}\{g_1, \ldots, g_n\} := \sum (-1)^{\epsilon}\, \gamma(f; \mathrm{id}, \ldots, \mathrm{id}, g_1, \mathrm{id}, \ldots, \mathrm{id}, g_n, \mathrm{id}, \ldots, \mathrm{id})$$

where the summation runs over all possible substitutions of $g_1, \ldots, g_n$ into f in the prescribed order and $\epsilon := \sum_{p=1}^{n} |g_p| i_p$, i_p is the total number of inputs in front of g_p. The multilinear braces $\{f\}\{g_1, \ldots, g_n\}$ are homogeneous of degree $-n$. Moreover, they satisfy the following identities.

• Higher pre-Jacobi identities:

$$\begin{aligned} &\{f\}\{g_1, \ldots, g_m\}\{h_1, \ldots, h_n\} \\ &= \sum_{0 \leq i_1 \leq \cdots \leq i_m \leq n} (-1)^{\epsilon}\, \{f\}\{h_1, \ldots, h_{i_1}, \{g_1\}\{h_{i_1+1}, \ldots, h_{j_1}\}, h_{j_1+1}, \ldots, h_{i_m}, \\ &\qquad\qquad \{g_m\}\{h_{i_m+1}, \ldots, h_{j_m}\}, h_{j_m+1}, \ldots, h_n\}, \end{aligned}$$

where $\epsilon := \sum_{p=1}^{m} \left(|g_p| \sum_{q=1}^{i_p} |h_q|\right)$.

One also assumes the following conventions in an operad:

$$\{f\}\{\ \} := f \quad \text{and} \quad f \circ g := \{f\}\{g\}.$$

2.2. Remark. The higher pre-Jacobi identities imply that

$$[f, g] = f \circ g - (-1)^{|f||g|} g \circ f, \quad \text{for } f, g \in \mathcal{O}, \tag{3}$$

defines a degree -1 graded Lie bracket on $\mathcal{O}$.

2.3. Definition. A multiplication on an operad $\mathcal{O}$ is an element $m \in \mathcal{O}(2)$ such that $m \circ m = 0$.

If m is a multiplication on an operad $\mathcal{O}$, then the dot product

$$f \cdot g = (-1)^{|f|+1}\{m\}\{f, g\}, \quad f, g \in \mathcal{O},$$

defines a graded associative algebra structure on $\mathcal{O}$. Moreover, the degree one map $d : \mathcal{O} \to \mathcal{O}$, $f \mapsto m \circ f - (-1)^{|f|} f \circ m$ is a differential on $\mathcal{O}$ and the triple $(\mathcal{O}, \cdot, d)$ is a differential graded associative algebra [5]. Moreover, the following identities hold.

• Distributivity:

$$\{f \cdot g\}\{h_1, \ldots, h_n\} = \sum_{k=0}^{n} (-1)^{\epsilon}\, (\{f\}\{h_1, \ldots, h_k\}) \cdot (\{g\}\{h_{k+1}, \ldots, h_n\}), \quad \text{where } \epsilon = |g| \sum_{p=1}^{k} |h_p|.$$

• Higher homotopies:

$$d(\{f\}\{g_1,\ldots,g_{n+1}\}) - \{df\}\{g_1,\ldots,g_{n+1}\} - (-1)^{|f|}\sum_{i=1}^{n+1}(-1)^{|g_1|+\cdots+|g_{i-1}|}\{f\}\{g_1,\ldots,dg_i,\ldots,g_{n+1}\}$$

$$=(-1)^{|f||g_1|+1}g_1\cdot(\{f\}\{g_2,\ldots,g_{n+1}\}) + (-1)^{|f|}\sum_{i=1}^{n}(-1)^{|g_1|+\cdots+|g_{i-1}|}\{f\}\{g_1,\ldots,g_i\cdot g_{i+1},\ldots,g_{n+1}\}$$

$$-\{f\}\{g_1,\ldots,g_n\}\cdot g_{n+1}.$$

Summarizing the properties of braces and multiplications on an operad, one gets the following algebraic structures [5,6].

2.4. Definition. A brace algebra is a graded vector space $\mathcal{O} = \mathcal{O}(n)$ together with a collection of braces $\{f\}\{g_1,\ldots,g_n\}$ of degree $-n$ satisfying the higher pre-Jacobi identities.

A brace algebra as above may be denoted by $(\mathcal{O} = \mathcal{O}(n),\ \{-\}\{-,\ldots,-\})$.

2.5. Definition. A homotopy G-algebra is a brace algebra $(\mathcal{O} = \mathcal{O}(n),\ \{-\}\{-,\ldots,-\})$ endowed with a differential graded associative algebra structure $(\mathcal{O} = \mathcal{O}(n), \cdot, d)$ satisfying the distributivity and higher homotopies. A homotopy G-algebra is denoted by $(\mathcal{O} = \mathcal{O}(n),\ \{-\}\{-,\ldots,-\}), \cdot, d)$.

As a summary, we get the following [5].

2.6. Theorem. *A multiplication on an operad $\mathcal{O}$ defines the structure of a homotopy G-algebra on $\mathcal{O} = \oplus\mathcal{O}(n)$.*

Next, we recall Gerstenhaber algebras (G-algebras in short).

2.7. Definition. A (left) Gerstenhaber algebra is a graded commutative associative algebra $(\mathcal{A} = \oplus\mathcal{A}^i, \cdot)$ together with a degree -1 graded Lie bracket $[-,-]$ on $\mathcal{A}$ satisfying the following Leibniz rule

$$[a, b\cdot c] = [a,b]\cdot c + (-1)^{|a|(|b|+1)}b\cdot[a,c],$$

for all homogeneous elements $a, b, c \in \mathcal{A}$.

2.8. Remark. Given a homotopy G-algebra $(\mathcal{O} = \mathcal{O}(n),\ \{-\}\{-,\ldots,-\}), \cdot, d)$, the product $\cdot$ induces a graded commutative associative product $\cdot$ on the cohomology $H^\bullet(\mathcal{O}, d)$. The degree -1 graded Lie bracket as defined in (3) also passes on to the cohomology $H^\bullet(\mathcal{O}, d)$. Moreover, the induced product and the bracket on the cohomology satisfy the graded Leibniz rule to becomes a Gerstenhaber algebra [5].

3. Hom-associative algebras

In this section, we first recall hom-associative algebras and their Hochschild cohomology. Then we show that the Hochschild complex of hom-associative algebras carries a natural operad structure together with a multiplication. Finally, we deduce a Gerstenhaber algebra structure on the cohomology.

3.1. Definition. A hom-associative algebra over $\mathbb{K}$ is a triple (A, μ, α) consists of a $\mathbb{K}$-vector space A together with a $\mathbb{K}$-bilinear map $\mu : A \times A \to A$ and a $\mathbb{K}$-linear map $\alpha : A \to A$ satisfying $\alpha(\mu(a,b)) = \mu(\alpha(a), \alpha(b))$ and

$$\mu(\alpha(a), \mu(b,c)) = \mu(\mu(a,b), \alpha(c)), \quad \text{for all } a, b, c \in A. \tag{4}$$

In [1] the authors called such a hom-associative algebra 'multiplicative'. By a hom-associative algebra, they mean a triple (A, μ, α) of a vector space A, a bilinear map $\mu : A \times A \to A$ and a linear map $\alpha : A \to A$ satisfying condition (4). See [1,10] for examples of hom-associative algebras.

When $\alpha =$ identity, in any case, one gets the definition of a classical associative algebra. Next, we recall the definition of Hochschild-type cohomology for hom-associative algebras. Like in the classical case, this cohomology theory controls the deformation of hom-associative algebras [1].

Let (A, μ, α) be a hom-associative algebra. For each $n \geq 1$, we define a $\mathbb{K}$-vector space $C^n_\alpha(A, A)$ consisting of all multilinear maps $f : A^{\otimes n} \to A$ satisfying $\alpha \circ f = f \circ \alpha^{\otimes n}$, that is,

$$(\alpha \circ f)(a_1,\ldots,a_n) = f(\alpha(a_1),\ldots,\alpha(a_n)), \quad \text{for all } a_i \in A.$$

Define $\delta_\alpha : C^n_\alpha(A, A) \to C^{n+1}_\alpha(A, A)$ by the following

$$\begin{aligned}(\delta_\alpha f)(a_1, a_2, \ldots, a_{n+1}) = {} & \mu\big(\alpha^{n-1}(a_1), f(a_2, \ldots, a_{n+1})\big) \\ & + \sum_{i=1}^{n} (-1)^i f\big(\alpha(a_1), \ldots, \alpha(a_{i-1}), \mu(a_i, a_{i+1}), \alpha(a_{i+2}), \ldots, \alpha(a_{n+1})\big) \\ & + (-1)^{n+1} \mu(f(a_1, \ldots, a_n), \alpha^{n-1}(a_{n+1})).\end{aligned}$$

Then we have $\delta^2_\alpha = 0$. The cohomology of this complex is called the Hochschild cohomology of the hom-associative algebra (A, μ, α). The cohomology groups are denoted by $H^n_\alpha(A, A)$, $n \geq 2$. When $\alpha =$ identity, one recovers the classical Hochschild cohomology of associative algebras.

Operad structure: Let A be a vector space and $\alpha : A \to A$ be a linear map. For each $k \geq 1$ define $C^k_\alpha(A, A)$ to be the space of all multilinear maps $f : A^{\otimes k} \to A$ satisfying

$$(\alpha \circ f)(a_1, \ldots, a_k) = f(\alpha(a_1), \ldots, \alpha(a_k)), \quad \text{for all } a_i \in A.$$

We define an operad structure on $\mathcal{O} = \{\mathcal{O}(k) | \ k \geq 1\}$ where $\mathcal{O}(k) = C^k_\alpha(A, A)$, for $k \geq 1$. Define partial compositions $\circ_i : \mathcal{O}(m) \otimes \mathcal{O}(n) \to \mathcal{O}(m + n - 1)$ by

$$(f \circ_i g)(a_1, \ldots, a_{m+n-1}) = f(\alpha^{n-1} a_1, \ldots, \alpha^{n-1} a_{i-1}, g(a_i, \ldots, a_{i+n-1}), \alpha^{n-1} a_{i+n}, \ldots, \alpha^{n-1} a_{m+n-1}),$$

for $f \in \mathcal{O}(m)$, $g \in \mathcal{O}(n)$ and $a_1, \ldots, a_{m+n-1} \in A$. In view of (2), the compositions

$$\gamma_\alpha : \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \to \mathcal{O}(n_1 + \cdots + n_k)$$

are given by

$$\begin{aligned}& \gamma_\alpha(f; g_1, \ldots, g_k)(a_1, \ldots, a_{n_1+\cdots+n_k}) \\ & = f\big(\alpha^{\sum_{l=2}^k |g_l|} g_1(a_1, \ldots, a_{n_1}), \ldots, \ \alpha^{\sum_{l=1, l\neq i}^k |g_l|} g_i(a_{n_1+\cdots+n_{i-1}+1}, \ldots, a_{n_1+\cdots+n_i}), \\ & \qquad \ldots, \ \alpha^{\sum_{l=1}^{k-1} |g_l|} g_k(a_{n_1+\cdots+n_{k-1}+1}, \ldots, a_{n_1+\cdots+n_k})\big),\end{aligned}$$

for $f \in \mathcal{O}(k)$, $g_i \in \mathcal{O}(n_i)$ and $a_1, \ldots, a_{n_1+\cdots+n_k} \in A$.

3.2. Proposition. *The partial compositions $\circ_i$ (or compositions γ_α) defines a non-$\sum$ operad structure on $C^\bullet_\alpha(A, A)$ with the identity element given by the identity map* id $\in C^1_\alpha(A, A)$.

Proof. For $f \in C^m_\alpha(A, A)$, $g \in C^n_\alpha(A, A)$, $h \in C^p_\alpha(A, A)$ and $1 \leq i \leq m$, $1 \leq j \leq n$, we have

$$\begin{aligned}& ((f \circ_i g) \circ_{i+j-1} h)(a_1, \ldots, a_{m+n+p-2}) \\ & = (f \circ_i g)\big(\alpha^{p-1} a_1, \ldots, \alpha^{p-1} a_{i+j-2}, \ h(a_{i+j-1}, \ldots, a_{i+j+p-2}), \ldots, \alpha^{p-1} a_{m+n+p-2}\big) \\ & = f\big(\alpha^{n+p-2} a_1, \ldots, \alpha^{n+p-2} a_{i-1}, g\big(\alpha^{p-1} a_i, \ldots, h(a_{i+j-1}, \ldots, a_{i+j+p-2}), \ldots, \alpha^{p-1} a_{i+n+p-2}\big), \\ & \qquad \ldots, \alpha^{n+p-2} a_{m+n+p-2}\big) \\ & = f\big(\alpha^{n+p-2} a_1, \ldots, \alpha^{n+p-2} a_{i-1}, (g \circ_j h)(a_i, \ldots, a_{i+n+p-2}), \ldots, \alpha^{n+p-2} a_{m+n+p-2}\big) \\ & = (f \circ_i (g \circ_j h))(a_1, \ldots, a_{m+n+p-2}).\end{aligned}$$

Similarly, for $1 \leq i < j \leq m$, we have $((f \circ_i g) \circ_{j+n-1} h) = ((f \circ_j h) \circ_i g)$. It is also easy to see that the identity map id is the identity element of the operad. Hence, the proof. □

3.3. Remark. When $\alpha : A \to A$ is the identity map, one recovers the endomorphism operad on the vector space A.

Note that the corresponding braces on $C^\bullet_\alpha(A, A)$ are given by

$$\{f\}\{g_1, \ldots, g_n\} := \sum (-1)^\epsilon \ \gamma_\alpha(f; \mathrm{id}, \ldots, \mathrm{id}, g_1, \mathrm{id}, \ldots, \mathrm{id}, g_n, \mathrm{id}, \ldots, \mathrm{id}).$$

Therefore, the degree -1 graded Lie bracket on $C^\bullet_\alpha(A, A)$ is given by

$$[f, g] = f \circ g - (-1)^{(m-1)(n-1)} g \circ f, \tag{5}$$

where

$$(f \circ g)(a_1, \ldots, a_{m+n-1}) = \sum_{i=1}^{m} (-1)^{(n-1)(i-1)} f(\alpha^{n-1}a_1, \ldots, g(a_i, \ldots, a_{i+n-1}), \ldots, \alpha^{n-1}a_{m+n-1}),$$

for $f \in C^m_\alpha(A, A)$, $g \in C^n_\alpha(A, A)$ and $a_1, \ldots, a_{m+n-1} \in A$. See also [1,2].

Next, let (A, μ, α) be a hom-associative algebra. Then $\mu \in C^2_\alpha(A, A)$. Moreover, we have

$$\begin{aligned}\{\mu\}\{\mu\}(a, b, c) &= \gamma_\alpha(\mu; \mu, \mathrm{id})(a, b, c) - \gamma_\alpha(\mu; \mathrm{id}, \mu)(a, b, c) \\ &= \mu(\mu(a, b), \alpha(c)) - \mu(\alpha(a), \mu(b, c)) = 0, \quad \text{for all } a, b, c \in A.\end{aligned}$$

Therefore, μ defines a multiplication on the operad structure on $C^\bullet_\alpha(A, A)$. The corresponding dot product on $C^\bullet_\alpha(A, A)$ is given by

$$(f \cdot g)(a_1, \ldots, a_{m+n}) = (-1)^{mn}\, \mu(f(\alpha^{n-1}a_1, \ldots, \alpha^{n-1}a_m), g(\alpha^{m-1}a_{m+1}, \ldots, \alpha^{m-1}a_{m+n})),$$

for $f \in C^m_\alpha(A, A)$, $g \in C^n_\alpha(A, A)$ and $a_1, \ldots, a_{m+n} \in A$. We remark that this dot product on $C^\bullet_\alpha(A, A)$ is same as (up to sign) the cup-product on $C^\bullet_\alpha(A, A)$ defined in [2]. Moreover, the differential d is given by

$$df = \mu \circ f - (-1)^{|f|} f \circ \mu = (-1)^{|f|+1} \delta_\alpha(f).$$

The last equality follows from a straightforward calculation [2].

Thus, in view of Theorem 2.6 and Remark 2.8, we get the following.

3.4. Theorem. *Let (A, μ, α) be a hom-associative algebra. Then its Hochschild cochain complex $C^\bullet_\alpha(A, A)$ inherits a homotopy G-algebra structure. Hence, its Hochschild cohomology $H^\bullet_\alpha(A, A)$ carries a Gerstenhaber algebra structure.*

3.5. Remark. A direct proof of the existence of a Gerstenhaber algebra structure on the cohomology $H^\bullet_\alpha(A, A)$ has been carried out by the author in [2]. More precisely, the author defined a cup-product $\cup_\alpha$ on $C^\bullet_\alpha(A, A)$ by

$$(f \cup_\alpha g)(a_1, \ldots, a_{m+n}) = \mu(f(\alpha^{n-1}a_1, \ldots, \alpha^{n-1}a_m), g(\alpha^{m-1}a_{m+1}, \ldots, \alpha^{m-1}a_{m+n})),$$

which is compatible with the Hochschild differential δ_α. Therefore, it induces a cup-product on the cohomology $H^\bullet_\alpha(A, A)$, which turns out to be graded commutative associative. Moreover, the degree -1 graded Lie bracket on $C^\bullet_\alpha(A, A)$ as defined in (5) induces a degree -1 graded Lie bracket on the cohomology. The induced cup-product and degree -1 graded Lie bracket give rise to a (right) Gerstenhaber algebra structure on the cohomology $H^\bullet_\alpha(A, A)$.

The dot product $\cdot$ and the differential d on $C^\bullet_\alpha(A, A)$ induced from the operad structure on $C^\bullet_\alpha(A, A)$ are the same as (up to some signs) the cup-product and Hochschild differential on $C^\bullet_\alpha(A, A)$. Due to the presence of signs, we get here a (left) Gerstenhaber algebra structure on the cohomology $H^\bullet_\alpha(A, A)$.

4. Hom-dialgebras

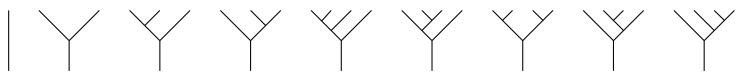
The notion of a (diassociative) dialgebra was introduced by Loday as a generalization of associative algebra and Leibniz algebra [8]. The hom-analogue of dialgebra is given by the following [13].

4.1. Definition. A hom-dialgebra is a vector space D together with two bilinear maps $\dashv, \vdash : D \otimes D \to D$ and a linear map $\alpha : D \to D$ satisfying $\alpha(a \dashv b) = \alpha(a) \dashv \alpha(b)$ and $\alpha(a \vdash b) = \alpha(a) \vdash \alpha(b)$ and such that the following axioms hold

$$\begin{aligned}&\alpha(a) \dashv (b \dashv c) = (a \dashv b) \dashv \alpha(c) = \alpha(a) \dashv (b \vdash c), \\ &(a \vdash b) \dashv \alpha(c) = \alpha(a) \vdash (b \dashv c), \\ &(a \dashv b) \vdash \alpha(c) = \alpha(a) \vdash (b \vdash c) = (a \vdash b) \vdash \alpha(c), \quad \text{for all } a, b, c \in D.\end{aligned}$$

A hom-dialgebra as above is denoted by $(D, \dashv, \vdash, \alpha)$. When $\alpha =$ identity, one gets the notion of a dialgebra. A hom-associative algebra (A, μ, α) is a hom-dialgebra where $\dashv\ = \mu =\ \vdash$.

Dialgebra cohomology with coefficients was introduced by Frabetti using planar binary trees [3]. Next, we introduce the cohomology of a hom-dialgebra with coefficients in itself. A planar binary tree with n-vertices (in short, an n-tree) is a planar tree with $(n+1)$ leaves, one root and each vertex trivalent. Let Y_n denote the set of all n-trees (see the figure below) and let Y_0 be the singleton set consisting



of a root only. Therefore, in the above trees, Y_0 consists of the first tree, Y_1 consists of the second tree, Y_2 consists of the third and the fourth tree, Y_3 consists of the rest of the trees.

For each $y \in Y_n$, the $(n+1)$ leaves are labelled by $\{0, 1, \ldots, n\}$ from left to right, the vertices are labelled by $\{1, \ldots, n\}$ so that the i-th vertex is between the leaves $(i-1)$ and i. The only element in Y_0 is denoted by [0] and the only element in Y_1 is denoted by [1]. The grafting of a p-tree y_1 and q-tree y_2 is a $(p+q+1)$-tree denoted by $y_1 \vee y_2$, and is obtained by joining the roots of y_1 and y_2 and creating a new root from that vertex. This is denoted by $[y_1 \quad p+q+1 \quad y_2]$ with the convention that all zeros are deleted except for the element in Y_0. With this notation, the trees in the above figure (from left to right) are [0], [1], [12], [21], [123], [213], [131], [312], [321].

For any fixed $n \geq 1$, there are maps $d_i : Y_n \to Y_{n-1}$ $(0 \leq i \leq n)$, $y \mapsto d_i y$, where $d_i y$ is obtained from y by deleting the i-th leaf. These maps are called face maps and satisfy the relations $d_i d_j = d_{j-1} d_i$, for all $i < j$.

Before we introduce the cohomology of a hom-dialgebra, we need the following notations. For any $0 \leq i \leq n+1$, the maps $\bullet_i : Y_{n+1} \to \{\dashv, \vdash\}$ are defined by

$$\bullet_0(y) = \bullet_0^y := \begin{cases} \dashv & \text{if } y \text{ is of the form } | \vee y_1 \text{for some } n\text{-tree } y_1, \\ \vdash & \text{otherwise,} \end{cases}$$

$$\bullet_i(y) = \bullet_i^y := \begin{cases} \dashv & \text{if the } i\text{th leaf of } y \text{ is oriented like } '\backslash', \\ \vdash & \text{if the } i\text{th leaf of } y \text{ is oriented like } '/', \end{cases}$$

for $1 \leq i \leq n$, and

$$\bullet_{n+1}(y) = \bullet_{n+1}^y := \begin{cases} \vdash & \text{if } y \text{ is of the form } y_1 \vee |, \text{for some } n\text{-tree } y_1, \\ \dashv & \text{otherwise.} \end{cases}$$

Let $(D, \dashv, \vdash, \alpha)$ be a hom-dialgebra. For any $n \geq 1$, the cochain group $CY_\alpha^n(D, D)$ consists of all linear maps

$$f : K[Y_n] \otimes D^{\otimes n} \to D, \quad y \otimes a_1 \otimes \cdots \otimes a_n \mapsto f(y; a_1, \ldots, a_n)$$

satisfying

$$(\alpha \circ f)(y; a_1, \ldots, a_n) = f(y; \alpha(a_1), \ldots, \alpha(a_n)), \quad \text{for all } y \in Y_n, a_i \in D.$$

The coboundary map $\delta_\alpha : CY_\alpha^n(D, D) \to CY_\alpha^{n+1}(D, D)$ defined by

$$\begin{aligned}(\delta_\alpha f)(y; a_1, \ldots, a_{n+1}) = \; & \alpha^{n-1}(a_1) \bullet_0^y f(d_0 y; a_2, \ldots, a_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i \, f(d_i y; \; \alpha(a_1), \ldots, a_i \bullet_i^y a_{i+1}, \ldots, \alpha(a_{n+1})) \\ & + (-1)^{n+1} f(d_{n+1} y; \; a_1, \ldots, a_n) \bullet_{n+1}^y \alpha^{n-1}(a_{n+1}),\end{aligned}$$

for $y \in Y_{n+1}$ and $a_1, \ldots, a_{n+1} \in D$. Similar to the hom-associative case [1], one can prove the following.

4.2. Proposition. *The coboundary map satisfies* $\delta_\alpha^2 = 0$.

The cohomology of the complex $(CY_\alpha^\bullet(D, D), \delta_\alpha)$ is called the cohomology of the hom-dialgebra $(D, \dashv, \vdash, \alpha)$ and the cohomology groups are denoted by $HY_\alpha^n(D, D)$, for $n \geq 2$.

4.3. Remark. When $(D, \dashv, \vdash, \alpha)$ is a dialgebra, that is, $\alpha = id$, one recovers the known dialgebra cohomology [8]. When $(D, \dashv, \vdash, \alpha)$ is a hom-associative algebra, that is, $\dashv \; = \mu = \; \vdash$, one recovers the cohomology of a hom-associative algebra.

We show that the cochain groups $CY_\alpha^\bullet(D, D)$ carries a homotopy G-algebra structure. Hence, the cohomology $HY_\alpha^\bullet(D, D)$ inherits a Gerstenhaber algebra structure.

Operad structure: Let D be a vector space and $\alpha : D \to D$ be a linear map. For each $k \geq 1$ define $CY_\alpha^k(D, D)$ to be the space of all multilinear maps $f : K[Y_k] \otimes D^{\otimes k} \to D$ satisfying

$$(\alpha \circ f)(y; a_1, \ldots, a_k) = f(y; \alpha(a_1), \ldots, \alpha(a_k)), \quad \text{for all } y \in Y_k \text{ and } a_i \in D.$$

Our aim is to define an operad structure on $\mathcal{O} = \{\mathcal{O}(k) | \; k \geq 1\}$ where $\mathcal{O}(k) = CY_\alpha^k(D, D)$, for $k \geq 1$. For this, we closely follow [9].

For any $k, n_1, \ldots, n_k \geq 1$, we define maps $\mathcal{R}_0(k; n_1, \ldots, n_k) : Y_{n_1+\cdots+n_k} \to Y_k$ by

$$\mathcal{R}_0(k; n_1, \ldots, n_k) := d_1 \cdots d_{n_1-1} d_{n_1+1} \cdots d_{n_1+n_2-1} d_{n_1+n_2+1} \cdots d_{n_1+\cdots+n_{k-1}-1} d_{n_1+\cdots+n_{k-1}+1} \cdots d_{n_1+\cdots n_k-1}.$$

Moreover, for any $1 \le i \le k$, there are maps $\mathcal{R}_i(k; n_1, \dots, n_k) : Y_{n_1+\dots+n_k} \to Y_{n_i}$ defined by

$$\mathcal{R}_i(k; n_1, \dots, n_k) := d_0 d_1 \cdots d_{n_1+\dots+n_{i-1}-1} d_{n_1+\dots+n_i+1} \cdots d_{n_1+\dots+n_k}.$$

In other words, the function $\mathcal{R}_0(k; n_1, \dots, n_k)$ misses $d_0,\ d_{n_1},\ d_{n_1+n_2}, \dots, d_{n_1+\dots+n_k}$ and the function $\mathcal{R}_i(k; n_1, \dots, n_k)$ misses $d_{n_1+\dots+n_{i-1}},\ d_{n_1+\dots+n_{i-1}+1}, \dots, d_{n_1+\dots+n_i}$.

Then the collection

$$\mathcal{R} = \{\mathcal{R}_0(k; n_1, \dots, n_k),\ \mathcal{R}_i(k; n_1, \dots, n_k) |\ k, n_1, \dots, n_k \ge 1 \text{ and } 1 \le i \le k\}$$

satisfies the following relations of a pre-operadic system [12]:

- $\mathcal{R}_0(k; \underbrace{1, \dots, 1}_{k \text{ times}}) = id_{Y_k}, \quad \text{for each } k \ge 1,$
- $\mathcal{R}_0(k; n_1, \dots, n_k)\mathcal{R}_0(n_1 + \dots + n_k;\ m_1, \dots, m_{n_1+\dots+n_k})$
  $= \mathcal{R}_0(k;\ m_1 + \dots + m_{n_1},\ m_{n_1+1} + \dots + m_{n_1+n_2}, \dots,\ m_{n_1+\dots+n_{k-1}+1} + \dots + m_{n_1+\dots+n_k}),$
- $\mathcal{R}_i(k; n_1, \dots, n_k)\mathcal{R}_0(n_1 + \dots + n_k;\ m_1, \dots, m_{n_1+\dots+n_k}) = \mathcal{R}_0(n_i; m_{n_1+\dots+n_{i-1}+1}, \dots, m_{n_1+\dots+n_i})$
  $\mathcal{R}_i(k;\ m_1 + \dots + m_{n_1},\ m_{n_1+1} + \dots + m_{n_1+n_2}\ , \dots, m_{n_1+\dots+n_{k-1}+1} + \dots + m_{n_1+\dots+n_k}),$
- $\mathcal{R}_{n_1+\dots+n_{i-1}+j}(n_1 + \dots + n_k;\ m_1, \dots, m_{n_1+\dots+n_k}) = \mathcal{R}_j(n_i; m_{n_1+\dots+n_{i-1}+1}, \dots, m_{n_1+\dots+n_i})$
  $\mathcal{R}_i(k;\ m_1 + \dots + m_{n_1},\ m_{n_1+1} + \dots + m_{n_1+n_2}\ , \dots, m_{n_1+\dots+n_{k-1}+1} + \dots + m_{n_1+\dots+n_k}),$

for any $m_1, \dots, m_{n_1+\dots+n_k} \ge 1$.

Now, we are in a position to define an operad structure on $\mathcal{O}$. Define partial compositions $\circ_i : \mathcal{O}(m) \otimes \mathcal{O}(n) \to \mathcal{O}(m + n - 1)$ by

$$(f \circ_i g)(y; a_1, \dots, a_{m+n-1}) = f\Big(\mathcal{R}_0(m; \overbrace{1, \dots, 1, \underbrace{n}_{i\text{-th place}}, 1, \dots, 1}^{m\text{-tuple}})y;\ \alpha^{n-1}a_1, \dots, \alpha^{n-1}a_{i-1},$$
$$g\big(\mathcal{R}_i(m; \overbrace{1, \dots, 1, \underbrace{n}_{i\text{-th place}}, 1, \dots, 1}^{m\text{-tuple}})y;\ a_i, \dots, a_{i+n-1}\big), \alpha^{n-1}a_{i+n}, \dots, \alpha^{n-1}a_{m+n-1}\Big),$$

for $f \in CY^m_\alpha(D, D),\ g \in CY^n_\alpha(D, D),\ y \in Y_{m+n-1}$ and $a_1, \dots, a_{m+n-1} \in D$. Therefore, by using (2) and the pre-operadic identities, it follows that the compositions

$$\gamma_\alpha : \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \dots \otimes \mathcal{O}(n_k) \to \mathcal{O}(n_1 + \dots + n_k)$$

are given by

$$\begin{aligned}
&\gamma_\alpha(f; g_1, \dots, g_k)(y; a_1, \dots, a_{n_1+\dots+n_k}) \\
&= f\big(\mathcal{R}_0(k; n_1, \dots, n_k)y;\ \alpha^{\sum_{l=2}^k |g_l|}\ g_1\big(\mathcal{R}_1(k; n_1, \dots, n_k)y;\ a_1, \dots, a_{n_1}\big), \dots, \\
&\qquad \alpha^{\sum_{l=1, l\neq i}^k |g_l|}\ g_i\big(\mathcal{R}_i(k; n_1, \dots, n_k)y;\ a_{n_1+\dots+n_{i-1}+1}, \dots, a_{n_1+\dots+n_i}\big), \dots, \\
&\qquad \alpha^{\sum_{l=1}^{k-1} |g_l|}\ g_k\big(\mathcal{R}_k(k; n_1, \dots, n_k)y;\ a_{n_1+\dots+n_{k-1}+1}, \dots, a_{n_1+\dots+n_k}\big)\big),
\end{aligned}$$

for all $y \in Y_{n_1+\dots+n_k}$ and $a_1, a_2, \dots, a_{n_1+\dots+n_k} \in D$.

We also consider the identity map $\mathrm{id} \in CY^1_\alpha(D, D)$ defined by $\mathrm{id}([1]; a) = a$, for all $a \in D$.

Using the pre-operadic identities of $\mathcal{R}$, we can prove the following. The proof is similar to Proposition 3.2; hence, we omit the details.

4.4. Proposition. *The partial compositions $\circ_i$ (or compositions γ_α) defines a non-$\sum$ operad structure on $CY^\bullet_\alpha(D, D)$ with the identity element given by the identity map $\mathrm{id} \in CY^1_\alpha(D, D)$.*

4.5. Remark. When $\alpha : D \to D$ is the identity map (dialgebra case), one recovers the operad considered in [9].

Note that, the corresponding braces are given by

$$\{f\}\{g_1, \dots, g_n\} := \sum (-1)^\epsilon\ \gamma_\alpha(f; \mathrm{id}, \dots, \mathrm{id}, g_1, \mathrm{id}, \dots, \mathrm{id}, g_n, \mathrm{id}, \dots, \mathrm{id}).$$

The degree -1 graded Lie bracket on $CY_\alpha^\bullet(D,D)$ is given by

$$[f,g] = f \circ g - (-1)^{(m-1)(n-1)} g \circ f,$$

where

$$(f \circ g)(y; a_1, \dots, a_{m+n-1})$$
$$= \sum_{i=1}^{m} (-1)^{(i-1)(n-1)} f\Big(\mathcal{R}_0(m; \overbrace{1,\dots,1, \underbrace{n}_{i\text{-th place}}, 1, \dots, 1}^{m\text{-tuple}})y;\ \alpha^{n-1}a_1, \dots, \alpha^{n-1}a_{i-1},$$
$$g\big(\mathcal{R}_i(m; \overbrace{1,\dots,1, \underbrace{n}_{i\text{-th place}}, 1, \dots, 1}^{m\text{-tuple}})y;\ a_i, \dots, a_{i+n-1}\big), \alpha^{n-1}a_{i+n}, \dots, \alpha^{n-1}a_{m+n-1}\Big),$$

for $f \in CY_\alpha^m(D,D)$, $g \in CY_\alpha^n(D,D)$ and $a_1, \dots, a_{m+n-1} \in D$.

Next, let $(D, \dashv, \vdash, \alpha)$ be a hom-dialgebra. Consider the operad structure on $CY_\alpha^\bullet(D,D)$ as defined above. Define an element $\pi \in CY_\alpha^2(D,D)$ by the following

$$\pi(y; a, b) := \begin{cases} a \dashv b & \text{if } y = [21], \\ a \vdash b & \text{if } y = [12], \end{cases}$$

for all $a, b \in D$. An easy calculation shows that

$$\{\pi\}\{\pi\}(y; a, b, c) = \begin{cases} (a \vdash b) \vdash \alpha(c) \ - \ \alpha(a) \vdash (b \vdash c) & \text{if } y = [123], \\ (a \dashv b) \vdash \alpha(c) \ - \ \alpha(a) \vdash (b \vdash c) & \text{if } y = [213], \\ (a \vdash b) \dashv \alpha(c) \ - \ \alpha(a) \vdash (b \dashv c) & \text{if } y = [131], \\ (a \dashv b) \dashv \alpha(c) \ - \ \alpha(a) \dashv (b \vdash c) & \text{if } y = [312], \\ (a \dashv b) \dashv \alpha(c) \ - \ \alpha(a) \dashv (b \dashv c) & \text{if } y = [321]. \end{cases}$$

Hence, it follows from the hom-dialgebra condition that $\{\pi\}\{\pi\}(y; a, b, c) = 0$, for all $y \in Y_3$ and $a, b, c \in D$. Therefore, π defines a multiplication on the operad $CY_\alpha^\bullet(D,D)$. The corresponding dot product on $CY_\alpha^\bullet(D,D)$ is given by

$$(f \cdot g)(y; a_1, \dots, a_{m+n}) = (-1)^m \{\pi\}\{f, g\}(y; a_1, \dots, a_{m+n})$$
$$= (-1)^{mn} \pi\big(\mathcal{R}_0(2; m, n)y; \alpha^{n-1} f(\mathcal{R}_1(2; m, n)y; a_1, \dots, a_m), \alpha^{m-1} g(\mathcal{R}_2(2; m, n)y; a_{m+1}, \dots, a_{m+n})\big),$$

for $f \in CY_\alpha^m(D,D)$, $g \in CY_\alpha^n(D,D)$, $y \in Y_{m+n}$ and $a_1, \dots, a_{m+n} \in D$. Like in the hom-associative case, the differential here is given by

$$df = (-1)^{|f|+1} \delta_\alpha(f).$$

Thus, in view of Theorem 2.6 and Remark 2.8, we get the following.

4.6. Theorem. *Let $(D, \dashv, \vdash, \alpha)$ be a hom-dialgebra. Then the cochain complex $CY_\alpha^\bullet(D,D)$ inherits a homotopy G-algebra structure. Hence, its cohomology $HY_\alpha^\bullet(D,D)$ carries a Gerstenhaber algebra structure.*

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References

[1] F. Ammar, Z. Ejbehi, A. Makhlouf, Cohomology and deformations of hom-algebras, J. Lie Theory 21 (4) (2011) 813–836.
[2] A. Das, Gerstenhaber algebra structure on the cohomology of a hom-associative algebra, arXiv:1805.01207.
[3] A. Frabetti, Dialgebra (co)homology with coefficients, in: Dialgebras and Related Operads, in: Lecture Notes in Math., vol. 1763, Springer, Berlin, 2001, pp. 67–103.
[4] M. Gerstenhaber, The cohomology structure of an associative ring, Ann. of Math. (2) 78 (1963) 267–288.
[5] M. Gerstenhaber, A.A. Voronov, Homotopy G-algebras and moduli space operad, Int. Math. Res. Not. 1995 (3) (1995) 141–153.
[6] E. Getzler, J.D.S. Jones, Operads, homotopy algebra and iterated integrals for double loop spaces, preprint, arXiv:hep-th/9403055.
[7] J.T. Hartwig, D. Larsson, S.D. Silvestrov, Deformations of Lie algebras using σ-derivations, J. Algebra 295 (2006) 314–361.

[8] J.-L. Loday, A. Frabetti, F. Chapoton, F. Goichot, Dialgebras, in: Dialgebras and Related Operads, in: Lecture Notes in Math., vol. 1763, Springer, Berlin, 2001, pp. 7–66.
[9] A. Majumder, G. Mukherjee, Dialgebra cohomology as a G-algebra, Trans. Amer. Math. Soc. 356 (6) (2004) 2443–2457.
[10] A. Makhlouf, S. Silvestrov, Hom-algebra structures, J. Gen. Lie Theory Appl. 2 (2) (2008) 51–64.
[11] A. Makhlouf, S. Silvestrov, Notes on 1-parameter formal deformations of hom-associative and hom-Lie algebras, Forum Math. 22 (4) (2010) 715–739.
[12] D. Yau, Gerstenhaber structure and Deligne's conjecture for Loday algebras, J. Pure Appl. Algebra 209 (3) (2007) 739–752.
[13] D. Yau, Enveloping algebras of hom-Lie algebras, J. Gen. Lie Theory Appl. 2 (2) (2008) 95–108.