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Partial differential equations

Solutions to a nonlinear Neumann problem in three-dimensional exterior domains

Solutions d'un problème de Neumann non linéaire dans des domaines extérieurs de dimension 3

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A B S T R A C T

We prove the existence of multipeak solutions to a nonlinear elliptic Neumann problem involving nearly critical Sobolev exponent, in three-dimensional exterior domains.

R É S U M É

Nous démontrons, pour un problème elliptique de Neumann avec non-linéarité presque critique, dans un domaine extérieur de dimension trois, l'existence de solutions qui se concentrent en plusieurs points de la frontière lorsque la non-linéarité devient critique.

To Prof. Norman Dancer

1. Introduction and results

There have been innumerable articles devoted, over the last three decades, to the study of elliptic partial differential equations of second order with critical nonlinearity. One thing, however, is to be noticed: virtually all articles consider problems in bounded domains. A work like Yan's one [15] is an exception. In that paper, Yan considers the following Neumann problem:

$$\begin{cases} -\Delta u = u^{2^*-1-\varepsilon} \quad, u>0 & \text{in } \mathbb{R}^N \setminus \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

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where Ω is a smooth and bounded domain in $\mathbb{R}^N$, $N \geq 3$, such that $R^N \setminus \Omega$ is connected, $2^* = 2N/(N-2)$ is the limiting Sobolev exponent for the embedding of the Sobolev space $W^{1,2}(\Omega)$ into the $L^p(\Omega)$-spaces, and ε is a strictly positive number, assumed to be small. Although formally, the problem is subcritical, it is asymptotically critical, and the techniques to be used to study it as ε goes to zero are the same as those to be used in the case of critical nonlinearities.

Considering exterior domains is all but arbitrary. Indeed, various models lead to study the equation $-\Delta u = u^p$ in domains with small holes. As the size of these holes goes to zero, the limiting problem which is obtained through a rescaling centered on one of the holes looks precisely as (1.1).

In order to state the results proved by Yan, as well as those that we propose to establish, some notations have first to be introduced. For $\lambda \in \mathbb{R}^*_+$ and $x \in \mathbb{R}^N$, we denote by $U_{\lambda,x}$ the function defined in $\mathbb{R}^N$ by

$$U_{\lambda,x}(y) = [N(N-2)]^{\frac{N-2}{4}} \frac{\lambda^{\frac{N-2}{2}}}{\left(1+\lambda^2|y-x|^2\right)^{\frac{N-2}{2}}}. \tag{1.2}$$

The $U_{\lambda,x}$'s are the only nontrivial solutions to the equation $-\Delta U = U^{2^*-1}$, $U \geq 0$ in $\mathbb{R}^N$ (see for example [2], [13] or [7]) and induce, as λ goes to infinity, a lack of compactness of the embedding of $W^{1,2}$ into L^{2^*}. In the following, $D^{1,2}(\mathbb{R}^N \setminus \Omega)$ refers to the completion of the set of smooth functions with compact support in $\mathbb{R}^N \setminus \Omega$ for the norm

$$\|u\| = < u, u >^{1/2}$$

with

$$< u, v > = \int_{R^N \setminus \Omega} \nabla u \cdot \nabla v.$$

Lastly, we denote by $H(y)$ the mean curvature of $\partial\Omega$ at a point y of this boundary. Yan proves:

Theorem 1.1. [15]. *Assume that $N \geq 4$.*

(1) [Case of a positive local maximum of H.] *Suppose that S is a connected subset of $\partial\Omega$ satisfying: $H(y) = H_m > 0$ for any $y \in S$; and there exists $\delta > 0$ such that $H(y) < H_m$ for any $y \in S_\delta \setminus S$, and H has no critical point in $S_\delta \setminus S$, with $S_\delta = \{y \in \partial\Omega$ s.t. $d(x, y) \leq \delta\}$. Then, for any positive integer k, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon_0 \in (0, \varepsilon_0)$, (1.1) has a solution*

$$u_\varepsilon = \sum_{i=1}^{k} \alpha_{\varepsilon,i} U_{\lambda_{\varepsilon,i}, x_{\varepsilon,i}} + v_\varepsilon \tag{1.3}$$

where, as ε goes to zero

$$\alpha_{\varepsilon,i} \to 1$$
$$\varepsilon\lambda_{\varepsilon,i} \to c^* H_m \quad c^* \text{ a positive constant depending on } N \text{ only}$$
$$x_{\varepsilon,i} \in S_\delta \quad \text{and} \quad x_{\varepsilon,i} \to x_i \in S$$

for any i, $i \leq i \leq k$, and

$$v_\varepsilon \to 0 \quad \text{in} \quad D^{1,2}(\mathbb{R}^N \setminus \Omega).$$

(2) [Case of a positive local minimum of H.] *Suppose that S is a connected subset of $\partial\Omega$ satisfying: $H(y) = H_m > 0$ for any $y \in S$; and there exists $\delta > 0$ such that $H(y) > H_m$ for any $y \in S_\delta \setminus S$, and H has no critical point in $S_\delta \setminus S$, with $S_\delta = \{y \in \partial\Omega \quad s.t. \quad d(x, y) \leq \delta\}$. Then, there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon_0 \in (0, \varepsilon_0)$, (1.1) has a solution*

$$u_\varepsilon = \alpha_\varepsilon U_{\lambda_\varepsilon, x_\varepsilon} + v_\varepsilon \tag{1.4}$$

where $\alpha_{\varepsilon,i} \to 1$, $\varepsilon\lambda_\varepsilon \to c^* H_m$, $x_\varepsilon \to x_0 \in S$ and $v_\varepsilon \to 0$ in $D^{1,2}(\mathbb{R}^N \setminus \Omega)$ as ε goes to zero.

The arguments developed by Yan to prove Theorem 1.1 allow him to consider also the problem

$$\begin{cases} -\Delta u + \mu u = u^{2^*-1} \quad , u > 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.5}$$

where μ is a positive number assumed to be large. Yan proves, for $N \geq 5$ and a positive local minimum of H, an equivalent of Theorem 1.1 (1) as μ goes to infinity ($1/\mu$ plays a role similar to that of ε previously). Such a result has been extended to the cases $N = 3, 4$ by Wei and Yan in [14]. On the other hand, it has never been demonstrated until now that the statement of Theorem 1.1 itself is valid in the case $N = 3$. As Brezis and Nirenberg [6] have shown, there are problems involving elliptic equations with critical nonlinearity where cases $N = 3$ and $N \geq 4$ are qualitatively different. This is not the case here, and we are able to prove the following theorem.

Theorem 1.2. *Let $N = 3$.*

(1) [Case of a positive local maximum of H.] *Suppose that $S \subset \partial\Omega$ is such that $0 < \sup_{y\in\partial S} H(y) < \sup_{y\in S} H(y) = H_m$. Then, for any positive integer k, there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon_0 \in (0, \varepsilon_0)$, (1.1) has a solution*

$$u_\varepsilon = \sum_{i=1}^{k} \alpha_{\varepsilon,i} U_{\lambda_{\varepsilon,i},x_{\varepsilon,i}} + v_\varepsilon \tag{1.6}$$

where, as ε goes to zero

$$\begin{aligned} &\alpha_{\varepsilon,i} \to 1 \\ &\frac{\ln \lambda_{\varepsilon,i}}{\varepsilon\lambda_{\varepsilon,i}} \to \frac{\pi}{16H_m} \quad \big(i.e.\ \lambda_{\varepsilon,i} \sim \frac{16H_m}{\pi\varepsilon} \ln \frac{1}{\varepsilon}\big) \\ &x_{\varepsilon,i} \to x_i \in S \quad \text{such that} \quad H(x_i) = \max_{y\in S} H(y) \end{aligned}$$

for any i, $1 \leq i \leq k$, and

$$v_\varepsilon \to 0 \quad \text{in} \quad D^{1,2}(\mathbb{R}^N \setminus \Omega).$$

(2) [Case of a positive local minimum of H.] *Suppose that $S \subset \partial\Omega$ is such that $0 < \inf_{y\in S} H(y) < \inf_{y\in\partial S} H(y) = H_m$. Then, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon_0 \in (0, \varepsilon_0)$, (1.1) has a solution*

$$u_\varepsilon = \alpha_\varepsilon U_{\lambda_\varepsilon,x_\varepsilon} + v_\varepsilon \tag{1.7}$$

where $\alpha_{\varepsilon,i} \to 1$, $\frac{\ln \lambda_{\varepsilon,i}}{\varepsilon\lambda_{\varepsilon,i}} \to \frac{\pi}{16H_m}$, $x_\varepsilon \to x_0 \in S$ such that $H(x_i) = \min_{y\in S} H(y)$ and $v_\varepsilon \to 0$ in $D^{1,2}(\mathbb{R}^N \setminus \Omega)$ as ε goes to zero.

Remark 1. As ∂S is closed and bounded, $\sup_{y\in\partial S} H(y)$ and $\inf_{y\in\partial S} H(y)$ are achieved. The same holds for $\sup_{y\in S} H(y)$ in case (1) and $\inf_{y\in S} H(y)$ in case (2). Indeed, let us consider in case (1) a maximizing sequence (y_n) in S for H. A subsequence of (y_n) converges to some limit $\bar{y}$ that satisfies $H(\bar{y}) = H_m$. As, by assumption, $\sup_{y\in\partial S} H(y) < H_m$, $\bar{y}$ has to be in the interior of S. Consequently, there exist one or several points $y \in S$ such that $H(y) = H_m$, and which are local maxima of H. We conclude in the same way in case (2).

Remark 2. As Ω is assumed to be bounded in $\mathbb{R}^N$, H has a strictly positive maximum on $\partial\Omega$, and case (1) always occurs (in the particular case of a ball, we can take $S = \partial\Omega$, $\partial S = \varnothing$).

Remark 3. The method we use to prove Theorem 1.2 allows us to get rid of the unessential assumption in the statement of Theorem 1.1 that H has no critical point in $S_\delta \setminus S$. Our argument to eliminate such an assumption applies as well to the case $N \geq 4$.

Problem (1.1) can be formulated in a variational way: formally, $u \in D^{1,2}(\mathbb{R}^N \setminus \Omega)$ solves (1.1) if and only if u is a nontrivial critical point of the functional

$$I_\varepsilon = \frac{1}{2} \int_{R^N\setminus\Omega} |\nabla u|^2 - \frac{1}{2^* - \varepsilon} \int_{R^N\setminus\Omega} (u^+)^{2^*-\varepsilon} \tag{1.8}$$

with $u^+ = \max(u, 0)$. Indeed, a critical point of I_ε satisfies

$$-\Delta u = (u^+)^{2^*-\varepsilon} \text{ in } R^N \setminus \Omega, \ ; \ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{1.9}$$

Multiplying the equation by $u^- = \max(-u, 0)$ and integrating on $R^N \setminus \Omega$, we see that $u^- \equiv 0$, whence $u \geq 0$ in $R^N \setminus \Omega$. If $u \not\equiv 0$, the strong maximum principle implies that $u > 0$ in $R^N \setminus \Omega$; as a consequence, u is a solution to (1.1).

However, a difficulty arises: I_ε is not defined in whole $D^{1,2}(\mathbb{R}^N \setminus \Omega)$ for, as $R^N \setminus \Omega$ is not bounded, this space does not embed into $L^{2^*-\varepsilon}(R^N \setminus \Omega)$. Our first task will therefore be, in the next section, to build a functional $\tilde{I_\varepsilon}$ well defined in $D^{1,2}(\mathbb{R}^N \setminus \Omega)$, and whose critical points that write as in Theorem 1.1 or 1.2 are solutions to (1.1).

In Section 3, we shall perform a parametrization of the problem in a neighborhood of the solutions of type (1.3) (1.6) we look for, in order to obtain a functional depending on the α_i's, λ_i's, x_i's and v.

In Section 4, an optimization of that functional with respect to the α_i's and v will provide us with a function now dependent only on the λ_i's and the x_i's. Then we will be able, in the last section (Section 5), to deduce from the assumptions of Theorem 1.2 that the reduced function has a critical point – whence, by construction, the existence of a solution to (1.1) with the properties specified in Theorem 1.2.

The proof of a number of technical results necessary for the exposition of the main argument is given in Appendix.

2. The variational formulation

Although this article is intended to establish Theorem 1.2, and is therefore concerned only with space dimension $N = 3$, we consider in this section any dimension $N \geq 3$ – insofar the argument is identical in all dimensions. Moreover, we shall make explicit some points that are sketched in [15]. To obtain from I_ε defined by (1.8) a functional $\widetilde{I_\varepsilon}$ well defined in $D^{1,2}(R^N \setminus \Omega)$, the nonlinearity has to be truncated at infinity. By adapting Yan's strategy in [15], we proceed as follows.

We choose $R > 0$ such that $\Omega \subset B_{R/2}(0)$, and $\tau > 0$. Let $\varphi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $0 \leq \varphi \leq 1$, φ smooth in $\mathbb{R}_+ \times \mathbb{R}_+$, such that

$$\begin{cases} \varphi(s,t) = 0 & \text{if } t < 0 \\ \varphi(s,t) = 1 & \text{if } 0 \leq s \leq R \text{ and } t \geq 0, \text{ or } s \geq R \text{ and } 0 \leq s^{N-2} t \leq \tau \\ \varphi(s,t) = 0 & \text{if } s \geq R+1 \text{ and } s^{N-2} t \geq \tau + 1. \end{cases} \tag{2.1}$$

Then, we define $g_\varepsilon : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ as

$$g_\varepsilon(y,t) = |t|^{2^*-1-\varepsilon} \varphi(|y|, t) \tag{2.2}$$

which, for ε small enough, is C^1, and we consider the problem

$$\begin{cases} -\Delta u = g_\varepsilon(y,u) & \text{in } R^N \setminus \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

By construction, $g_\varepsilon(y,u) = (u^+)^{2^*-1-\varepsilon}$ for $|y| \leq R$, or $|y| \geq R$ and $|y|^{N-2}u^+ \leq \tau$; $g_\varepsilon(y,u) = 0$ for $|y| \geq R+1$ and $|y|^{N-2}u^+ \geq \tau + 1$. We notice that, according to definition (1.2) of the $U_{\lambda,x}$'s, when $x \in \partial\Omega$ (whence $|x| < R$), $|y|^{N-2} U_{\lambda,x}(y)$ goes to zero as λ goes to infinity, uniformly with respect to y, $|y| \geq R$. Consequently, when the x_i's belong to $\partial\Omega$, the α_i's are close to 1 and the λ_i's are large enough, we have

$$g_\varepsilon\big(y, \sum_{i=1}^k \alpha_i U_{\lambda_i,x_i}(y)\big) = \Big(\sum_{i=1}^k \alpha_i U_{\lambda_i,x_i}(y)\Big)^{2^*-1-\varepsilon}$$

in all $R^N \setminus \Omega$: the truncation does not affect the nonlinearity when $u = \sum_{i=1}^k \alpha_i U_{\lambda_i,x_i}$, in the neighborhood of which we look for a solution to (1.1). We set now

$$\widetilde{I_\varepsilon}(u) = \frac{1}{2} \int_{R^N \setminus \Omega} |\nabla u|^2 - \int_{R^N \setminus \Omega} G_\varepsilon(y,u) \tag{2.4}$$

with

$$G_\varepsilon(y,u) = \int_0^u g_\varepsilon(y,t)\,\mathrm{d}t.$$

According to the definition of g_ε, $G_\varepsilon(y,u) = 0$ if $u \leq 0$, $G_\varepsilon(y,u) = \frac{1}{2^*-\varepsilon} u^{2^*-\varepsilon}$ if $u \geq 0$ and $|y| \leq R$, and $|G_\varepsilon(y,u)| \leq \frac{1}{2^*-\varepsilon}|u|^{2^*-\varepsilon}$ everywhere. Moreover, (2.1) and (2.2) imply that if $|y| \geq R+1$ and $|y|^{N-2}|u| \geq \tau + 1$, $g_\varepsilon(y,u) = 0$, and if $|y| \geq R+1$ and $|y|^{N-2}|u| \leq \tau + 1$

$$\left| \frac{g_\varepsilon(y,u)}{u} \right| \leq |u|^{2^*-2-\varepsilon} \leq \frac{(\tau+1)^{\frac{4}{N-2}-\varepsilon}}{|y|^{4-(N-2)\varepsilon}}. \tag{2.5}$$

Then, using Young's inequality, we can write, for $|y| \geq R+1$,

$$0 \leq G_\varepsilon(y,u) \leq \frac{u^2 (\tau+1)^{\frac{4}{N-2}-\varepsilon}}{2|y|^{4-(N-2)\varepsilon}} \leq (\tau+1)^{\frac{4}{N-2}-\varepsilon} \left(\frac{|u|^{2^*}}{2^*} + \frac{1}{N|y|^{2N-\frac{N(N-2)}{2}\varepsilon}} \right).$$

As $D^{1,2}(R^N \setminus \Omega)$ embeds into $L^{2^*}(R^N \setminus \Omega)$, we see that $\widetilde{I_\varepsilon}$ is well defined in this functional space.

Arguing as we did before for the solutions to (1.8), we know that a nontrivial solution to (2.3) is strictly positive in $R^N \setminus \Omega$. Let us show that a solution u_ε to (2.3), writing as (1.3) (1.6), is actually a solution to (1.1). To this end, we shall show that $|y|^{N-2}u_\varepsilon(y)$ goes uniformly to 0 in $\mathbb{R}^N \setminus B_R(0)$ as ε goes to zero – this entails, by definition of g_ε, that $g_\varepsilon(y,u_\varepsilon) = u_\varepsilon^{2^*-1-\varepsilon}$ in $R^N \setminus \Omega$, whence the result. Setting

$$w_\varepsilon(y) = \frac{1}{|y|^{N-2}} u_\varepsilon\left(\frac{y}{|y|^2}\right)$$

this amounts to showing that w_ε goes uniformly to 0 in $B_{1/R}(0)$. As u_ε is assumed to solve (2.3), w_ε satisfies, in $B_{2/R}(0) \setminus \{0\}$

$$-\Delta w_\varepsilon = \frac{1}{|y|^{N+2}} \Delta u_\varepsilon\left(\frac{y}{|y|^2}\right) = \frac{1}{|y|^{N+2}} g_\varepsilon\left(\frac{y}{|y|^2}, |y|^{N-2} w_\varepsilon(y)\right).$$

We may also write

$$-\Delta w_\varepsilon = a_\varepsilon(y) w_\varepsilon \quad \text{in } B_{2/R}(0) \setminus \{0\} \tag{2.6}$$

where, according to the definition (2.1) (2.2) of g_ε, a_ε satisfies

$$0 < a_\varepsilon(y) \le \frac{1}{|y|^{(N-2)\varepsilon}} w_\varepsilon^{2^*-2-\varepsilon} \quad \text{in } B_{2/R}(0) \tag{2.7}$$

and also, because of (2.5)

$$0 < a_\varepsilon(y) \le \frac{C}{|y|^{(N-2)\varepsilon}} \quad \text{in } B_{1/(R+1)}(0) \tag{2.8}$$

where C is a constant depending only on τ and N.

Actually, we can check that (2.6) holds in the whole $B_{2/R}(0)$. To this end, we consider $\psi \in C^\infty(\mathbb{R}^N, \mathbb{R})$ such that $\psi(y) = 0$ if $|y| \le 1/2$ and $\psi(y) = 1$ if $|y| \ge 1$, and we set, for $n \in \mathbb{N}^*$, $\psi_n(y) = \psi(ny)$. For any $\varphi \in C_0^\infty(B_{2/R}(0))$, $\psi_n\varphi \in C_0^\infty(B_{2/R}(0) \setminus \{0\})$, and (2.6) yields

$$\langle -\Delta w_\varepsilon, \psi_n \varphi\rangle = \int\limits_{B_{2/R}(0)} a_\varepsilon(y) w_\varepsilon \psi_n \varphi.$$

Through dominated convergence, it is easily checked that the right-hand side goes to $\int_{B_{2/R}(0)} a_\varepsilon(y) w_\varepsilon \varphi$ as n goes to infinity. On the left-hand side, we have

$$\langle -\Delta w_\varepsilon, \psi_n\varphi\rangle = -\langle w_\varepsilon, \psi_n \Delta\varphi + 2\nabla\psi_n . \nabla\varphi + \varphi\Delta\psi_n\rangle$$

and

$$\begin{aligned} -\langle w_\varepsilon, \varphi\Delta\psi_n\rangle &= \int\limits_{1/2n \le |y| \le 1/n} w_\varepsilon \varphi \Delta\psi_n \\ &= \mathrm{O}\left(n^2 \left(\int\limits_{1/2n \le |y| \le 1/n} w_\varepsilon^{2^*}\right)^{\frac{1}{2^*}} \left(\int\limits_{1/2n \le |y| \le 1/n} \mathrm{d}y\right)^{\frac{N+2}{2N}}\right). \end{aligned}$$

As

$$\int\limits_{1/2n \le |y| \le 1/n} w_\varepsilon^{2^*} = \int\limits_{n \le |y| \le 2n} \frac{1}{|y|^{2N}} w_\varepsilon^{2^*}\left(\frac{y}{|y|^2}\right) = \int\limits_{n \le |y| \le 2n} u_\varepsilon^{2^*}$$

we see that

$$-\langle w_\varepsilon, \varphi\Delta\psi_n\rangle = \mathrm{o}(n^{-\frac{N-2}{2}}).$$

The term involving $\nabla\psi_n . \nabla\varphi$ may be treated in the same way, so that

$$\lim_{n\to\infty} \langle -\Delta w_\varepsilon, \psi_n\varphi\rangle = -\lim_{n\to\infty} \langle w_\varepsilon, \psi_n\Delta\varphi\rangle = -\langle w_\varepsilon, \Delta\varphi\rangle = \langle -\Delta w_\varepsilon, \varphi\rangle$$

and (2.6) holds in whole $B_{2/R}(0)$, as announced.

Now, in view of (2.7) and (2.8), for any $y_0 \in B_{3/(2R)}(0)$ and any δ, $0<\delta<\frac{1}{4R}$, we can write

$$\begin{aligned}\int\limits_{B_\delta(y_0)} a_\varepsilon^{N/2} &= \int\limits_{B_\delta(y_0)\cap B_{1/(R+1)}(0)} a_\varepsilon^{N/2} + \int\limits_{B_\delta(y_0)\setminus B_{1/(R+1)}(0)} a_\varepsilon^{N/2}\\ &\le C_1 \int\limits_{B_\delta(y_0)} |y|^{-\frac{N(N-2)}{2}\varepsilon}\mathrm{d}y + C_2(R+1)^{\frac{N(N-2)}{2}\varepsilon}\int\limits_{B_{7/(4R)}(0)} w_\varepsilon^{\frac{N}{2}(2^*-2-\varepsilon)}\\ &\le C_3\delta^{N-\frac{N(N-2)}{2}\varepsilon} + C_4\Bigg(\int\limits_{B_{7/(4R)}(0)} w_\varepsilon^{2^*}\Bigg)^{1-\frac{N-2}{4}\varepsilon}\end{aligned}$$

where $C_1,\dots,C_4$ are constants depending only on τ, R and N. Furthermore, we have

$$\int\limits_{B_{7/(4R)}(0)} w_\varepsilon^{2^*} = \int\limits_{\mathbb{R}^N\setminus B_{4R/7}(0)} u_\varepsilon^{2^*} \tag{2.9}$$

and, when u_ε write as (1.3) (1.6), the last integral goes to zero as ε goes to zero (remember that the $U_{\lambda_{\varepsilon,i},x_{\varepsilon,i}}$'s concentrate at points located on the boundary of Ω, with $\Omega\subset B_{R/2}(0)$, and v_ε goes to zero in $D^{1,2}(R^N\setminus\Omega)$ whence also in $L^{2^*}(R^N\setminus\Omega)$). Consequently, we see that, for every $\eta>0$, we can choose $\delta>0$ such that, for ε small enough,

$$\int\limits_{B_\delta(y_0)} a_\varepsilon^{N/2} < \eta \tag{2.10}$$

uniformly with respect to $y_0\in B_{3/(2R)}(0)$. This will imply that w_ε goes to zero in $L^\infty(B_{1/R}(0))$.

Indeed, let us consider δ', $0<\delta'<\delta$, and $\psi\in C^\infty(\mathbb{R}^N)$ such that $0\le\psi\le 1$ and $\psi\equiv 1$ in $B_{\delta'}(0)$, $\psi\equiv 0$ in $\mathbb{R}^N\setminus B_\delta(0)$. For $y_0\in B_{3/(2R)}(0)$, let $\psi_{y_0}\in C^\infty(\mathbb{R}^3)$ be the function defined by $\psi_{y_0}(y)=\psi(y-y_0)$. Multiplying (2.6) by $\psi_{y_0}^2 w_\varepsilon^\gamma$, $\gamma>1$, and integrating in $B_\delta(y_0)$, we obtain

$$-\int\limits_{B_\delta(y_0)} \Delta w_\varepsilon.\psi_{y_0}^2 w_\varepsilon^\gamma = \int\limits_{B_\delta(y_0)} a_\varepsilon\psi_{y_0}^2 w_\varepsilon^{\gamma+1} \tag{2.11}$$

provided that the integrals are well defined. On the one hand, integrating by parts, we have

$$\begin{aligned}-\int\limits_{B_\delta(y_0)} \Delta w_\varepsilon.\psi_{y_0}^2 w_\varepsilon^\gamma = \frac{4\gamma}{(\gamma+1)^2}\int\limits_{B_\delta(y_0)} \big|\nabla(\psi_{y_0}w_\varepsilon^{\frac{\gamma+1}{2}})\big|^2 &- \frac{4(\gamma-1)}{(\gamma+1)^2}\int\limits_{B_\delta(y_0)} w_\varepsilon^{\frac{\gamma+1}{2}}\nabla\psi_{y_0}.\nabla(\psi_{y_0}w_\varepsilon^{\frac{\gamma+1}{2}})\\ &- \frac{4}{(\gamma+1)^2}\int\limits_{B_\delta(y_0)} \big|\nabla\psi_{y_0}\big|^2 w_\varepsilon^{\gamma+1}.\end{aligned}$$

On the other hand, Hölder's inequality yields

$$\int\limits_{B_\delta(y_0)} a_\varepsilon\psi_{y_0}^2 w_\varepsilon^{\gamma+1} \le \Bigg(\int\limits_{B_\delta(y_0)} a_\varepsilon^{N/2}\Bigg)^{\frac{2}{N}}\Bigg(\int\limits_{B_\delta(y_0)} \psi_{y_0}^{2^*} w_\varepsilon^{\frac{N}{N-2}(\gamma+1)}\Bigg)^{\frac{N-2}{N}}.$$

Coming back to (2.11), we deduce from (2.10) and the Cauchy–Schwarz inequality

$$\begin{aligned}&\frac{4\gamma}{(\gamma+1)^2}\int\limits_{B_\delta(y_0)} \big|\nabla(\psi_{y_0}w_\varepsilon^{\frac{\gamma+1}{2}})\big|^2\\ &\quad - \frac{4(\gamma-1)}{(\gamma+1)^2}\Bigg(\int\limits_{B_\delta(y_0)} \big|\nabla\psi_{y_0}\big|^2 w_\varepsilon^{\gamma+1}\Bigg)^{\frac12}\Bigg(\int\limits_{B_\delta(y_0)} \big|\nabla(\psi_{y_0}w_\varepsilon^{\frac{\gamma+1}{2}})\big|^2\Bigg)^{\frac12}\\ &\qquad\le \frac{4}{(\gamma+1)^2}\int\limits_{B_\delta(y_0)} \big|\nabla\psi_{y_0}\big|^2 w_\varepsilon^{\gamma+1} + \eta^{\frac{2}{N}}\Bigg(\int\limits_{B_\delta(y_0)} \psi_{y_0}^{2^*} w_\varepsilon^{\frac{N}{N-2}(\gamma+1)}\Bigg)^{\frac{N-2}{2}}.\end{aligned}$$

Therefore, there exists a constant C such that

$$\int_{B_\delta(y_0)} \left|\nabla(\psi_{y_0} w_\varepsilon^{\frac{\gamma+1}{2}})\right|^2 \le C\left[\int_{B_\delta(y_0)} w_\varepsilon^{\gamma+1} + \eta^{\frac{2}{N}}\left(\int_{B_\delta(y_0)} \psi_{y_0}^{2^*} w_\varepsilon^{\frac{N}{N-2}(\gamma+1)}\right)^{\frac{N-2}{2}}\right].$$

Still assuming that the integrals are well defined, $\psi_{y_0} w_\varepsilon^{\frac{\gamma+1}{2}} \in W_0^{1,2}(B_\delta(y_0))$, and as $W_0^{1,2}(B_\delta(y_0))$ embeds continuously into $L^{2^*}(B_\delta(y_0))$, we have

$$\left(\int_{B_\delta(y_0)} \psi_{y_0}^{2^*} w_\varepsilon^{\frac{N}{N-2}(\gamma+1)}\right)^{\frac{N-2}{2}} \le \frac{1}{S}\int_{B_\delta(y_0)} \left|\nabla(\psi_{y_0} w_\varepsilon^{\frac{\gamma+1}{2}})\right|^2$$

where S is a strictly positive constant (which depends only on N). We see that, choosing η small enough, there exists a constant C', depending only on δ and N, such that

$$\left(\int_{B_\delta(y_0)} \psi_{y_0}^{2^*} w_\varepsilon^{\frac{N}{N-2}(\gamma+1)}\right)^{\frac{N-2}{2}} \le C'\int_{B_\delta(y_0)} w_\varepsilon^{\gamma+1}. \tag{2.12}$$

With $\gamma = 2^* - 1$, we obtain

$$\left(\int_{B_{\delta'}(y_0)} w_\varepsilon^{\frac{N}{N-2}2^*}\right)^{\frac{N-2}{2}} \le C'\int_{B_\delta(y_0)} w_\varepsilon^{2^*}. \tag{2.13}$$

The right-hand-side integral is well defined, which proves that the left-hand-side one is also well defined. (Actually, to make the previous argument perfectly rigorous, we should have multiplied (2.11) by $\psi_{y_0}^2 w_{\varepsilon,n}^\gamma$, where, for $n \in \mathbb{N}^*$, $w_{\varepsilon,n}(y) = w_\varepsilon(y)$ if $w_\varepsilon(y) \le n$, $w_{\varepsilon,n}(y) = n$ otherwise. Then, all the integrals in the previous computations are well defined and, letting n go to infinity, we obtain (2.13).)

The right-hand side of (2.13) goes to zero as ε goes to zero, as (2.9) proves. As a consequence, w_ε goes to zero in $L^{\frac{N}{N-2}2^*}(B_{\delta'}(y_0))$, uniformly with respect to y_0, and thus in $L^{\frac{N}{N-2}2^*}(B_{3/(2R)}(0))$. Iterating the process, we find through (2.12) that w_ε goes to zero in every $L^p(B_{3/(2R)}(0))$, $p < \infty$. This implies, taking into account (2.7), that a_ε goes to zero in $L^q(B_{3/(2R)}(0))$ for some $q > N/2$ (actually, for any $q < \infty$). Then, standard theory for elliptic equations (see, *e.g.*, Theorem 8.24 in [9]) ensures that w_ε goes to zero in $L^\infty(B_{1/R}(0))$. Therefore, $g_\varepsilon(y, u_\varepsilon) = u_\varepsilon^{2^*-1-\varepsilon}$ in $R^N \setminus \Omega$ and u_ε is a solution to (1.1).

3. Parametrization of the variational problem

In this section, we shall still consider the general case $N \ge 3$. Let $k \in \mathbb{N}^*$. We set

$$\mathcal{D}_\varepsilon = \left\{(\Lambda, X) \in (\mathbb{R}_+^*)^k \times (\partial\Omega)^k \text{ s.t. } f_1(\varepsilon) < \lambda_i < f_2(\varepsilon) \text{ and } |x_i - x_j| > h(\varepsilon), 1 \le i, j \le k, i \ne j\right\} \tag{3.1}$$

where $\Lambda = (\lambda_1, \dots, \lambda_n)$, $X = (x_1, \dots, x_n)$ and f_1, f_2, h are positive functions such that $f_1(\varepsilon)$ and $f_2(\varepsilon)$ go to infinity, $h(\varepsilon)$ goes to zero as ε goes to zero, and whose precise expression will be determined later. We define also, for $(\Lambda, X) \in (\mathbb{R}_+^*)^k \times (\partial\Omega)^k$

$$E_{\Lambda,X} = \left\{v \in D^{1,2}(R^N \setminus \Omega) \text{ s.t. } \langle v, U_i\rangle = \left\langle v, \frac{\partial U_i}{\partial \lambda_i}\right\rangle = \left\langle v, \frac{\partial U_i}{\partial \tau_{i,j}}\right\rangle = 0,\ \ 1 \le i \le k, 1 \le j \le N-1\right\} \tag{3.2}$$

where $U_i = U_{\lambda_i, x_i}$, and $(\tau_{i,1}, \dots, \tau_{i,j})$ is an orthogonal system of coordinates of the tangent space to $\partial\Omega$ at x_i. Lastly, for $\delta > 0$, we define

$$\begin{aligned}\mathcal{M}_{\varepsilon,\delta} = \Big\{&(A, \Lambda, X, v) \in \mathbb{R}^k \times (\mathbb{R}_+^*)^k \times (\partial\Omega)^k \times D^{1,2}(R^N \setminus \Omega)\\ &\text{s.t. } |\alpha_i - 1| < \delta, 1 \le i \le k\,; (\Lambda, X) \in \mathcal{D}_\varepsilon\,; v \in E_{\Lambda,X}, \|v\| < \delta\Big\}\end{aligned} \tag{3.3}$$

where $A = (\alpha_1, \dots, \alpha_k)$, and we consider on $\mathcal{M}_{\varepsilon,\delta}$ the functional

$$J_\varepsilon(A, \Lambda, X, v) = \widetilde{I}_\varepsilon\left(\sum_{i=1}^k \alpha_i U_i + v\right). \tag{3.4}$$

According to [3,4,10], we know that, for ε and δ small enough, (A, Λ, X, v) is a critical point of J_ε in $\mathcal{M}_{\varepsilon,\delta}$ if and only if $u = \sum_{i=1}^k \alpha_i U_i + v$ is a critical point of $\widetilde{I_\varepsilon}$ in $D^{1,2}(R^N \setminus \Omega)$. Let us remark that, in consideration of (3.2), (A, Λ, X, v) is a critical point of J_ε in $\mathcal{M}_{\varepsilon,\delta}$ if and only if there exists Lagrange multipliers A_i, B_i, C_{ij}, $1 \le i \le k$, $1 \le j \le N-1$ such that

$$\frac{\partial J_\varepsilon}{\partial \alpha_i} = 0 \tag{3.5}$$

$$\frac{\partial J_\varepsilon}{\partial \lambda_i} = B_i \Big\langle \frac{\partial^2 U_i}{\partial \lambda_i^2}, v \Big\rangle + \sum_{\ell=1}^{N-1} C_{i\ell} \Big\langle \frac{\partial^2 U_i}{\partial \lambda_i \partial \tau_{ij}}, v \Big\rangle \tag{3.6}$$

$$\frac{\partial J_\varepsilon}{\partial \tau_{ij}} = B_i \Big\langle \frac{\partial^2 U_i}{\partial \lambda_i \partial \tau_{ij}}, v \Big\rangle + \sum_{\ell=1}^{N-1} C_{i\ell} \Big\langle \frac{\partial^2 U_i}{\partial \tau_{ij} \partial \tau_{i\ell}}, v \Big\rangle \tag{3.7}$$

$$\frac{\partial J_\varepsilon}{\partial v} = \sum_{i=1}^k A_i U_i + \sum_{i=1}^k B_i \frac{\partial U_i}{\partial \lambda_i} + \sum_{i=1}^k \sum_{\ell=1}^{N-1} C_{i\ell} \frac{\partial U_i}{\partial \tau_{i\ell}}. \tag{3.8}$$

In order to find critical points of J_ε in $\mathcal{M}_{\varepsilon,\delta}$ we shall proceed in two steps. Firstly we shall eliminate the non-significant parameters: the α_i's and v. More precisely, we shall prove the existence of a C^1-map that with every $(\Lambda, X) \in \mathcal{D}_\varepsilon$ associates $A_\varepsilon(\Lambda, X) \in \mathbb{R}^k$, each $\alpha_{\varepsilon,i}$ close to 1, and $v_\varepsilon \in E_{\Lambda,X}$, $\|v_\varepsilon\|$ close to zero, such that (3.5) and (3.8) are satisfied. It will be left to us, in a second time, to show, using a topological argument, that the function $(\Lambda, X) \mapsto J_\varepsilon\big(A_\varepsilon(\Lambda, X), \Lambda, X, v_\varepsilon(\Lambda, X)\big)$ has a critical point in $\mathcal{D}_\varepsilon$.

4. The reduced functional

We first perform a change of variables concerning the parameters α_i's, setting

$$A' = A - \mathbf{1} \tag{4.1}$$

where $\mathbf{1} = (1, \ldots, 1)$, that is $A' = (\alpha'_1, \ldots, \alpha'_k)$, with

$$\alpha'_i = \alpha_i - 1, \quad 1 \le i \le k. \tag{4.2}$$

We define also, on $\mathbb{R}^k \times D^{1,2}(R^N \setminus \Omega)$, the scalar product

$$\ll (A, f), (B, g) \gg = \sum_{i=1}^k a_i b_i + \int_{R^N \setminus \Omega} \nabla f \cdot \nabla g \tag{4.3}$$

and $|||\cdot|||$ will denote the associated norm. Let us expand $J_\varepsilon(A, \Lambda, X, v)$ with respect to $w = (A', v)$ in a neighborhood of $(\mathbf{1}, 0)$ in $\mathbb{R}^k \times E_{\Lambda,X}$. We can write, using Riesz's representation theorem

$$J_\varepsilon(A, \Lambda, X, v) = J_\varepsilon(\mathbf{1}, \Lambda, X, 0) + \ll f_{\varepsilon,\Lambda,X}, w \gg + \frac{1}{2} \ll Q_{\varepsilon,\Lambda,X} w, w \gg + R_{\varepsilon,\Lambda,X}(w) \tag{4.4}$$

where $f_{\varepsilon,\Lambda,X} \in \mathbb{R}^k \times E_{\Lambda,X}$ is such that

$$\ll f_{\varepsilon,\Lambda,X}, w \gg = \sum_{i=1}^k \Big[\Big\langle \sum_{j=1}^k U_j, U_i \Big\rangle - \int_{R^N \setminus \Omega} U_i g_\varepsilon\big(y, \sum_{j=1}^k U_j\big) \Big] \alpha'_i - \int_{R^N \setminus \Omega} g_\varepsilon\big(y, \sum_{j=1}^k U_j\big) v \tag{4.5}$$

$Q_{\varepsilon,\Lambda,X}$ is an endomorphism of $\mathbb{R}^k \times E_{\Lambda,X}$ such that

$$\begin{aligned} \ll Q_{\varepsilon,\Lambda,X} w, w \gg \; = & \sum_{i,j=1}^k \Big[\langle U_i, U_j \rangle - \int_{R^N \setminus \Omega} U_i U_j \frac{\partial g_\varepsilon}{\partial t}\big(y, \sum_{\ell=1}^k U_\ell\big) \Big] \alpha'_i \alpha'_j \\ & - \sum_{i=1}^k \Big(\int_{R^3 \setminus \Omega} \frac{\partial g_\varepsilon}{\partial t}\big(y, \sum_{\ell=1}^k U_\ell\big) U_i v \Big) \alpha'_i + \|v\|^2 - \int_{R^N \setminus \Omega} \frac{\partial g_\varepsilon}{\partial t}\big(y, \sum_{\ell=1}^k U_\ell\big) v^2 \end{aligned} \tag{4.6}$$

and $R_{\varepsilon,\Lambda,X}$ satisfies

$$R^{(m)}_{\varepsilon,\Lambda,X}(w) = O\big(\|w\|^{\min(2^*-\varepsilon,3)-m}\big), \quad m = 0, 1, 2. \tag{4.7}$$

(In the special case $N=3$, $2^*=6$ and $R^{(m)}_{\varepsilon,\Lambda,X}(w)=O(\|w\|^{3-m})$, $m=0,1,2$.) The fact that equations (3.5) and (3.8) are satisfied is equivalent to

$$f_{\varepsilon,\Lambda,X}+Q_{\varepsilon,\Lambda,X}w+R'_{\varepsilon,\Lambda,X}=0. \tag{4.8}$$

In the special case $N=3$, we choose in the definition (3.1) of $\mathcal{D}_\varepsilon$ (for reasons that will become clear later)

$$f_1(\varepsilon)=\frac{a_1}{\varepsilon}\ln\frac{1}{\varepsilon},\quad f_2(\varepsilon)=\frac{a_2}{\varepsilon}\ln\tfrac{1}{\varepsilon},\quad h(\varepsilon)=b\big(\ln\frac{1}{\varepsilon}\big)^{-3/4} \tag{4.9}$$

where a_1, a_2, b, with $a_1<a_2$, are strictly positive constants that will be determined later. (For $N\ge 4$, following [15], we would choose $f_1(\varepsilon)=a_1/\varepsilon$, $f_2(\varepsilon)=a_2/\varepsilon$, $h(\varepsilon)=\varepsilon^{1-\frac{1+\theta}{N-2}}$ with θ a small positive number.) Then it is proved in Appendix B.1 that

$$|||f_{\varepsilon,\Lambda,X}|||=O\Big(\varepsilon^{1/2}\big(\ln\frac{1}{\varepsilon}\big)^{-1/2}\Big). \tag{4.10}$$

Moreover, it is proved in Appendix B.2 that $Q_{\varepsilon,\Lambda,X}$ is invertible and there exists ρ, independent of ε and $(\Lambda,X)\in\mathcal{D}_\varepsilon$, such that

$$\Big|\Big|\Big|Q^{-1}_{\varepsilon,\Lambda,X}\Big|\Big|\Big|\le\rho. \tag{4.11}$$

Let us consider now the map $F_{\varepsilon,\Lambda,X}$, from a neighborhood of $(0,0)$ in $(\mathbb{R}^k\times E_{\Lambda,X})^2$ to $\mathbb{R}^k\times E_{\Lambda,X}$ defined by

$$F_{\varepsilon,\Lambda,X}(f,w)=f+Q_{\varepsilon,\Lambda,X}w+R'_{\varepsilon,\Lambda,X}(w).$$

We have

$$F_{\varepsilon,\Lambda,X}(0,0)=0\quad\text{and}\quad\frac{\partial F_{\varepsilon,\Lambda,X}}{\partial w}=Q_{\varepsilon,\Lambda,X}+R''_{\varepsilon,\Lambda,X}(w).$$

From (4.11) and (4.7), we deduce the existence of $\eta>0$ such that for ε small enough and $|||w|||<\eta$, $\frac{\partial F_{\varepsilon,\Lambda,X}}{\partial w}$ is invertible. Then, taking into account (4.10), the implicit function theorem allows us to state the following proposition.

Proposition 4.1. *There exists $\varepsilon_0>0$ such that for every $\varepsilon\in(0,\varepsilon_0)$, a C^1-map exists, which with every $(\Lambda,X)\in\mathcal{D}_\varepsilon$ associates $w_{\varepsilon,\Lambda,X}=(A'_{\varepsilon,\Lambda,X},v_{\varepsilon,\Lambda,X})\in\mathbb{R}^k\times E_{\Lambda,X}$ so that $F_{\varepsilon,\Lambda,X}(f_{\varepsilon,\Lambda,X},w_{\varepsilon,\Lambda,X})=0$. This means that (3.5) and (3.8) are satisfied when the α^i's and v are such that $A=\mathbf{1}+A'_{\varepsilon,\Lambda,X}$ and $v=v_{\varepsilon,\Lambda,X}$. Moreover, when $N=3$,*

$$|\alpha'_{i;\varepsilon,\Lambda,X}|=|\alpha_{i;\varepsilon,\Lambda,X}-1|=O\big(\varepsilon^{1/2}\big(\ln\tfrac{1}{\varepsilon}\big)^{-1/2}\big),\quad 1\le i\le k, \tag{4.12}$$

$$\|v_{\varepsilon,\Lambda,X}\|=O\big(\varepsilon^{1/2}\big(\ln\tfrac{1}{\varepsilon}\big)^{-1/2}\big) \tag{4.13}$$

as ε goes to zero.

Remark. Actually, estimate (4.12) could be improved: we could prove that $|\alpha'_i|=O(\varepsilon\ln\frac{1}{\varepsilon})$. However, (4.12) is sufficient for our purposes.

We consider now the reduced functional

$$\widetilde{J}_\varepsilon(\Lambda,X)=J_\varepsilon\big(\alpha_\varepsilon(\Lambda,X),\Lambda,X,v_\varepsilon(\Lambda,X)\big) \tag{4.14}$$

in $\mathcal{D}_\varepsilon$. For $\widetilde{J}_\varepsilon$ and its derivatives with respect to the λ_i's, the following expansion holds.

Proposition 4.2. *Let $N=3$. For $\varepsilon\in(0,\varepsilon_0)$ and $(\Lambda,X)\in\mathcal{D}_\varepsilon$, we have:*

$$\widetilde{J}_\varepsilon(\Lambda,X)=kK_{1,\varepsilon}+\sum_{i=1}^k\Big(K_2H(x_i)\frac{\ln\lambda_i}{\lambda_i}+K_3\varepsilon\ln\lambda_i\Big)-K_4\sum_{\substack{i,j=1\\ i\ne j}}^k\frac{1}{\lambda_i^{1/2}\lambda_j^{1/2}|x_i-x_j|}+O\big(\varepsilon\big(\ln\tfrac{1}{\varepsilon}\big)^{-1}\big), \tag{4.15}$$

$$\frac{\partial\widetilde{J}_\varepsilon}{\partial\lambda_i}(\Lambda,X)=-K_2H(x_i)\frac{\ln\lambda_i}{\lambda_i^2}+K_3\frac{\varepsilon}{\lambda_i}+O\big(\varepsilon^2\big(\ln\tfrac{1}{\varepsilon}\big)^{-5/4}\big) \tag{4.16}$$

where $K_{\varepsilon,1}$, K_2, K_3, K_4 are strictly positive constants.

This proposition is proved in Appendix A. We are now able to prove Theorem 1.2.

5. Proof of Theorem 1.2

For the sake of simplicity, we always assume now on that $N = 3$. We shall concentrate our attention on part (1) of Theorem 1.2 (the proof of part (2) will then be straightforward). We look for a critical point $(\Lambda, X) \in \mathcal{D}_\varepsilon$ of $\widetilde{J}_\varepsilon$. The x_i's will be supposed to belong to the subset S which, according to the assumptions of Theorem 1.2, is such that

$$0 < \max_{y\in\partial S} H(y) < \max_{y\in S} H(y) = H_m.$$

More precisely, the x_i's will be supposed to converge, as ε goes to zero, to points $\bar{x}_i$ that satisfy $H(\bar{x}_i) = H_m$. Then, according to (4.16), the λ_i's should be close to $\lambda(\varepsilon)$ defined by

$$\frac{\ln\lambda(\varepsilon)}{\lambda(\varepsilon)} = \frac{K_3\varepsilon}{K_2 H_m}. \tag{5.1}$$

A simple computation yields the expansions

$$\lambda(\varepsilon) = \frac{K_2H_m}{K_3\varepsilon}\ln\frac{1}{\varepsilon} + \frac{K_2H_m}{K_3\varepsilon}\ln\ln\frac{1}{\varepsilon} + \mathrm{o}\Big(\frac{1}{\varepsilon}\Big), \tag{5.2}$$

$$\ln\lambda(\varepsilon) = \ln\frac{1}{\varepsilon} + \ln\ln\frac{1}{\varepsilon} + \mathrm{O}(1). \tag{5.3}$$

Now, we are able to fix a_1 and a_2 in the definition (4.9) of the functions f_1 and f_2 that occur in (3.1): we choose a_1 and a_2 such that

$$0 < a_1 < \frac{K_2H_m}{K_3} < a_2. \tag{5.4}$$

(For example, in view of (A.23), we may take $a_1 = 15H_m/\pi$ and $a_2 = 17H_m/\pi$.)

In order to prove the theorem, we are going to argue by contradiction. The general strategy is the following: we define two levels, $c_{\varepsilon,1}$ and $c_{\varepsilon,2}$, $c_{\varepsilon,1} < c_{\varepsilon,2}$, and assume that $\widetilde{J}_\varepsilon$ has no critical value between them. Then we use the gradient flow to deform the level sets of $\widetilde{J}_\varepsilon$ corresponding to $c_{\varepsilon,2}$ and $c_{\varepsilon,1}$ one into the other. The difference of topology between the two level sets provides us with a contradiction.

Let us make the argument precise. Firstly, we define the two levels $c_{\varepsilon,1}$ and $c_{\varepsilon,2}$ as follows. In view of (4.15), we set

$$c_{1,\varepsilon} = kK_{\varepsilon,1} + \sum_{i=1}^{k}\left(K_2H_m\frac{\ln\lambda(\varepsilon)}{\lambda(\varepsilon)} + K_3\varepsilon\ln\lambda(\varepsilon)\right) - \varepsilon\Big(\ln\frac{1}{\varepsilon}\Big)^{-1/4}. \tag{5.5}$$

Taking into account (5.1), we can also write

$$c_{1,\varepsilon} = kK_{\varepsilon,1} + kK_3\varepsilon\big(\ln\lambda(\varepsilon) + 1\big) - \varepsilon\Big(\ln\frac{1}{\varepsilon}\Big)^{-1/4}. \tag{5.6}$$

Concerning $c_{2,\varepsilon}$, we just set $c_{2,\varepsilon} = kK_{\varepsilon,1} + \tau$ where τ is some small positive constant. It is clear that for ε small enough, $c_{2,\varepsilon} > c_{1,\varepsilon}$. Next, we have to define the subset of $\mathcal{D}_\varepsilon$ on which the deformation argument will be implemented. We set

$$S' = \left\{x \in S \ \text{ s.t. } \ H(x) > H_m - a_0\Big(\ln\frac{1}{\varepsilon}\Big)^{-1/4}\right\} \tag{5.7}$$

where a_0 is a strictly positive constant to be determined later,

$$M = \left\{X \in (\partial\Omega)^k \ \text{ s.t. } \ x_i \in S' \ \text{ and } \ |x_i - x_j| > b\Big(\ln\frac{1}{\varepsilon}\Big)^{-3/4}, \ \ 1 \le i, j \le k, i \ne j\right\} \tag{5.8}$$

(b, which already occurs in the definition (4.9), will be determined later), and T is the interval

$$T = \left]\left(1 - D\Big(\ln\frac{1}{\varepsilon}\Big)^{-1/4}\right)\lambda(\varepsilon), \left(1 + D\Big(\ln\frac{1}{\varepsilon}\Big)^{-1/4}\right)\lambda(\varepsilon)\right[\tag{5.9}$$

where D is a large constant that will be chosen later. We note that for ε small enough, every $\lambda \in T$ satisfies $\frac{a_1}{\varepsilon}\ln\frac{1}{\varepsilon} < \lambda < \frac{a_2}{\varepsilon}\ln\frac{1}{\varepsilon}$, and $T^k \times M \subset \mathcal{D}_\varepsilon$. We note also that for ε small enough, $T^k \times M \subset \widetilde{J}_\varepsilon^{\,c_{\varepsilon,2}}$, where $\widetilde{J}_\varepsilon^{\,\omega}$, $\omega \in \mathbb{R}$, is the level set $\{(\Lambda, X) \in \mathcal{D}_\varepsilon \ \text{ s.t. } \ \widetilde{J}_\varepsilon(\Lambda, X) < \omega\}$.

We consider now the gradient flow

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\big(\Lambda(t), X(t)\big) = -\nabla\widetilde{J}_\varepsilon\big(\Lambda(t), X(t)\big) \\ \big(\Lambda(0), X(0)\big) \in (T^k \times M) \end{cases} \tag{5.10}$$

Arguing by contradiction, we assume:

(H) $\widetilde{J}_\varepsilon$ has no critical value between $c_{\varepsilon,1}$ and $c_{\varepsilon,2}$ in $T^k \times M$.

The following holds.

Lemma 5.1. *The flow line $\big(\Lambda(t), X(t)\big)$ defined by (5.8) and starting from a point of $T^k \times M$ does not leave $T^k \times M$ before it reaches $\widetilde{J}_\varepsilon^{\,c_{\varepsilon,1}}$.*

Proof. We have to check that, for any $(\Lambda, X) \in \partial(T^k \times M)$, either $-\nabla \widetilde{J}_\varepsilon$ points inwards, or $\widetilde{J}_\varepsilon$ is less than $c_{\varepsilon,1}$.

Case 1. $X \in \partial M$.

Case 1a. There is some j such that $H(x_j) = H_m - a_0\big(\ln \frac{1}{\varepsilon}\big)^{-1/4}$.

According to (4.15), we have

$$\widetilde{J}_\varepsilon(\Lambda, X) \le kK_{1,\varepsilon} + \sum_{i=1}^{k}\Big(K_2 H(x_i)\frac{\ln\lambda_i}{\lambda_i} + K_3\varepsilon\ln\lambda_i\Big) + K_2\big(H(x_j) - H_m\big)\frac{\ln\lambda_j}{\lambda_j} + \mathrm{O}\Big(\varepsilon\big(\ln\tfrac{1}{\varepsilon}\big)^{-1}\Big). \tag{5.11}$$

When $\lambda_i = \lambda(\varepsilon)(1+\zeta_i)$, ζ_i small, a simple computation yields

$$K_2 H_m \frac{\ln\lambda_i}{\lambda_i} + K_3\varepsilon\ln\lambda_i = K_2 H_m\frac{\ln\lambda(\varepsilon)}{\lambda(\varepsilon)} + K_3\varepsilon\ln\lambda(\varepsilon) + \mathrm{O}\big(\tfrac{\zeta_i}{\lambda(\varepsilon)} + \varepsilon\zeta_i^2\big)$$

and, if $\zeta_i = \mathrm{O}\Big(\big(\ln\frac{1}{\varepsilon}\big)^{-1/4}\Big)$, as it is the case when $\lambda_i \in T$,

$$\tfrac{\zeta_i}{\lambda(\varepsilon)} + \varepsilon\zeta_i^2 = \mathrm{O}\Big(\varepsilon\big(\ln\tfrac{1}{\varepsilon}\big)^{-1/2}\Big).$$

Moreover, we have, using (5.9) and (5.1) (5.2),

$$\frac{\ln\lambda_j}{\lambda_j} = \frac{\ln\lambda(\varepsilon)}{\lambda(\varepsilon)} + \mathrm{O}\Big(\zeta_j\frac{\ln\lambda(\varepsilon)}{\lambda(\varepsilon)}\Big) = \frac{K_3\varepsilon}{K_2 H_m} + \mathrm{O}\Big(\varepsilon\big(\ln\tfrac{1}{\varepsilon}\big)^{-1/4}\Big).$$

Therefore, taking into account (5.5),

$$\widetilde{J}_\varepsilon(\Lambda, X) \le c_{\varepsilon,1} + \varepsilon\big(\ln\tfrac{1}{\varepsilon}\big)^{-1/4} - a_0\frac{K_3}{H_m}\varepsilon\big(\ln\tfrac{1}{\varepsilon}\big)^{-1/4} + \mathrm{O}\Big(\varepsilon\big(\ln\tfrac{1}{\varepsilon}\big)^{-1/2}\Big)$$

so that $\widetilde{J}_\varepsilon(\Lambda, X) < c_{\varepsilon,1}$ for ε small enough, provided that $a_0 > \frac{H_m}{K_3}$. For example, we set in (5.7)

$$a_0 = 2\frac{H_m}{K_3} \tag{5.12}$$

(or, in view of (A.23), $a_0 = \frac{64}{\sqrt{3}\pi^2}H_m$).

Case 1b. There is some i, j, $i \neq j$, such that $|x_i - x_j| = b\big(\ln\frac{1}{\varepsilon}\big)^{-3/4}$.

According to (4.15) and using the previous computations, we have in that case

$$\widetilde{J}_\varepsilon(\Lambda, X) \le c_{\varepsilon,1} + \varepsilon\big(\ln\tfrac{1}{\varepsilon}\big)^{-1/4} - \frac{K_4}{b\lambda_i^{1/2}\lambda_j^{1/2}}\varepsilon\big(\ln\tfrac{1}{\varepsilon}\big)^{3/4} + \mathrm{O}\Big(\varepsilon\big(\ln\tfrac{1}{\varepsilon}\big)^{-1/2}\Big).$$

As we have also, from (5.9)

$$\frac{1}{\lambda_i^{1/2}\lambda_j^{1/2}} = \frac{1}{\lambda(\varepsilon)}\Big(1 + \mathrm{O}\Big(\big(\ln\tfrac{1}{\varepsilon}\big)^{-1/4}\Big)\Big)$$

and from (5.2)

$$\frac{1}{\lambda(\varepsilon)} = \frac{K_3\varepsilon}{K_2 H_m}\left[\frac{1}{\ln\frac{1}{\varepsilon}} + \mathrm{O}\Big(\frac{\ln\ln\frac{1}{\varepsilon}}{(\ln\frac{1}{\varepsilon})^2}\Big)\right] \tag{5.13}$$

we obtain

$$\widetilde{J}_\varepsilon(\Lambda, X) \le c_{\varepsilon,1} + \varepsilon\big(\ln\tfrac{1}{\varepsilon}\big)^{-1/4} - \frac{K_3K_4}{bK_2H_m}\varepsilon\big(\ln\tfrac{1}{\varepsilon}\big)^{-1/4} + \mathrm{o}\Big(\varepsilon\big(\ln\tfrac{1}{\varepsilon}\big)^{-1/4}\Big).$$

Therefore, $\widetilde{J}_\varepsilon(\Lambda, X) < c_{\varepsilon,1}$ for ε small enough, provided that $b < \frac{K_3K_4}{K_2H_m}$. For example, we set in (4.9) and (5.8)

$$b = \frac{K_3K_4}{2K_2H_m} \tag{5.14}$$

(or, in view of (A.23), $b = \frac{\sqrt{3}\pi^2}{32H_m}$).

Case 2. $\Lambda \in \partial T$.

Case 2a. There is some j such that $\lambda_j = \Big(1 + D\big(\ln\frac{1}{\varepsilon}\big)^{-1/4}\Big)\lambda(\varepsilon)$.

In view of (4.16), we compute, using the fact that $H(x_i) \le H_m$, (5.1)–(5.3) and (5.13)

$$\begin{aligned}-K_2H(x_i)\frac{\ln\lambda_j}{\lambda_j^2} + K_3\frac{\varepsilon}{\lambda_j} &\ge -K_2H_m\frac{\ln\lambda_j}{\lambda_j^2} + K_3\frac{\varepsilon}{\lambda_j}\\ &\ge \frac{K_3^2}{K_2H_m}D\varepsilon^2\big(\ln\frac{1}{\varepsilon}\big)^{-5/4} + \mathrm{o}\Big(\varepsilon^2\big(\ln\tfrac{1}{\varepsilon}\big)^{-5/4}\Big).\end{aligned}$$

From (4.16), we know that there exists a constant C, independent of D, such that

$$\frac{\partial\widetilde{J}_\varepsilon}{\partial\lambda_i}(\Lambda, X) \ge -K_2H(x_i)\frac{\ln\lambda_i}{\lambda_i^2} + K_3\frac{\varepsilon}{\lambda_i} - C\big(\varepsilon^2\big(\ln\tfrac{1}{\varepsilon}\big)^{-5/4}\big).$$

Therefore, if D is chosen large enough, so that $\frac{K_3^2}{K_2H_m}D > C$, the inequality implies $\frac{\partial\widetilde{J}_\varepsilon}{\partial\lambda_j}(\Lambda, X) > 0$, so that $-\nabla\widetilde{J}_\varepsilon(\Lambda, X)$ is directed toward the interior of $T^k \times M$.

Case 2b. There is some j such that $\lambda_j = \Big(1 - D\big(\ln\frac{1}{\varepsilon}\big)^{-1/4}\Big)\lambda(\varepsilon)$.

According to (4.16), (5.7) and (5.12), we have

$$\frac{\partial\widetilde{J}_\varepsilon}{\partial\lambda_j}(\Lambda, X) \le -K_2H_m\Big(1 - \frac{2}{K_3}\big(\ln\frac{1}{\varepsilon}\big)^{-1/4}\Big)\frac{\ln\lambda_j}{\lambda_j^2} + K_3\frac{\varepsilon}{\lambda_j} + \mathrm{O}\Big(\varepsilon^2\big(\ln\tfrac{1}{\varepsilon}\big)^{-5/4}\Big)$$

and the same kind of computations as in the previous subcase yield

$$\frac{\partial\widetilde{J}_\varepsilon}{\partial\lambda_j}(\Lambda, X) \le -\frac{K_3^2}{K_2H_m}\big(D - \frac{2}{K_3}\big)\varepsilon^2\big(\ln\frac{1}{\varepsilon}\big)^{-5/4} + \mathrm{O}\Big(\varepsilon^2\big(\ln\tfrac{1}{\varepsilon}\big)^{-5/4}\Big)$$

where once again the last term denotes a quantity whose absolute value is less than $C\varepsilon^2\big(\ln\frac{1}{\varepsilon}\big)^{-5/4}$ for some constant C independent of D. Consequently, if D is chosen large enough, $\frac{\partial\widetilde{J}_\varepsilon}{\partial\lambda_j}(\Lambda, X) < 0$, so that $-\nabla\widetilde{J}_\varepsilon(\Lambda, X)$ is directed toward the interior of $T^k \times M$. $\square$

We can now complete the proof of Theorem 1.2. We define $S'' \subset S'$ and $M' \subset M$ as

$$S'' = \Big\{x \in S \text{ s.t. } H(x) > H_m - \frac{H_m}{4kK_3}\big(\ln\frac{1}{\varepsilon}\big)^{-1/4}\Big\} \tag{5.15}$$

and

$$M' = \Big\{X \in (\partial\Omega)^k \text{ s.t. } x_i \in S'' \text{ and } |x_i - x_j| > 4k(k-1)\frac{K_3K_4}{K_2H_m}\big(\ln\frac{1}{\varepsilon}\big)^{-3/4}, 1 \le i, j \le k, i \ne j\Big\}. \tag{5.16}$$

We claim that, for ε small enough,

$$\forall(\Lambda, X) \in T^k \times M' \quad \widetilde{J}_\varepsilon(\Lambda, X) > c_{\varepsilon,1}. \tag{5.17}$$

Let us assume that (5.17) is true. For $X \in M'$, we take $(\Lambda(\varepsilon), X)$ as an initial value in (5.10) – with $\Lambda(\varepsilon) = \big(\lambda(\varepsilon), \dots, \lambda(\varepsilon)\big)$. Lemma 5.1 and (5.17) imply that the flow line has to meet $\partial M'$ before $\big(\Lambda(t), X(t)\big)$ reaches $\widetilde{J}_\varepsilon^{\,c_{\varepsilon,1}}$. Consequently, the flow, projected onto the X-variable, provides us with a deformation of M' onto $\partial M'$. However, M' is topologically different from $\partial M'$ – see Proposition B1 in [8] – hence a contradiction. Consequently, assumption (H) is not true – that is, $\widetilde{J}_\varepsilon$ has a critical point in $T^k \times M$, which provides us with a solution to (1.1) satisfying the statements of Theorem 1.2 (1).

It only remains to prove (5.17). Let $(\Lambda, X) \in T^k \times M'$. According to (4.15), we have

$$\widetilde{J}_\varepsilon(\Lambda, X) = kK_{1,\varepsilon} + \sum_{i=1}^{k} \Big(K_2 H_m \frac{\ln \lambda_i}{\lambda_i} + K_3 \varepsilon \ln \lambda_i \Big) - \sum_{i=1}^{k} K_2 \big(H_m - H(x_i) \big) \frac{\ln \lambda_i}{\lambda_i}$$
$$- K_4 \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \frac{1}{\lambda_i^{1/2} \lambda_j^{1/2} |x_i - x_j|} + \mathrm{O}\Big(\varepsilon \big(\ln \tfrac{1}{\varepsilon} \big)^{-1} \Big).$$

We remark that, for $\lambda_i \in T$,

$$\frac{\ln \lambda_i}{\lambda_i} = \frac{K_3 \varepsilon}{K_2 H_m} + \mathrm{O}\Big(\varepsilon \big(\ln \tfrac{1}{\varepsilon} \big)^{-1/4} \Big)$$

so that, using (5.15),

$$\sum_{i=1}^{k} K_2 \big(H_m - H(x_i) \big) \frac{\ln \lambda_i}{\lambda_i} \leq \frac{\varepsilon}{4} \big(\ln \frac{1}{\varepsilon} \big)^{-1/4} + \mathrm{O}\Big(\varepsilon \big(\ln \tfrac{1}{\varepsilon} \big)^{-1/2} \Big).$$

We have also, using (5.16) and (5.2),

$$K_4 \sum_{\substack{i,j=1 \\ i \neq j}}^{k} \frac{1}{\lambda_i^{1/2} \lambda_j^{1/2} |x_i - x_j|} \leq \frac{\varepsilon}{4} \big(\ln \frac{1}{\varepsilon} \big)^{-1/4} + \mathrm{O}\Big(\varepsilon \big(\ln \tfrac{1}{\varepsilon} \big)^{-5/4} \ln \ln \tfrac{1}{\varepsilon} \Big).$$

Lastly, we remark that, in view of (5.1) and (5.9),

$$K_2 H_m \frac{\ln \lambda_i}{\lambda_i} + K_3 \varepsilon \ln \lambda_i = K_2 H_m \frac{\ln \lambda(\varepsilon)}{\lambda(\varepsilon)} + K_3 \varepsilon \ln \lambda(\varepsilon) + \mathrm{O}\Big(\varepsilon \big(\ln \tfrac{1}{\varepsilon} \big)^{-1/4} \Big).$$

Then, taking into account definition (5.5) of $c_{\varepsilon,1}$, we obtain

$$\widetilde{J}_\varepsilon(\Lambda, X) \geq c_{\varepsilon,1} + \frac{\varepsilon}{2} \big(\ln \frac{1}{\varepsilon} \big)^{-1/4} + \mathrm{O}\Big(\varepsilon \big(\ln \tfrac{1}{\varepsilon} \big)^{-1/2} \Big)$$

and (5.17) follows. This ends the proof of the first part of Theorem 1.2.

The proof of the second part is straightforward. Indeed, in that case, the only thing we have to do is to minimize $\widetilde{J}_\varepsilon(\lambda, x)$ in $T \times S'$. One can easily deduce from the previous computations that such a minimum cannot lie on the boundary of $T \times S'$ – whence the existence of a critical point of $\widetilde{J}_\varepsilon$ that provides us with the desired solution to (1.1).

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Appendix A

We begin this appendix by a number of integral estimates, which will be useful in establishing Proposition 4.2.

A.1. Integral estimates

We recall that U_i denotes the function U_{λ_i, x_i} defined by (1.2), where x_i is assumed to belong to the boundary of Ω. In the first place, we state some results concerning integrals involving only U_i and its derivatives with respect to λ_i and x_i.

Lemma A.1. *As $\lambda_i \to \infty$, we have*

$$\|U_i\|^2 = \frac{3\sqrt{3}\pi^2}{8} + \sqrt{3}\pi H(x_i) \frac{\ln \lambda_i}{\lambda_i} + \mathrm{O}\Big(\frac{1}{\lambda_i} \Big), \tag{A.1}$$

$$\int_{R^3 \setminus \Omega} U_i^6 = \frac{3\sqrt{3}\pi^2}{8} + \frac{3\sqrt{3}\pi}{4} \frac{H(x_i)}{\lambda_i} + \mathrm{O}\Big(\frac{1}{\lambda_i^2} \Big), \tag{A.2}$$

$$\int_{R^3\setminus\Omega} U_i^{6-\varepsilon} = \frac{3\sqrt{3}\pi^2}{8} - \frac{3\sqrt{3}\pi^2}{16}\varepsilon\ln\lambda_i + \frac{3\sqrt{3}\pi^2}{32}(4\ln 2 - 1 - \ln 3)\varepsilon$$
$$+ \frac{3\sqrt{3}\pi^2}{4}\frac{H(x_i)}{\lambda_i} + O\Big(\varepsilon^2\ln^2\lambda_i + \frac{1}{\lambda_i^2}\Big) \text{ as } \varepsilon\ln\lambda_i \to 0, \tag{A.3}$$

$$\langle U_i, \frac{\partial U_i}{\partial\lambda_i}\rangle = -\frac{\sqrt{3}\pi}{2}H(x_i)\frac{\ln\lambda_i}{\lambda_i^2} + O\Big(\frac{1}{\lambda_i^2}\Big), \tag{A.4}$$

$$\langle U_i, \frac{\partial U_i}{\partial\tau_{i\ell}}\rangle = -\frac{\sqrt{3}\pi}{2}\frac{\partial H}{\partial\tau_{i\ell}}(x_i)\frac{\ln\lambda_i}{\lambda_i} + O\Big(\frac{1}{\lambda_i}\Big), \tag{A.5}$$

$$\Big\|\frac{\partial U_i}{\partial\lambda_i}\Big\|^2 = \frac{15\sqrt{3}\pi^2}{128\lambda_i^2} + O\Big(\frac{\ln\lambda_i}{\lambda_i^3}\Big), \tag{A.6}$$

$$\langle\frac{\partial U_i}{\partial\lambda_i}, \frac{\partial U_i}{\partial\tau_{i\ell}}\rangle = O\Big(\frac{\ln\lambda_i}{\lambda_i^2}\Big), \tag{A.7}$$

$$\langle\frac{\partial U_i}{\partial\tau_{i\ell}}, \frac{\partial U_i}{\partial\tau_{im}}\rangle = \frac{15\sqrt{3}\pi^2}{128}\lambda_i^2\delta_{\ell m} + O\Big(\lambda_i\ln\lambda_i\Big). \tag{A.8}$$

Proof. (A.1) and (A.2) are proved in Appendix C of [12]. Let us prove (A.3). Outside of $B_\tau(x_i)$, where $\tau > 0$ is some fixed number, $U_i(y)$ behaves as $\lambda_i^{-1/2}|y-x_i|^{-1}$, whence

$$\int_{(R^3\setminus\Omega)\cap B_\tau(x_i)^c} U_i^{6-\varepsilon} = O\Big(\frac{1}{\lambda_i^3}\Big).$$

In $B_\tau(x_i)$, we can write

$$U_i^{-\varepsilon}(y) = \exp\big(-\varepsilon\ln U_i(y)\big) = 1 - \frac{\varepsilon}{2}\ln\lambda_i + \frac{\varepsilon}{2}\ln\big(1+\lambda_i^2|y-x_i|^2\big) - \frac{\varepsilon}{4}\ln 3 + O\big(\varepsilon^2(\ln\lambda_i)^2\big).$$

On the one hand,

$$\int_{(R^3\setminus\Omega)\cap B_\tau(x_i)} U_i^6 = \int_{R^3\setminus\Omega} U_i^6 + O\Big(\frac{1}{\lambda_i^3}\Big)$$

and the integral of U_i^6 over $R^3\setminus\Omega$ is given by (A.2). On the other hand,

$$\int_{(R^3\setminus\Omega)\cap B_\tau(x_i)} U_i^6\ln\big(1+\lambda_i^2|y-x_i|^2\big) = \int_{R^3\setminus\Omega} U_i^6\ln\big(1+\lambda_i^2|y-x_i|^2\big) + O\Big(\frac{1}{\lambda_i^3}\Big)$$

and similarly to (A.2)

$$\int_{R^3\setminus\Omega} U_i^6\ln\big(1+\lambda_i^2|y-x_i|^2\big) = \frac{1}{2}\int_{\mathbb{R}^3} U_i^6\ln\big(1+\lambda_i^2|y-x_i|^2\big) + O\Big(\frac{1}{\lambda_i}\Big).$$

Lastly, using the residue theorem, we compute

$$\int_{\mathbb{R}^3} U_i^6\ln\big(1+\lambda_i^2|y-x_i|^2\big) = \frac{3\sqrt{3}\pi^2}{8}(4\ln 2 - 1).$$

Gathering these results, we obtain (A.3).

Estimates (A.4)–(A.8) follow from Appendix D of [11] – although in that paper N is assumed to be larger than 5, the computations carried out therein extend straightforwardly to the case $N = 3$, with the minor needed modifications. □

In the next lemma, we collect the estimates relative to integrals involving both U_i and U_j, $i \neq j$. Following [3], we set

$$\varepsilon_{ij} = \Big(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|x_i - x_j|^2\Big)^{-1/2}. \tag{A.9}$$

Lemma A.2. *Let $i \neq j$. As $\lambda_i, \lambda_j \to \infty$ and $\varepsilon_{ij} \to 0$, we have*

$$\int_{R^3\setminus\Omega} U_i^5 U_j = 2\sqrt{3}\pi\varepsilon_{ij} + \mathrm{O}\Big(\frac{\varepsilon_{ij}}{\lambda_i^{1/2}} + \varepsilon_{ij}^3\Big) \text{ as } \varepsilon_{ij} \to 0, \tag{A.10}$$

$$\int_{\partial\Omega} \frac{\partial U_i}{\partial \nu} U_j = \mathrm{O}\Big(\frac{\big|\ln|x_i - x_j|\big|}{\lambda_i^{1/2}\lambda_j^{1/2}}\Big) \text{ as } \lambda_i|x_i - x_j|, \lambda_j|x_i - x_j| \to \infty, \tag{A.11}$$

$$\langle U_i, U_j\rangle = 2\sqrt{3}\pi\varepsilon_{ij} + \mathrm{O}\Big(\frac{\big|\ln|x_i - x_j|\big|}{\lambda_i^{1/2}\lambda_j^{1/2}} + \frac{\varepsilon_{ij}}{\lambda_i^{1/2}} + \varepsilon_{ij}^3\Big), \quad \text{as } \lambda_i|x_i - x_j|, \lambda_j|x_i - x_j| \to \infty, \tag{A.12}$$

$$\Big\langle U_i, \frac{\partial U_j}{\partial \lambda_j}\Big\rangle = \mathrm{O}\Big(\frac{\varepsilon_{ij}}{\lambda_j} + \frac{\big|\ln|x_i - x_j|\big|}{\lambda_i^{1/2}\lambda_j^{3/2}}\Big), \tag{A.13}$$

$$\Big\langle U_i, \frac{\partial U_j}{\partial \tau_{j\ell}}\Big\rangle = \mathrm{O}\Big(\lambda_j\varepsilon_{ij} + \frac{\lambda_j^{1/2}}{\lambda_i^{1/2}}\big|\ln|x_i - x_j|\big|\Big), \tag{A.14}$$

$$\Big\langle \frac{\partial U_i}{\partial \lambda_i}, \frac{\partial U_j}{\partial \lambda_j}\Big\rangle = \mathrm{O}\Big(\frac{\varepsilon_{ij}}{\lambda_i\lambda_j} + \frac{\big|\ln|x_i - x_j|\big|}{\lambda_i^{3/2}\lambda_j^{3/2}}\Big), \tag{A.15}$$

$$\Big\langle \frac{\partial U_i}{\partial \lambda_i}, \frac{\partial U_j}{\partial \tau_{j\ell}}\Big\rangle = \mathrm{O}\Big(\frac{\lambda_j\varepsilon_{ij}}{\lambda_i} + \frac{\lambda_j^{1/2}}{\lambda_i^{3/2}}\big|\ln|x_i - x_j|\big|\Big), \tag{A.16}$$

$$\Big\langle \frac{\partial U_i}{\partial \tau_{i\ell}}, \frac{\partial U_j}{\partial \tau_{jm}}\Big\rangle = \mathrm{O}\Big(\lambda_i\lambda_j\varepsilon_{ij} + \lambda_i^{1/2}\lambda_j^{1/2}\big|\ln|x_i - x_j|\big|\Big). \tag{A.17}$$

Proof. We know, from formula (E.1) in [3], that for $N \geq 3$

$$\int_{\mathbb{R}^N} U_i^{\frac{N+2}{N-2}} U_j = \big(N(N-2)\big)^{N/2} C_N \varepsilon_{ij} + \mathrm{O}\Big(\varepsilon_{ij}^{\frac{N}{N-2}}\Big)$$

with

$$C_N = \int_{\mathbb{R}^N} \frac{\mathrm{d}x}{\big(1 + |x|^2\big)^{\frac{N+2}{2}}} = \frac{\sigma_{N-1}}{N}.$$

Therefore, for $N = 3$,

$$\int_{\mathbb{R}^3} U_i^5 U_j = 4\sqrt{3}\pi\varepsilon_{ij} + \mathrm{O}\big(\varepsilon_{ij}^3\big).$$

It is easily checked, through a rescaling, that the integral of $U_i^5 U_j$ over $R^3 \setminus \Omega$ is equal to half of the integral over the whole space, to within a small amount which can be estimated proceeding as in Appendix C of [12]. (A.10) follows.

We turn to (A.11). Up to a translation and a rotation of the coordinates in $\mathbb{R}^3$, we can assume that $x_i = 0$, and that for $\tau > 0$ small enough

$$\Omega \cap B_\tau(0) = \Big\{ y = (y', y_3) \in \mathbb{R}^2 \times \mathbb{R} \text{ s.t. } |y| < R, y_3 > f(y') \Big\} \tag{A.18}$$

where f is a smooth function such that $f(0) = 0$, $f'(0) = 0$. We have, for $y \in \partial\Omega \cap B_\tau(0)$,

$$\frac{\partial U_i}{\partial \nu}(y) = \sum_{\ell=1,2} \frac{\partial U_i}{\partial x_{i\ell}}(y)\frac{\partial f}{\partial y_\ell}(y') - \frac{\partial U_i}{\partial x_{i3}} = \mathrm{O}\Big(\frac{\lambda_i^{5/2}|y'|^2}{\big(1 + \lambda_i^2|y'|^2\big)^{3/2}}\Big).$$

Let us assume first that $d = |x_i - x_j| > \frac{\tau}{2}$. In $B_\tau(0)$, $U_j = \mathrm{O}\big(\lambda_j^{-1/2}\big)$, so that

$$\int_{\partial\Omega\cap B_\tau(0)} \frac{\partial U_i}{\partial \nu} U_j = \mathrm{O}\Big(\frac{1}{\lambda_j^{1/2}} \int_0^\tau \frac{\lambda_i^{5/2} r^3 \mathrm{d}r}{(1 + \lambda_i^2 r^2)^{3/2}}\Big) = \mathrm{O}\Big(\frac{1}{\lambda_i^{1/2}\lambda_j^{1/2}}\Big).$$

Outside of $B_\tau(0)$, $\frac{\partial U_i}{\partial \nu} = O(\lambda_i^{-1/2})$, so that

$$\int\limits_{\partial\Omega\setminus B_\tau(0)} \frac{\partial U_i}{\partial \nu} U_j = O\Big(\frac{1}{\lambda_i^{1/2}} \int\limits_{\partial\Omega\setminus B_\tau(0)} \frac{\lambda_j^{1/2}}{(1+\lambda_j^2|y-x_j|^2)^{1/2}} \mathrm{d}y\Big)$$
$$= O\Big(\frac{1}{\lambda_i^{1/2}\lambda_j^{1/2}} \int\limits_{\partial\Omega\setminus B_\tau(0)} \frac{\mathrm{d}y}{|y-x_j|}\Big) = O\Big(\frac{1}{\lambda_i^{1/2}\lambda_j^{1/2}}\Big).$$

Let us assume now that $d = |x_i - x_j|$ goes to zero. In $B_{d/2}(0)$, $U_j = O(\lambda_j^{-1/2} d^{-1})$, so that

$$\int\limits_{\partial\Omega\cap B_{d/2}(0)} \frac{\partial U_i}{\partial \nu} U_j = O\Big(\frac{1}{\lambda_j^{1/2} d} \int_0^{d/2} \frac{\lambda_i^{5/2} r^3 \mathrm{d}r}{(1+\lambda_i^2 r^2)^{3/2}}\Big) = O\Big(\frac{1}{\lambda_i^{1/2}\lambda_j^{1/2}}\Big).$$

In the same way, as $\frac{\partial U_i}{\partial \nu} = O(\lambda_i^{-1/2} d^{-1})$ in $\partial\Omega \cap B_{d/2}(x_j)$, we have

$$\int\limits_{\partial\Omega\cap B_{d/2}(x_j)} \frac{\partial U_i}{\partial \nu} U_j = O\Big(\frac{1}{\lambda_i^{1/2}\lambda_j^{1/2}} \int\limits_{\partial\Omega\cap B_{d/2}(x_j)} \frac{\mathrm{d}y}{|y-x_j|}\Big) = O\Big(\frac{1}{\lambda_i^{1/2}\lambda_j^{1/2}}\Big)$$

and the same estimate holds for the integral over $\partial\Omega \setminus B_\tau(0)$, since outside of $B_\tau(0)$, $\frac{\partial U_i}{\partial \nu} = O(\lambda_i^{-1/2})$. Finally we notice that in $\omega = \big(\partial\Omega \cap B_\tau(0)\big) \setminus \big(B_{d/2}(0) \cup B_{d/2}(x_j)\big)$, $|y - x_j| \le \frac{1}{3}|y|$, and

$$\frac{\partial U_i}{\partial \nu} U_j = O\Big(\frac{\lambda_i^{5/2}|y'|^2}{\big(1+\lambda_i^2|y'|^2\big)^{3/2}} \frac{\lambda_i^{1/2}}{\big(1+\lambda_j^2|y'|^2\big)^{1/2}}\Big) = O\Big(\frac{1}{\lambda_i^{1/2}\lambda_j^{1/2}|y'|^2}\Big).$$

Consequently,

$$\int\limits_{\omega} \frac{\partial U_i}{\partial \nu} U_j = O\Big(\frac{1}{\lambda_i^{1/2}\lambda_j^{1/2}} \int_{d/2}^{\tau} \frac{\mathrm{d}r}{r}\Big) = O\Big(\frac{|\ln d|}{\lambda_i^{1/2}\lambda_j^{1/2}}\Big)$$

and (A.11) is proved.

(A.12) follows from (A.10) and (A.11), since

$$\langle U_i, U_j\rangle = \int\limits_{R^3\setminus\Omega} U_i^5 U_j - \int\limits_{\partial\Omega} \frac{\partial U_i}{\partial \nu} U_j$$

(with ν the outward normal to Ω). (A.13)–(A.17) follow from computations quite similar to those above (some of these estimates could be improved, but they are sufficient for our purposes). □

We are now able to prove Proposition 4.2.

A.2. Proof of Proposition 4.2

From now on, in accordance with the hypotheses of Proposition 4.2, we assume that $\varepsilon \in (0, \varepsilon_0)$ and $(\Lambda, X) \in \mathcal{D}_\varepsilon$. This means, in view of (3.1) and (4.9),

$$\frac{a_1}{\varepsilon} \ln \frac{1}{\varepsilon} < \lambda_i < \frac{a_2}{\varepsilon} \ln \frac{1}{\varepsilon}, 1 \le i \le k\,; \quad |x_i - x_j| > b\Big(\ln \frac{1}{\varepsilon}\Big)^{-3/4}. \tag{A.19}$$

We notice that, in that case, ε_{ij} defined by (A.9) is such that

$$\varepsilon_{i,j} = \frac{1}{\lambda_i^{1/2}\lambda_j^{1/2}|x_i - x_j|} + O\Big(\varepsilon^3\big(\ln\tfrac{1}{\varepsilon}\big)^{-3/4}\Big); \quad \varepsilon_{i,j} = O\Big(\varepsilon\big(\ln\tfrac{1}{\varepsilon}\big)^{-1/4}\Big). \tag{A.20}$$

Proof of (4.15)**.** From (4.14) and Proposition 4.1, we have

$$\widetilde{J}_\varepsilon(\Lambda, X) = J_\varepsilon(\mathbf{1}, \Lambda, X, 0) + O\Big(\varepsilon\big(\ln\tfrac{1}{\varepsilon}\big)^{-1}\Big).$$

Next we have

$$J_\varepsilon(\mathbf{1},\Lambda,X,0)=\frac{1}{2}\int\limits_{R^3\setminus\Omega}\Big|\nabla\big(\sum_{i=1}^{k}U_i\big)\Big|^2-\frac{1}{6-\varepsilon}\int\limits_{R^3\setminus\Omega}\big(\sum_{i=1}^{k}U_i\big)^{6-\varepsilon}$$

since, in accordance with the definition of g_ε, $g_\varepsilon\big(\sum_{i=1}^k U_i\big)=\big(\sum_{i=1}^k U_i\big)^{5-\varepsilon}$ provided that ε is small enough (implying, through (A.19), that the λ_i's are large enough). On the one hand,

$$\int\limits_{R^3\setminus\Omega}\Big|\nabla\big(\sum_{i=1}^{k}U_i\big)\Big|^2=\sum_{i=1}^{k}\int\limits_{R^3\setminus\Omega}|\nabla U_i|^2+\sum_{\substack{1\le i,j\le k\\ i\ne j}}\int\limits_{R^3\setminus\Omega}\nabla U_i.\nabla U_j$$

– quantities that are estimated by (A.1) and (A.12). On the other hand,

$$\int\limits_{R^3\setminus\Omega}\big(\sum_{i=1}^{k}U_i\big)^{6-\varepsilon}=\sum_{i=1}^{k}\int\limits_{R^3\setminus\Omega}U_i^{6-\varepsilon}+(6-\varepsilon)\sum_{\substack{1\le i,j\le k\\ i\ne j}}\int\limits_{R^3\setminus\Omega}U_i^{5-\varepsilon}U_j+\mathrm{O}\Big(\sum_{\substack{1\le i,j\le k\\ i\ne j}}\int\limits_{R^3\setminus\Omega}U_i^{4-\varepsilon}U_j^2\Big).\tag{A.21}$$

The first integral on the right-hand side is estimated by (A.3). Concerning the second one, we remark that $U_i^{-\varepsilon}=1+\mathrm{O}(\varepsilon\ln\lambda_i)$ as $\varepsilon\ln\lambda_i$ goes to zero. Therefore, (A.19) yields $U_i^{-\varepsilon}=1+\mathrm{O}\big(\varepsilon\big(\ln\frac{1}{\varepsilon}\big)^{-1}\big)$, and

$$\int\limits_{R^3\setminus\Omega}U_i^{5-\varepsilon}U_j=\Big(1+\mathrm{O}\Big(\varepsilon\big(\ln\tfrac{1}{\varepsilon}\big)^{-1}\Big)\Big)\int\limits_{R^3\setminus\Omega}U_i^5U_j$$

the last integral being estimated by (A.10). Lastly, we know from formula (E.3) in [3] that, for $N\ge 3$, $i\ne j$,

$$\int\limits_{\mathbb{R}^N}U_i^aU_j^b=\Big(\varepsilon_{ij}\big(\ln\tfrac{1}{\varepsilon_{ij}}\big)^{\frac{N-2}{N}}\Big)^{\min(a,b)},\quad a,b>1;\,a+b=\frac{2N}{N-2}\tag{A.22}$$

as $\varepsilon_{i,j}$ goes to zero. In our case, with $N=3$, $a=4$, $b=2$, we obtain, using (A.19),

$$\int\limits_{R^3\setminus\Omega}U_i^{4-\varepsilon}U_j^2=\mathrm{O}\Big(\int\limits_{\mathbb{R}^3}U_i^4U_j^2\Big)=\mathrm{O}\Big(\varepsilon^2\big(\ln\tfrac{1}{\varepsilon}\big)^{1/2}\Big).$$

Gathering these results, we obtain (4.15), with

$$\begin{cases}K_{\varepsilon,1}=\dfrac{\sqrt{3}\pi^2}{8}\Big(1-\big(\dfrac{1}{12}+\dfrac{1}{8}(4\ln 2-1-\ln 3)\big)\varepsilon\Big)\\ K_2=\dfrac{\sqrt{3}\pi}{2},\quad K_3=\dfrac{\sqrt{3}\pi^2}{32},\quad K_4=\sqrt{3}\pi.\quad\square\end{cases}\tag{A.23}$$

Proof of (4.16). As $\widetilde{J}_\varepsilon(\Lambda,X)=J_\varepsilon\big(A_\varepsilon(\Lambda,X),\Lambda,X,v_\varepsilon(\Lambda,X)\big)$, and $\frac{\partial J_\varepsilon}{\partial\alpha_i}=0$ for all j,

$$\frac{\partial\widetilde{J}_\varepsilon}{\partial\lambda_i}(\Lambda,X)=\frac{\partial J_\varepsilon}{\partial\lambda_i}(\Lambda,X)+\langle\frac{\partial J_\varepsilon}{\partial v},\frac{\partial v_\varepsilon}{\partial\lambda_i}\rangle.\tag{A.24}$$

Let us first compute $\frac{\partial J_\varepsilon}{\partial\lambda_i}(\Lambda,X)$. According to (2.4) and (3.4), we have

$$\frac{1}{\alpha_i}\frac{\partial J_\varepsilon}{\partial\lambda_i}(\Lambda,X)=\int\limits_{R^3\setminus\Omega}\nabla\big(\sum_{j=1}^{k}\alpha_jU_j\big).\nabla\frac{\partial U_i}{\partial\lambda_i}-\int\limits_{R^3\setminus\Omega}g_\varepsilon\big(y,\sum_{j=1}^{k}\alpha_jU_j+v_\varepsilon\big)\frac{\partial U_i}{\partial\lambda_i}.$$

The first integral on the right-hand side is estimated by (A.4) and (A.13). Concerning the second one, we remark that, according to the definition of g_ε,

$$\int\limits_{(R^3\setminus\Omega)\cap B_R(0)} g_\varepsilon\big(y,\sum_{j=1}^k \alpha_j U_j + v_\varepsilon\big)\frac{\partial U_i}{\partial\lambda_i}$$

$$= \int\limits_{(R^3\setminus\Omega)\cap B_R(0)} \Big(\big(\sum_{j=1}^k \alpha_j U_j + v_\varepsilon\big)^+\Big)^{5-\varepsilon}\frac{\partial U_i}{\partial\lambda_i}$$

$$= \int\limits_{(R^3\setminus\Omega)\cap B_R(0)} \Big[\big(\sum_{j=1}^k \alpha_j U_j\big)^{5-\varepsilon} + (5-\varepsilon)\big(\sum_{j=1}^k \alpha_j U_j\big)^{4-\varepsilon} v_\varepsilon + \mathrm{O}\Big(\big(\sum_{j=1}^k \alpha_j U_j\big)^3 v_\varepsilon^2 + |v_\varepsilon|^{5-\varepsilon}\Big)\Big]\frac{\partial U_i}{\partial\lambda_i}.$$

We notice that $\frac{\partial U_i}{\partial\lambda_i} = \mathrm{O}\big(\frac{1}{\lambda_i}U_i\big)$. Then we have

$$\int\limits_{(R^3\setminus\Omega)\cap B_R(0)} \big(\sum_{j=1}^k \alpha_j U_j\big)^{5-\varepsilon}\frac{\partial U_i}{\partial\lambda_i} = \alpha_i^{5-\varepsilon}\int\limits_{(R^3\setminus\Omega)\cap B_R(0)} U_i^{5-\varepsilon}\frac{\partial U_i}{\partial\lambda_i} + \mathrm{O}\Big(\frac{1}{\lambda_i}\sum_{\substack{1\le i,j\le k\\ i\ne j}}\int\limits_{R^3\setminus\Omega}(U_i^5 U_j + U_j^5 U_i)\Big) \tag{A.25}$$

(we recall that $\varepsilon\ln\lambda_j \to 0$ and $U_j^{-\varepsilon} = \mathrm{O}(1)$ for all j). The first integral on the right-hand side may be estimated exactly in the same way as (A.3), and we obtain

$$\int\limits_{(R^3\setminus\Omega)\cap B_R(0)} U_i^{5-\varepsilon}\frac{\partial U_i}{\partial\lambda_i} = -\frac{\sqrt{3}\pi^2}{32}\frac{\varepsilon}{\lambda_i} + \mathrm{O}\Big(\frac{1}{\lambda_i^2} + \varepsilon^2\frac{\ln^2\lambda_i}{\lambda_i}\Big).$$

The remaining terms are estimated through (A.10). Next, we write

$$\big(\sum_{j=1}^k \alpha_j U_j\big)^{4-\varepsilon} = \alpha_i^{4-\varepsilon}U_i^{4-\varepsilon} + \mathrm{O}\Big(\varepsilon\lambda_i U_i^4 + \big(\sum_{\substack{1\le j\le k\\ j\ne i}} U_i^3 U_j + U_j^4\big)\Big)$$

and, as $-\Delta\frac{\partial U_i}{\partial\lambda_i} = 5U_i^4\frac{\partial U_i}{\partial\lambda_i}$ in $\mathbb{R}^3$, and $\big\langle\frac{\partial U_i}{\partial\lambda_i}, v_\varepsilon\big\rangle = 0$

$$\int\limits_{(R^3\setminus\Omega)\cap B_R(0)} U_i^4\frac{\partial U_i}{\partial\lambda_i}v_\varepsilon = \int\limits_{R^3\setminus\Omega} U_i^4\frac{\partial U_i}{\partial\lambda_i}v_\varepsilon + \mathrm{O}\Big(\frac{1}{\lambda_i}\int\limits_{B_R^c(0)} U_i^5|v_\varepsilon|\Big) = -\frac{1}{5}\int\limits_{\partial\Omega}\frac{\partial^2 U_i}{\partial\lambda_i\partial\nu}v_\varepsilon + \mathrm{O}\Big(\frac{\|v\|}{\lambda_i^{7/2}}\Big).$$

Since $D^{1,2}(R^3\setminus\Omega)$ embeds into $L^4(\partial\Omega)$,

$$\int\limits_{\partial\Omega}\frac{\partial^2 U_i}{\partial\lambda_i\partial\nu}v_\varepsilon = \mathrm{O}\Big(\Big(\int\limits_{\partial\Omega}\Big|\frac{\partial^2 U_i}{\partial\lambda_i\partial\nu}\Big|^{4/3}\Big)^{3/4}\|v_\varepsilon\|\Big).$$

Proceeding as in the proof of Lemma A.2, we remark that outside of $B_\tau(x_i)$, $\frac{\partial^2 U_i}{\partial\lambda_i\partial\nu} = \mathrm{o}\big(\frac{1}{\lambda_i^{3/2}}\big)$, and in $B_\tau(x_i)$, with the notation (A.18),

$$\frac{\partial^2 U_i}{\partial\lambda_i\partial\nu}(x) = \mathrm{O}\Big(\frac{\lambda_i^{3/2}|x'|^2}{(1+\lambda_i^2|x'|^2)^{3/2}}\Big).$$

Then we compute

$$\int\limits_{\partial\Omega\cap B_\tau(x_i)}\Big|\frac{\partial^2 U_i}{\partial\lambda_i\partial\nu}\Big|^{4/3} = \mathrm{O}\Big(\lambda_i^2\int_0^\tau\frac{r^{\frac{8}{3}+1}}{(1+\lambda_i^2 r^2)^2}\mathrm{d}r\Big) = \mathrm{O}\Big(\frac{1}{\lambda_i^2}\Big)$$

so that

$$\int\limits_{\partial\Omega}\frac{\partial^2 U_i}{\partial\lambda_i\partial\nu}v_\varepsilon = \mathrm{O}\Big(\frac{\|v_\varepsilon\|}{\lambda_i^{3/2}}\Big).$$

We have also

$$\varepsilon \ln\lambda_i \int_{R^3\setminus\Omega} U_i^4\Big|\frac{\partial U_i}{\partial\lambda_i}\Big||v_\varepsilon| = \mathrm{O}\Big(\varepsilon\frac{\ln\lambda_i}{\lambda_i}\|v_\varepsilon\|\Big)$$

and, for $j\neq i$,

$$\int_{R^3\setminus\Omega}(U_i^3U_j+U_j^4)\Big|\frac{\partial U_i}{\partial\lambda_i}\Big||v_\varepsilon| = \mathrm{O}\Big(\frac{1}{\lambda_i}\int_{R^3\setminus\Omega}(U_i^4U_j+U_iU_j^4)|v_\varepsilon|\Big)$$
$$= \mathrm{O}\Big(\frac{\|v_\varepsilon\|}{\lambda_i}\int_{R^3\setminus\Omega}(U_i^{20/5}U_j^{6/5}+U_i^{6/5}U_j^{20/5})|v_\varepsilon|\Big).$$

This last quantity may be estimated through (A.22). Lastly

$$\int_{(R^3\setminus\Omega)\cap B_R(0)}\Big(\big(\sum_{j=1}^k\alpha_jU_j\big)^3v_\varepsilon^2+|v_\varepsilon|^{5-\varepsilon}\Big)\frac{\partial U_i}{\partial\lambda_i} = \mathrm{O}\Big(\frac{\|v_\varepsilon\|^2}{\lambda_i}\Big).$$

Finally, in $B_{R+1}(0)\setminus B_R(0)$, we use the fact that $|g_\varepsilon(y,u)|\le|u|^{5-\varepsilon}$ and $U_j=\mathrm{O}\big(\frac{1}{\lambda_i^{1/2}}\big)$ for all j, and in $B^c_{R+1}(0)$, we use (2.5) and, taking into account (4.12) (4.13) and (A.19) (A.20), we obtain

$$\frac{\partial J_\varepsilon}{\partial\lambda_i}(A_\varepsilon,\Lambda,X,v_\varepsilon) = -\frac{\sqrt{3}\pi}{2}H(x_i)\frac{\ln\lambda_i}{\lambda_i^2}+\frac{\sqrt{3}\pi^2}{32}\frac{\varepsilon}{\lambda_i}+\mathrm{O}\Big(\varepsilon^2\big(\ln\tfrac{1}{\varepsilon}\big)^{-5/4}\Big). \tag{A.26}$$

Comparing this expansion with (4.16), it appears, in view of (A.24), that it only remains to prove that $\big\langle\frac{\partial J_\varepsilon}{\partial v},\frac{\partial v_\varepsilon}{\partial\lambda_i}\big\rangle = \mathrm{O}\Big(\varepsilon^2\big(\ln\frac{1}{\varepsilon}\big)^{-5/4}\Big)$. According to (3.8)

$$\Big\langle\frac{\partial J_\varepsilon}{\partial v},\frac{\partial v_\varepsilon}{\partial\lambda_i}\Big\rangle = \sum_{j=1}^k\Big\langle A_jU_j+B_j\frac{\partial U_j}{\partial\lambda_j}+\sum_{\ell=1,2}C_{j\ell}\frac{\partial U_j}{\partial\tau_{j\ell}},\frac{\partial v_\varepsilon}{\partial\lambda_i}\Big\rangle$$
$$= -B_i\Big\langle\frac{\partial^2U_i}{\partial\lambda_i^2},v_\varepsilon\Big\rangle-\sum_{\ell=1,2}C_{i\ell}\Big\langle\frac{\partial^2U_i}{\partial\lambda_i\partial\tau_{i\ell}},v_\varepsilon\Big\rangle$$

since $\langle U_j,v_\varepsilon\rangle = \big\langle\frac{\partial U_j}{\partial\lambda_j},v_\varepsilon\big\rangle = \big\langle\frac{\partial U_j}{\partial\tau_{j\ell}},v_\varepsilon\big\rangle = 0$ for all j and ℓ. On the one hand,

$$\Big\langle\frac{\partial^2U_i}{\partial\lambda_i^2},v_\varepsilon\Big\rangle = \mathrm{O}\Big(\frac{\|v_\varepsilon\|}{\lambda_i^2}\Big);\quad \Big\langle\frac{\partial^2U_i}{\partial\lambda_i\partial\tau_{i\ell}},v_\varepsilon\Big\rangle = \mathrm{O}\big(\|v_\varepsilon\|\big). \tag{A.27}$$

On the other hand, the multipliers A_i, B_i, $C_{i\ell}$ that occur in (3.5)–(3.8) can easily be estimated. Namely, let us take the scaler product of (3.8) with U_i, $\frac{\partial U_i}{\partial\lambda_i}$, $\frac{\partial U_i}{\partial\tau_{i\ell}}$, $1\le i\le k$, $\ell=1,2$, respectively. On one side, we find

$$\Big\langle\frac{\partial J_\varepsilon}{\partial v},U_i\Big\rangle = \frac{\partial J_\varepsilon}{\partial\alpha_i}=0\,;\quad \Big\langle\frac{\partial J_\varepsilon}{\partial v},\frac{\partial U_i}{\partial\lambda_i}\Big\rangle = \frac{1}{\alpha_i}\frac{\partial J_\varepsilon}{\partial\lambda_i}\,;\quad \Big\langle\frac{\partial J_\varepsilon}{\partial v},\frac{\partial U_i}{\partial\tau_{i\ell}}\Big\rangle = \frac{1}{\alpha_i}\frac{\partial J_\varepsilon}{\partial\tau_{i\ell}}.$$

$\frac{\partial J_\varepsilon}{\partial\lambda_i}$ is given by (A.26), and computations quite similar to those establishing (A.26) show that $\frac{\partial J_\varepsilon}{\partial\tau_{i\ell}}=\mathrm{O}(1)$. On the other side, we have linear equations involving the A_i's, B_i's, $C_{i\ell}$'s, whose coefficients are given by estimates (A.1), (A.4)–(A.8), and (A.12)–(A.17). Such a linear system is quasi-diagonal and invertible, and provides us with the estimates

$$B_i=\mathrm{O}\big(\ln\tfrac{1}{\varepsilon}\big)\,;\quad C_{i\ell}=\mathrm{O}\Big(\varepsilon^2\big(\ln\tfrac{1}{\varepsilon}\big)^{-2}\Big).$$

Then, we deduce from (A.27), (4.13), and (A.19) that

$$\Big\langle\frac{\partial J_\varepsilon}{\partial v},\frac{\partial v_\varepsilon}{\partial\lambda_i}\Big\rangle = \mathrm{O}\Big(\varepsilon^{5/2}\big(\ln\tfrac{1}{\varepsilon}\big)^{-3/2}\Big)$$

and the proof of (4.16) is complete. □

Appendix B

B.1. Proof of (4.10)

Let $f = f_{\varepsilon,\Lambda,X}$ be defined by (4.5), that is, for $w = (A', v) \in \mathbb{R}^k \times E_{\Lambda,X}$, $A' = (\alpha'_1, \dots, \alpha'_k)$,

$$\ll f, w \gg = \sum_{i=1}^{k}\Big[\big\langle \sum_{j=1}^{k} U_j, U_i\big\rangle - \int_{R^3\setminus\Omega} U_i g_\varepsilon\big(y, \sum_{j=1}^{k} U_j\big)\Big]\alpha'_i - \int_{R^3\setminus\Omega} g_\varepsilon\big(y, \sum_{j=1}^{k} U_j\big) v.$$

First, we remark that, according to the definition of g_ε, $g_\varepsilon\big(y, \sum_j U_j\big) = \big(\sum_j U_j\big)^{5-\varepsilon}$ everywhere, provided that ε is small enough, so that

$$\begin{aligned}\big\langle \sum_{j=1}^{k} U_j, U_i\big\rangle - \int_{R^3\setminus\Omega} U_i g_\varepsilon\big(y, \sum_{j=1}^{k} U_j\big) &= \sum_{j=1}^{k}\langle U_j, U_i\rangle - \int_{R^3\setminus\Omega} \big(\sum_{j=1}^{k} U_j\big)^{5-\varepsilon} U_i \\ &= \sum_{j=1}^{k}\Big[\langle U_j, U_i\rangle - \int_{R^3\setminus\Omega} U_j^5 U_i\Big] + O\Big(\varepsilon \ln \lambda_i + \sum_{\substack{1\le i,j\le k\\ i\ne j}} \int_{R^3\setminus\Omega} (U_j^5 + U_j U_i^4) U_i\Big)\end{aligned}$$

and, using (A.1) (A.2), (A.10), (A.12) and (A.19) (A.20), we obtain

$$\big\langle \sum_{j=1}^{k} U_j, U_i\big\rangle - \int_{R^3\setminus\Omega} U_i g_\varepsilon\big(y, \sum_{j=1}^{k} U_j\big) = O\Big(\varepsilon \ln \tfrac{1}{\varepsilon}\Big).$$

We turn now to the integral involving v. We have

$$\begin{aligned}\int_{R^3\setminus\Omega} g_\varepsilon\big(y, \sum_{j=1}^{k} U_j\big) v &= \int_{R^3\setminus\Omega} \big(\sum_{j=1}^{k} U_j\big)^{5-\varepsilon} v \\ &= \int_{R^3\setminus\Omega} \big(\sum_{j=1}^{k} U_j\big)^{5} v + O\Big(\varepsilon \ln(\max_j \lambda_j) \|v\|\Big)\end{aligned}$$

since $\big(\sum_j U_j\big)^{-\varepsilon} = 1 + O\big(\varepsilon \ln(\max_j \lambda_j)\big)$ and

$$\begin{aligned}\int_{R^3\setminus\Omega} \big(\sum_{j=1}^{k} U_j\big)^{5} |v| &\le k^4 \sum_{j=1}^{k} \int_{R^3\setminus\Omega} U_j^5 |v| \\ &\le k^4 \sum_{j=1}^{k} \Big(\int_{R^3\setminus\Omega} U_j^5\Big)^{\frac{5}{6}} \Big(\int_{R^3\setminus\Omega} v^6\Big)^{\frac{1}{6}} \le C\|v\|\end{aligned}$$

where C is a constant. Next, we have

$$\int_{R^3\setminus\Omega} \big(\sum_{j=1}^{k} U_j\big)^{5} v = \sum_{j=1}^{k} \int_{R^3\setminus\Omega} U_j^5 v + O\Big(\sum_{\substack{1\le i,j\le k\\ i\ne j}} \int_{R^3\setminus\Omega} U_j^4 U_i |v|\Big)$$

and

$$\int_{R^3\setminus\Omega} U_j^4 U_i |v| \le \Big(\int_{\mathbb{R}^3} U_j^{\frac{24}{5}} U_i^{\frac{6}{5}}\Big)^{\frac{5}{6}} \|v\|$$

with, according to (A.22) and (A.20)

$$\Big(\int_{\mathbb{R}^3} U_j^{\frac{24}{5}} U_i^{\frac{6}{5}}\Big)^{\frac{5}{6}} = O\Big(\varepsilon\big(\ln \tfrac{1}{\varepsilon}\big)^{1/2}\Big).$$

Lastly, we have

$$\int_{R^3\setminus\Omega} U_j^5 v = \int_{R^3\setminus\Omega} -\Delta U_j v = -\int_{\partial\Omega} \frac{\partial U_i}{\partial \nu} v$$

and we proceed, to estimate the last integral, as we did to estimate the integral over $\partial\Omega$ of $U_i^4 \frac{\partial U_i}{\partial \lambda_i} v_\varepsilon$. Namely, we observe that $D^{1,2}(R^3 \setminus \Omega)$ embeds into $L^4(\partial\Omega)$, from which we deduce

$$\int_{\partial\Omega} \frac{\partial U_i}{\partial \nu} v = \mathrm{O}\Big(\Big(\int_{\partial\Omega} \Big|\frac{\partial U_j}{\partial \nu}\Big|^{4/3}\Big)^{3/4} \|v\|\Big).$$

Far from x_i, $\frac{\partial U_i}{\partial \nu} = \mathrm{O}(\frac{1}{\lambda_i^{1/2}})$, whereas close to x_i we can write, with the notation of (A.18)

$$\frac{\partial U_j}{\partial \nu}(x) = \mathrm{O}\Big(\frac{\lambda_j^{5/2}|x'|^2}{\big(1+\lambda_j^2|x'|^2\big)^{3/2}}\Big)$$

so that

$$\int_{\partial\Omega\cap B_\tau(x_i)} \Big|\frac{\partial U_j}{\partial \nu}\Big|^{4/3} = \mathrm{O}\Big(\lambda_j^{\frac{10}{3}} \int_0^\tau \frac{r^{\frac{8}{3}+1}}{(1+\lambda_j^2 r^2)^2} \mathrm{d}r\Big) = \mathrm{O}\Big(\frac{1}{\lambda_j^{2/3}}\Big)$$

and

$$\int_{\partial\Omega} \frac{\partial U_i}{\partial \nu} v = \mathrm{O}\Big(\frac{\|v\|}{\lambda_j^{1/2}}\Big).$$

Then, (4.10) follows from (A.19).

B.2. Proof of the invertibility of $Q_{\varepsilon,\Lambda,X}$

Let $Q = Q_{\varepsilon,\Lambda,X}$ by defined by (4.6), that is, for $w = (A', v) \in \mathbb{R}^k \times E_{\Lambda,X}$

$$\begin{aligned}\ll Q w, w \gg\, = &\sum_{i,j=1}^k \Big[\langle U_i, U_j\rangle - (5-\varepsilon) \int_{R^3\setminus\Omega} \big(\sum_{\ell=1}^k U_\ell\big)^{4-\varepsilon} U_i U_j\Big]\alpha_i'\alpha_j' \\ &- (5-\varepsilon)\sum_{i=1}^k \Big(\int_{R^3\setminus\Omega} \big(\sum_{\ell=1}^k U_\ell\big)^{4-\varepsilon} U_i v\Big)\alpha_i' + \|v\|^2 - (5-\varepsilon)\int_{R^3\setminus\Omega} \big(\sum_{\ell=1}^k U_\ell\big) v^2\end{aligned}$$

since, as we previously noticed, $g_\varepsilon\big(y, \sum_j U_j\big) = \big(\sum_j U_j\big)^{5-\varepsilon}$ and $\frac{\partial g_\varepsilon}{\partial t}\big(y, \sum_j U_j\big) = (5-\varepsilon)\big(\sum_j U_j\big)^{4-\varepsilon}$ everywhere, provided that ε is small enough. Using (A.1) (A.2), (A.10), (A.12), and (A.19)–(A.22), we obtain

$$\langle U_i, U_j\rangle - (5-\varepsilon)\int_{R^3\setminus\Omega} \big(\sum_{\ell=1}^k U_\ell\big)^{4-\varepsilon} U_i U_j = \begin{cases} -\frac{3\sqrt{3}\pi^2}{2} + \mathrm{O}\Big(\varepsilon \ln \frac{1}{\varepsilon}\Big) & \text{if } i = j \\ \mathrm{O}\Big(\varepsilon\big(\ln \frac{1}{\varepsilon}\big)^{-1/4}\Big) & \text{if } i \neq j. \end{cases} \tag{B.1}$$

Next, we write

$$\begin{aligned}\int_{R^3\setminus\Omega} \big(\sum_{\ell=1}^k U_\ell\big)^{4-\varepsilon} U_i v &= \int_{R^3\setminus\Omega} \big(\sum_{\ell=1}^k U_\ell\big)^4 U_i v + \mathrm{O}\Big(\varepsilon \ln(\max_\ell \lambda_\ell)\|v\|\Big) \\ &= \int_{R^3\setminus\Omega} U_i^5 v + \mathrm{O}\Big(\varepsilon \ln \tfrac{1}{\varepsilon}\|v\| + \sum_{\substack{1\le i,j\le k \\ i\neq j}} \int_{R^3\setminus\Omega} (U_j^4 U_i + U_j U_i^4)|v|\Big).\end{aligned}$$

We just proved, in the previous subsection, that

$$\int_{R^3\setminus\Omega} U_i^5 v = O\Big(\frac{\|v\|}{\lambda_i^{1/2}}\Big).$$

Then, using (A.22) and (A.20), we find

$$\int_{R^3\setminus\Omega} \big(\sum_{\ell=1}^{k} U_\ell\big)^{4-\varepsilon} U_i v = O\Big(\varepsilon^{1/2}\big(\ln\tfrac{1}{\varepsilon}\big)^{-1/2}\|v\|\Big). \tag{B.2}$$

Lastly, we have to consider the term involving v^2 – that is, the quadratic form in v, which writes

$$\widetilde{Q}(v) = \|v\|^2 - (5-\varepsilon)\int_{R^3\setminus\Omega} \big(\sum_{\ell=1}^{k} U_\ell\big)^{4-\varepsilon} v^2 = \|v\|^2 - 5\sum_{\ell=1}^{k}\int_{R^3\setminus\Omega} U_\ell^4 v^2 + o\big(\|v\|^2\big).$$

We want to prove that $\widetilde{Q}$ is coercive, with a modulus of coercivity independent of ε and $(\Lambda, X) \in \mathcal{D}_\varepsilon$. Together with (B.1) and (B.2), this will prove that Q is invertible, and the existence of ρ independent of ε and $(\Lambda, X) \in \mathcal{D}_\varepsilon$ such that $|||Q^{-1}||| \le \rho$.

We begin by considering the eigenvalue and eigenvector problem in $H^1(\mathbb{R}^N)$, $N \ge 3$

$$-\Delta\omega = \mu U_{\lambda,x}^{\frac{4}{N-2}}\omega. \tag{B.3}$$

The spectrum and the eigenvectors of (B.3) are linked to the spectrum and the eigenvectors of $-\Delta$ of S^N, which are known [5]. Namely, the eigenvalues of $-\Delta_{S^N}$ are $\lambda_n = n(N+k-1)$, $n \in \mathbb{N}$, with multiplicity $m_n = \frac{(N+n-2)!(N+2n-1)}{n!(N-1)!}$, and the corresponding eigenvectors are harmonic polynomials of degree n: $\lambda_0 = 0$, $m_0 = 1$, $u_0 = 1$; $\lambda_1 = N$, $m_1 = N+1$, $u_{1,i} = x_i$, $1 \le i \le N+1$; $\lambda_2 = 2(N+1)$, ... For u a function defined on S^3, we define a function v in $\mathbb{R}^3$ by

$$u(x) = \big(1+|y|^2\big)^{\frac{N-2}{N}} v(y)$$

where $y = \Pi x$ is the stereographic projection of S^N, the unit sphere of $\mathbb{R}^{N+1}$, with respect to the north pole $x_i = 0$, $1 \le i \le N$, $x_{N+1} = 1$, onto $\mathbb{R}^N$ identified with the hyperplane of $\mathbb{R}^{N+1}$ defined by $x_{N+1} = 0$. u is an eigenvector of $-\Delta_{S^N}$ with eigenvalue λ_n if and only if v solves (B.3) with $(\lambda, x) = (1, 0)$ and $\mu = \mu_n = \frac{4\lambda_n}{N(N-2)} + 1$. In particular, the eigenvectors of (B.3) are $U_{1,0}$ for μ_0, $\frac{\partial U_{1,0}}{\partial x_i}$, $1 \le i \le N$, and $\frac{\partial U_{1,0}}{\partial \lambda}$ for μ_1. The orthogonality of v to $U_{1,0}$, $\frac{\partial U_{1,0}}{\partial \lambda}$, and $\frac{\partial U_{1,0}}{\partial x_i}$, $1 \le i \le N$, in $D^{1,2}(\mathbb{R}^N)$ means that u is orthogonal to 1 and the x_i's, $1 \le i \le N+1$, so that

$$\int_{S^N} |\nabla_{S_N} u|^2 \ge \lambda_2 \int_{S^N} u^2.$$

From such an inequality, we deduce, through straightforward computations,

$$\int_{\mathbb{R}^N} \big(|\nabla v|^2 - 5U_{1,0}^4 v^2\big) \ge \Big(1 - \frac{N(N+2)}{4\lambda_2 + N(N-2)}\Big)\int_{\mathbb{R}^N} |\nabla v|^2. \tag{B.4}$$

Through rescaling, we see that the same inequality holds replacing $U_{1,0}$ by $U_{\lambda,x}$, when u is assumed to be orthogonal to $U_{\lambda,x}$, $\frac{\partial U_{\lambda,x}}{\partial \lambda}$, and $\frac{\partial U_{\lambda,x}}{\partial x_i}$, $1 \le i \le N$, in $D^{1,2}(\mathbb{R}^N)$.

Let us consider now the eigenvalue and eigenvector problem in $H^1(\mathbb{R}^N_+)$,

$$-\Delta\omega = \mu U_{\lambda,0}^{\frac{4}{N-2}}\omega \text{ in } \mathbb{R}^N_+\,; \quad \frac{\partial\omega}{\partial\nu} = 0 \text{ on } \partial\mathbb{R}^N_+ \tag{B.5}$$

with $\mathbb{R}^N_+ = \{x = (x_1, \dots, x_N) \in \mathbb{R}^N \text{ s.t. } x_N > 0\}$. A symmetry argument shows that a solution to (B.5) provides us with a solution to (B.3). Therefore, for this new problem, the eigenvalues are the same as for (B.3), and the eigenvectors are the eigenvectors ω of (B.3) such that $\frac{\partial\omega}{\partial\nu} = 0$ on $\partial\mathbb{R}^N_+$. We notice that $U_{\lambda,0}$, $\frac{\partial U_{\lambda,0}}{\partial\lambda}$, and $\frac{\partial U_{\lambda,0}}{\partial x_i}$, $1 \le i \le N-1$ satisfy that condition. Consequently, for v orthogonal to $U_{\lambda,0}$, $\frac{\partial U_{\lambda,0}}{\partial\lambda}$, and $\frac{\partial U_{\lambda,0}}{\partial x_i}$, $1 \le i \le N-1$, in $D^{1,2}(\mathbb{R}^N)$, we have the same inequality as (B.4) replacing $\mathbb{R}^N$ by $\mathbb{R}^N_+$ (and $U_{1,0}$ by $U_{\lambda,0}$). From such an inequality, we can deduce, proceeding as in [1], that for $\rho < \mu_2$, $x \in \partial\Omega$ and λ large enough,

$$\int_{\mathbb{R}^N\setminus\Omega} |\nabla v|^2 - 5U_{\lambda,x}^4 v^2 \ge \rho \int_{\mathbb{R}^N\setminus\Omega} |\nabla v|^2$$

for any v orthogonal to $U_{\lambda,x}$, $\frac{\partial U_{\lambda,x}}{\partial \lambda}$, and $\frac{\partial U_{\lambda,x}}{\partial \tau_\ell}$, $1 \le \ell \le N$, in $D^{1,2}(\mathbb{R}^N)$. To complete the proof of the coercivity of $\widetilde{Q}$, we proceed as in [11]. We set

$$d = \min_{i \neq j} |x_i - x_j|, \quad \Omega_i = (R^3 \setminus \Omega) \cap B_{d/2}(x_i)$$

and, defining $v_i = v\mathbf{1}_{\Omega_i}$, we write

$$v_i = v_i^+ + v_i^-$$

with

$$v_i^- \in \mathrm{Span}\Big(U_i; \frac{\partial U_i}{\partial \lambda_i}; \frac{\partial U_i}{\partial \tau_{i\ell}}, \ell = 1, 2\Big)$$

and v_i^+ is orthogonal to v_i^- for the scalar product $\langle\ ,\ \rangle$ in B_i. The previous arguments imply that

$$\int_{\Omega_i} |\nabla v_i^+|^2 - 5U_i^4 v^2 \ge \rho \int_{\Omega_i} |\nabla v_i^+|^2$$

for λ_i large enough. On the other hand, multiplying the gradient of

$$v_i^- = a_i U_i + b_i \frac{\partial U_i}{\partial \lambda_i} + \sum_{\ell=1,2} c_{i\ell} \frac{\partial U_i}{\partial \tau_{i\ell}}$$

by the gradient of U_i, $\frac{\partial U_i}{\partial \lambda_i}$, $\frac{\partial U_i}{\partial \tau_{i\ell}}$ respectively, and integrating over Ω_i, we obtain a quasi-diagonal and invertible linear system that allows us to estimate a_i, b_i, c_{il} with respect to

$$\int_{\Omega_i} \nabla v_i^- \nabla U_i = \int_{\Omega_i} \nabla v \nabla U_i = -\int_{\Omega \setminus \Omega_i} \nabla v \nabla U_i = \mathrm{O}\Big(\big(\int_{\Omega \setminus \Omega_i} |\nabla U_i|^2\big)^{1/2} \|v\|\Big)$$

and similar formulas for the integrals involving $\frac{\partial U_i}{\partial \lambda_i}$ and $\frac{\partial U_i}{\partial \tau_{i\ell}}$. We check that

$$\int_{\Omega \setminus \Omega_i} |\nabla U_i|^2 = \mathrm{O}\Big(\frac{1}{\lambda_i d}\Big) = \mathrm{O}\Big(\varepsilon \big(\ln \tfrac{1}{\varepsilon}\big)^{-1/4}\Big)$$

and omitting here the details, we obtain

$$\int_{\Omega_i} |\nabla v_i^-|^2 = \mathrm{O}\Big(\varepsilon \big(\ln \tfrac{1}{\varepsilon}\big)^{-1/4} \|v\|^2\Big).$$

Consequently, we have

$$\int_{\Omega_i} |\nabla v|^2 - 5U_i^4 v^2 \ge \rho \int_{\Omega_i} |\nabla v|^2 + \mathrm{o}\big(\|v\|^2\big)$$

and

$$\begin{aligned} \widetilde{Q}(v) &= \|v\|^2 - \sum_{i=1}^k \int_{\Omega_i} |\nabla v|^2 + \sum_{i=1}^k \Big(\int_{\Omega_i} |\nabla v|^2 - 5 \int_{\Omega_i} U_i^4 v^2\Big) + 5 \sum_{i=1}^k \int_{(R^3 \setminus \Omega) \setminus \Omega_i} U_i^4 v^2 \\ &\ge \int_{(R^3 \setminus \Omega) \setminus \Omega_i} |\nabla v|^2 + \rho \sum_{i=1}^k \int_{\Omega_i} |\nabla v|^2 + \mathrm{o}\big(\|v\|^2\big) \\ &\ge \rho' \|v\|^2 \end{aligned}$$

for ε small enough and $\rho' > 0$ a suitable constant independent of ε.

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