Partial differential equations/Calculus of variations

Uniqueness of degree-one Ginzburg–Landau vortex in the unit ball in dimensions \( N \geq 7 \)

*Unicité du tourbillon de Ginzburg–Landau de degré un dans la boule unité en dimension \( N \geq 7 \)*

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**A B S T R A C T**

For \( \varepsilon > 0 \), we consider the Ginzburg–Landau functional for \( \mathbb{R}^N \)-valued maps defined in the unit ball \( B^N \subset \mathbb{R}^N \) with the vortex boundary data \( x \) on \( \partial B^N \). In dimensions \( N \geq 7 \), we prove that, for every \( \varepsilon > 0 \), there exists a unique global minimizer \( u_\varepsilon \) of this problem; moreover, \( u_\varepsilon \) is symmetric and of the form \( u_\varepsilon(x) = f_\varepsilon(|x|) \frac{x}{|x|} \) for \( x \in B^N \).

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**R É S U M É**

Nous considérons la fonctionnelle de Ginzburg–Landau pour les applications à valeurs dans \( \mathbb{R}^N \) définies dans la boule unité \( B^N \subset \mathbb{R}^N \) avec la donnée de tourbillon \( x \) au bord \( \partial B^N \). En dimension \( N \geq 7 \), nous montrons que, pour tout \( \varepsilon > 0 \), il existe un unique minimiseur global \( u_\varepsilon \) à ce problème; de plus, \( u_\varepsilon \) est symétrique de la forme \( u_\varepsilon(x) = f_\varepsilon(|x|) \frac{x}{|x|} \) pour \( x \in B^N \).

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1. Introduction and main results

In this note, we consider the following Ginzburg–Landau-type energy functional

\[ E_\epsilon(u) = \int_{\mathbb{R}^N} \left[ \frac{1}{2} |\nabla u|^2 + \frac{1}{2\epsilon^2} W(1-|u|^2) \right] \, dx. \]

where \( \epsilon > 0 \), \( B^N \) is the unit ball in \( \mathbb{R}^N \), \( N \geq 2 \), and the potential \( W \in C^1((-\infty, 1]; \mathbb{R}) \) satisfies

\[ W(0) = 0, \quad W(t) > 0 \text{ for all } t \in (-\infty, 1]\setminus\{0\}, \quad \text{and } W \text{ is convex}. \]  

We investigate the global minimizers of the energy \( E_\epsilon \) in the set

\[ \mathcal{A} := \{ u \in H^1(B^N; \mathbb{R}^N) : u(x) = x \text{ on } \partial B^N = \mathbb{S}^{N-1} \}. \]

The requirement that \( u(x) = x \) on \( \mathbb{S}^{N-1} \) is sometimes referred to in the literature as the vortex boundary condition.

We note that, in our analysis, the convexity of \( W \) needs not be strict; compare [7] where strict convexity is assumed.

The direct method in the calculus of variations yields the existence of a global minimizer \( u_\epsilon \) of \( E_\epsilon \) over \( \mathcal{A} \) for all range of \( \epsilon > 0 \). Moreover, any minimizer \( u_\epsilon \) belongs to \( C^1(\mathbb{R}^N; \mathbb{R}^N) \) and satisfies \( |u_\epsilon| \leq 1 \) and the system of PDEs (in the sense of distributions):

\[ -\Delta u_\epsilon = \frac{1}{\epsilon^2} u_\epsilon W'(1-|u_\epsilon|^2) \quad \text{in } B^N. \]  

The goal of this note is to give a short proof of the uniqueness and symmetry of the global minimizer of \( E_\epsilon \) in \( \mathcal{A} \) for all \( \epsilon > 0 \) in dimensions \( N \geq 7 \). We prove that, in these dimensions, the global minimizer is unique and given by the unique radially symmetric critical point of \( E_\epsilon \) defined by

\[ u_\epsilon(x) = f_\epsilon(|x|) \frac{x}{|x|} \quad \text{for all } x \in B^N, \]

where the radial profile \( f_\epsilon : [0, 1] \rightarrow \mathbb{R}_+ \) is the unique solution to

\[ \begin{cases} -f''_\epsilon - \frac{N-1}{r} f'_\epsilon + \frac{N-1}{r^2} f_\epsilon = \frac{1}{\epsilon^2} f_\epsilon W'(1-f_\epsilon^2) & \text{for } r \in (0, 1), \\ f_\epsilon(0) = 0, \quad f_\epsilon(1) = 1. \end{cases} \]

Moreover, \( f_\epsilon > 0 \) and \( f'_\epsilon > 0 \) in \( (0, 1) \) (see, e.g., [5]).

**Theorem 1.** Assume that \( W \) satisfies (1). If \( N \geq 7 \), then for every \( \epsilon > 0 \), \( u_\epsilon \) given in (3) is the unique global minimizer of \( E_\epsilon \) in \( \mathcal{A} \).

To our knowledge, the question about the uniqueness of minimizers/critical points of \( E_\epsilon \) in \( \mathcal{A} \) for any \( \epsilon > 0 \) was raised in dimension \( N = 2 \) in the book of Bethuel, Brézis and Hélein [1, Problem 10, page 139], and in general dimensions \( N \geq 2 \) and also for the blow-up limiting problem around the vortex (when the domain is the whole space \( \mathbb{R}^N \) and by rescaling, \( \epsilon \) can be assumed equal to 1) in an article of Brézis [2, Section 2].

It is well known that uniqueness is present for large enough \( \epsilon > 0 \) for any \( N \geq 2 \). Indeed, for any \( \epsilon > (W'(1)/\lambda_1)^{1/2} \)

where \( \lambda_1 \) is the first eigenvalue of \( -\Delta \) in \( B^N \) with zero Dirichlet boundary condition, \( E_\epsilon \) is strictly convex in \( \mathcal{A} \) and thus has a unique critical point in \( \mathcal{A} \) (that is the global minimizer of our problem).

For sufficiently small \( \epsilon > 0 \), all results regarding uniqueness question available in the literature are in the affirmative. In particular, we have:

(i) Pacard and Rivière [12, Theorem 10.2] showed in dimension \( N = 2 \) that, for small \( \epsilon > 0 \), \( E_\epsilon \) has in fact a unique critical point in \( \mathcal{A} \);

(ii) Mironescu [11] showed in dimension \( N = 2 \) that, when \( B^2 \) is replaced by \( \mathbb{R}^2 \) and \( \epsilon = 1 \), a local minimizer of \( E_\epsilon \) subjected to a degree-one boundary condition at infinity is unique (up to translation and suitable rotation). This was generalized to dimension \( N = 3 \) by Millot and Pisante [10] and dimensions \( N \geq 4 \) by Pisante [13], also in the case of the blow-up limiting problem on \( \mathbb{R}^N \) and \( \epsilon = 1 \).

These results should be compared to those for the limit problem on the unit ball obtained by sending \( \epsilon \to 0 \). In this limit, the Ginzburg–Landau problem 'converges' to the harmonic map problem from \( B^N \) to \( \mathbb{S}^{N-1} \). It is well known that the vortex boundary condition gives rise to a unique minimizing harmonic map \( x \mapsto \frac{x}{|x|} \) if \( N \geq 3 \); see Brezis, Coron and Lieb [3] in dimension \( N = 3 \), Jäger and Kaul [8] in dimensions \( N \geq 7 \), and Lin [9] in dimensions \( N \geq 3 \) (see also [4]).

We highlight that, in contrast to the above, our result holds for all \( \epsilon > 0 \), provided that \( N \geq 7 \). The method of our proof deviates somewhat from that in the aforementioned works. In fact, it is reminiscent of our recent work [7] on
the (non-)uniqueness and symmetry of minimizers of the Ginzburg–Landau functionals for \( \mathbb{R}^M \)-valued maps defined on \( N \)-dimensional domains, where \( M \) is not necessarily the same as \( N \). However, we note that the results in [7] do not directly apply to the present context, as in [7] it is required that \( W \) be strictly convex. Furthermore, a priori, it is not clear why non-strict convexity of the potential \( W \) is sufficient to ensure uniqueness of global minimizers.

We exploit the convexity of \( W \) to lower estimate the ‘excess’ energy by a suitable quadratic energy that can be handled by the factorization trick à la Hardy. Indeed, the positivity of the excess energy is then related to the validity of a Hardy-type inequality, which explains our restriction of \( N \geq 7 \). This echoes our observation made in [7] that a result of Jäger and Kaul [8] on the minimality of the equator map (for the harmonic map problem) in these dimensions is related to a certain inequality involving the sharp constant in the Hardy inequality; see Remark 3.

We expect that our result remains valid in dimensions \( 2 \leq N \leq 6 \), but this goes beyond the scope of this note and remains for further investigation.

2. Proof of Theorem 1

Theorem 1 will be obtained as a consequence of a stronger result on the uniqueness of global minimizers of the \( \mathbb{R}^M \)-valued Ginzburg–Landau functional with \( M \geq N \). By a slight abuse of notation, we consider the energy functional

\[
E_\varepsilon(u) = \int_{\mathbb{R}^N} \left[ \frac{1}{2} \nabla u |^2 + \frac{1}{2\varepsilon^2} W(1 - |u|^2) \right] \, dx,
\]

where \( u \) belongs to

\[
\mathcal{A} := \{ u \in H^1(B^N; \mathbb{R}^M) : u(x) = x \text{ on } \partial B^N = S^{N-1} \subset \mathbb{R}^M \}.
\]

**Theorem 2.** Assume that \( W \) satisfies (1). If \( M \geq N \geq 7 \), then for every \( \varepsilon > 0 \), \( u_\varepsilon \) given in (3) is the unique global minimizer of \( E_\varepsilon \) in \( \mathcal{A} \).

When \( W \) is strictly convex, the above theorem is proved in [7]; see [7, Theorem 1.7]. The argument therein uses the strict convexity in a crucial way.

**Proof.** The proof will be done in several steps. First, we consider the difference between the energies of the critical point \( u_\varepsilon \), defined in (3), and an arbitrary competitor \( u_\varepsilon + v \) and show that this difference is controlled from below by some quadratic energy functional \( F_\varepsilon(v) \). Second, we employ the positivity of the radial profile \( f_\varepsilon \) in (4) and apply the Hardy decomposition method in order to show that \( F_\varepsilon(v) \geq 0 \), which proves in particular that \( u_\varepsilon \) is a global minimizer of \( E_\varepsilon \). Finally, we characterise the situation when this difference is zero and conclude to the uniqueness of the global minimizer \( u_\varepsilon \).

**Step 1: Lower bound for energy difference.** For any \( v \in H^1_0(B^N; \mathbb{R}^M) \), we have

\[
E_\varepsilon(u_\varepsilon + v) - E_\varepsilon(u_\varepsilon) = \int_{\mathbb{R}^N} \left[ \nabla u_\varepsilon \cdot \nabla v + \frac{1}{2} \nabla v |^2 \right] \, dx + \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^N} \left[ W(1 - |u_\varepsilon + v|^2) - W(1 - |u_\varepsilon|^2) \right] \, dx.
\]

Using the convexity of \( W \), we have

\[
W(1 - |u_\varepsilon + v|^2) - W(1 - |u_\varepsilon|^2) \geq -W'(1 - |u_\varepsilon|^2)(|u_\varepsilon + v|^2 - |u_\varepsilon|^2).
\]

The last two relations imply that

\[
E_\varepsilon(u_\varepsilon + v) - E_\varepsilon(u_\varepsilon) \geq \int_{\mathbb{R}^N} \left[ \nabla u_\varepsilon \cdot \nabla v - \frac{1}{\varepsilon^2} W'(1 - f_\varepsilon^2) u_\varepsilon \cdot v \right] \, dx + \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^N} \left[ \frac{1}{2} |\nabla v|^2 - \frac{1}{2\varepsilon^2} W'(1 - f_\varepsilon^2) |v|^2 \right] \, dx.
\]

Moreover, by (2), we obtain

\[
E_\varepsilon(u_\varepsilon + v) - E_\varepsilon(u_\varepsilon) \geq \int_{\mathbb{R}^N} \left[ \frac{1}{2} |\nabla v|^2 - \frac{1}{2\varepsilon^2} W'(1 - f_\varepsilon^2) |v|^2 \right] \, dx =: \frac{1}{2} F_\varepsilon(v)
\]

for all \( v \in H^1_0(B^N; \mathbb{R}^M) \). (In the sequel, for simplicity, we will also write \( F_\varepsilon(v) \) for scalar \( v \in H^1_0(B^N; \mathbb{R}) \).

**Step 2: A rewriting of \( F_\varepsilon(v) \) using the decomposition \( v = f_\varepsilon w \) for every scalar test function \( v \in C^\infty_0(B^N \setminus \{0\}; \mathbb{R}) \).** We consider the operator

\[
L_\varepsilon := \frac{1}{2} \nabla L_\varepsilon = -\Delta - \frac{1}{\varepsilon^2} W'(1 - f_\varepsilon^2).
\]
Using the decomposition

\[ \nu = \varphi \theta \]

for the scalar function \( \nu \in C_c^\infty(B^N \setminus \{0\}; \mathbb{R}) \), we have (see, e.g., [6, Lemma A.1]):

\[
F_\varphi(\nu) = \int_{B^N} L_\varphi \varphi \cdot \varphi \, dx = \int_{B^N} w^2 L_\varphi \varphi \cdot \varphi \, dx + \int_{B^N} f_\varphi^2 |\nabla \varphi|^2 \, dx
\]

\[ = \int_{B^N} f_\varphi^2 \left( |\nabla \varphi|^2 - \frac{N-1}{r^2} \varphi^2 \right) \, dx, \]

because (4) yields \( L_\varphi \varphi \cdot \varphi = -\frac{N-1}{r^2} f_\varphi^2 \) in \( B^N \).

**Step 3:** We prove that \( F_\varphi(\nu) \geq 0 \) for every scalar test function \( \nu \in C_c^\infty(B^N \setminus \{0\}; \mathbb{R}) \). Within the notation \( \nu = \varphi \theta \) of Step 2 with \( \nu, \theta \in C_c^\infty(B^N \setminus \{0\}; \mathbb{R}) \), we use the decomposition

\[ \theta = \varphi g \]

with \( \varphi = |x|^{-\frac{N-2}{2}} \) being the first eigenfunction of the Hardy’s operator \(-\Delta - \frac{(N-2)^2}{4|x|^2}\) in \( \mathbb{R}^N \setminus \{0\} \) and \( g \in C_c^\infty(B^N \setminus \{0\}; \mathbb{R}) \). We compute

\[ |\nabla \theta|^2 = |\nabla \varphi|^2 g^2 + |\nabla g|^2 \varphi^2 + \frac{1}{2} \nabla \varphi \cdot \nabla (g^2). \]

As \( |\nabla \varphi|^2 = \frac{(N-2)^2}{4|x|^2} \varphi^2 \) and \( \varphi^2 \) is harmonic in \( B^N \setminus \{0\} \), integration by parts yields

\[
F_\varphi(\nu) = \int_{B^N} f_\varphi^2 \left( |\nabla g|^2 \varphi^2 + \frac{(N-2)^2}{4r^2} \varphi^2 g^2 - \frac{N-1}{r^2} \varphi^2 g^2 \right) \, dx - \frac{1}{2} \int_{B^N} \nabla \varphi \cdot \nabla (f_\varphi^2 g^2) \, dx
\]

\[ \geq \int_{B^N} f_\varphi^2 |\nabla g|^2 \varphi^2 \, dx + \left( \frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{f_\varphi^2}{r^2} \varphi^2 g^2 \, dx
\]

\[ \geq \left( \frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{v^2}{r^2} \, dx \geq 0, \]

where we have used \( N \geq 7 \) and \( \frac{1}{2} \nabla \varphi \cdot \nabla (f_\varphi^2) = 2 \varphi \nabla \varphi \cdot \nabla f_\varphi^2 \leq 0 \) in \( B^N \setminus \{0\} \).

**Step 4:** We prove that \( F_\varphi(\nu) \geq 0 \) for every \( \nu \in H^1_0(B^N; \mathbb{R}^M) \), meaning that \( u_\varphi \) is a global minimizer of \( E_\varphi \) over \( \mathscr{A} \); moreover, \( F_\varphi(\nu) = 0 \) if and only if \( \nu = 0 \). Let \( \nu \in H^1_0(B^N; \mathbb{R}^M) \). As a point has zero \( H^1 \) capacity in \( \mathbb{R}^N \), a standard density argument implies the existence of a sequence \( \nu_k \in C_c^\infty(B^N \setminus \{0\}; \mathbb{R}^M) \) such that \( \nu_k \rightharpoonup \nu \) in \( H^1(B^N; \mathbb{R}^M) \) and a.e. in \( B^N \). On the one hand, by definition (5) of \( F_\varphi \), since \( W'(1-f_\varphi^2) \in L^\infty \), we deduce that \( F_\varphi(\nu_k) \to F_\varphi(\nu) \) as \( k \to \infty \). On the other hand, by (6) and Fatou’s lemma, we deduce

\[
\liminf_{k \to \infty} F_\varphi(\nu_k) \geq \left( \frac{(N-2)^2}{4} - (N-1) \right) \liminf_{k \to \infty} \int_{B^N} \frac{v^2}{r^2} \, dx
\]

\[ \geq \left( \frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{v^2}{r^2} \, dx. \]

Therefore, we conclude that

\[ F_\varphi(\nu) \geq \left( \frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{v^2}{r^2} \, dx \geq 0, \quad \forall \nu \in H^1_0(B^N; \mathbb{R}^M), \]

implying by (5) that \( u_\varphi \) is a minimizer of \( E_\varphi \) over \( \mathscr{A} \). Moreover, \( F_\varphi(\nu) = 0 \) if and only if \( \nu = 0 \).

**Step 5:** Conclusion. We have shown that \( u_\varphi \) is a global minimizer. Assume that \( \bar{u}_\varphi \) is another global minimizer of \( E_\varphi \) over \( \mathscr{A} \). If \( \nu := \bar{u}_\varphi - u_\varphi \), then \( \nu \in H^1_0(B^N; \mathbb{R}^M) \) and by steps 1 and 4, we have that \( 0 = E_\varphi(\bar{u}_\varphi) - E_\varphi(u_\varphi) \geq F_\varphi(\nu) \geq 0 \), which yields \( F_\varphi(\nu) = 0 \). Step 4 implies that \( \nu = 0 \), i.e. \( \bar{u}_\varphi = u_\varphi \).
Remark 3. Recall that, in the case \( M \geq N \geq 7 \), Jäger and Kaul [8] proved the uniqueness of global minimizer for harmonic map problem

\[
\min_{u \in \mathcal{A}} \int_{B^N} |\nabla u|^2 \, dx,
\]

where \( \mathcal{A} = \{ u \in H^1(B^N; S^{M-1}) : u(x) = x \text{ on } \partial B^N = S^{N-1} \subset S^{M-1} \} \). This can also be seen by the method above, as observed in our earlier paper [7]. We give the argument here for readers’ convenience: take a perturbation \( v \in H^1_0(B^N, \mathbb{R}^M) \) of the harmonic map \( u_* (x) = \frac{x}{|x|} \) such that \( |u_* (x) + v(x)| = 1 \) a.e. in \( B^N \). Then, by [7, Proof of Theorem 5.1],

\[
\int_{B^N} \left( |\nabla (u_* + v)|^2 - |\nabla u_*|^2 \right) \, dx = \int_{B^N} \left( |\nabla v|^2 - |\nabla u_*|^2 |v|^2 \right) \, dx = \int_{B^N} \left( |\nabla v|^2 - (N-1) \frac{|v|^2}{|x|^2} \right) \, dx.
\]

Using Hardy’s inequality in dimension \( N \), we arrive at

\[
\int_{B^N} \left( |\nabla (u_* + v)|^2 - |\nabla u_*|^2 \right) \, dx \geq \left( \frac{(N-2)^2}{4} - (N-1) \right) \int_{B^N} \frac{|v|^2}{|x|^2} \, dx.
\]

The result follows since \( N \geq 7 \).

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References


