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Geodesic orbit metrics on compact simple Lie groups arising from flag manifolds

Métriques définies par les variétés de drapeaux sur les groupes de Lie compacts, simples, dont les géodésiques sont des orbites

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ABSTRACT

In this paper, we investigate left-invariant geodesic orbit metrics on connected simple Lie groups, where the metrics are formed by the structures of flag manifolds. We prove that all these left-invariant geodesic orbit metrics on simple Lie groups are naturally reductive.

RÉSUMÉ

Dans cet article, nous étudions les métriques à géodésiques homogènes, invariantes à gauche, sur des groupes de Lie simples connexes, où les métriques sont définies par les structures de variétés de drapeaux. Nous montrons que toutes ces métriques à géodésiques homogènes invariantes à gauche sur des groupes de Lie simples sont naturellement réductives.

1. Introduction

Consider a homogeneous Riemannian manifold $(M = G/H, g)$, where H is a compact subgroup of G and g is a G-invariant Riemannian metric on M. If every geodesic of M is the orbit of some 1-parameter subgroup of G, then M is called a *geodesic orbit space* (g.o. space), and the metric g is called a *geodesic orbit metric* (g.o. metric). A complete Riemannian manifold (M, g) is called *geodesic orbit* if it is a geodesic orbit space with respect to the isometry group. This terminology was introduced by O. Kowalski and L. Vanhecke in [9], where they started a systematic research program on geodesic orbit manifolds including the classification in dimensions $\leqq 6$.

After that, classifications were worked out under various settings. See [10], [13], [6] and their references.

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In [11], Nikonorov started to investigate g.o. metrics on compact simple Lie groups G with isometry group $G \times K$, where K is a compact subgroup of G. He obtained an equivalent algebraic condition for g.o. spaces. In [7], it was shown that all the g.o. metrics on compact Lie groups, arising from generalized Wallach spaces, are naturally reductive.

In this paper, we investigate all the geodesic orbit metrics on compact simple Lie groups G with the structure from flag manifolds. Using the structure of flag manifolds, we prove that all such g.o. metrics are naturally reductive with respect to $G \times K$.

This paper is organized as follows. In Section 2, we recall the definition and structure of flag manifolds, along with some basic facts on g.o. metrics on compact simple Lie groups. In Section 3, we prove that all these g.o. metrics are naturally reductive by using the structure of flag manifolds.

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2. Geodesic orbit metrics on compact simple Lie groups and flag manifolds

In this paper, the Lie groups G and K are always assumed to be connected.

We first recall some basic concepts. Let K be a closed subgroup of Lie group G, a G-invariant metric g on $M = G/K$ corresponds to an $Ad(K)$-invariant scalar product $(\ , \ )$ on $\mathfrak{m} = T_o M$ and vice versa. The metric g is called *standard* if the scalar product $(\ , \ )$ on $\mathfrak{m}$ is the restriction of B, where B is the negative of the Killing form of $\mathfrak{g}$. For a given non-degenerate $Ad(K)$-invariant scalar product $(\ , \ )$ on $\mathfrak{m}$, there exists an $Ad(K)$-invariant positive definite symmetric operator A on $\mathfrak{m}$ such that $(x, y) = B(Ax, y)$ for $x, y \in \mathfrak{m}$. Conversely, any such operator A determines an $Ad(K)$-invariant scalar product $(x, y) = B(Ax, y)$ on $\mathfrak{m}$. We call such A a *metric endomorphism*. A homogeneous Riemannian metric on $M = G/K$ is called *naturally reductive* if

$$([Z, X]_{\mathfrak{m}}, Y) + (X, [Z, Y]_{\mathfrak{m}}) = 0, \forall X, Y, Z \in \mathfrak{m}.$$

In [2], there is an equivalent algebraic description of g.o. metrics on $M = G/K$, which we recall below.

Theorem 2.1 *([2] Corollary 2). Let $(M = G/K, g)$ be a homogeneous Riemannian manifold. Then M is geodesic orbit space if and only if, for every $X \in \mathfrak{m}$, there exists an $a(X) \in \mathfrak{k}$ such that*

$$[a(X) + X, AX] \in \mathfrak{k},$$

where A is the metric endomorphism.

According to the Ochiai–Takahashi theorem [12], the full connected isometry group $\mathrm{Isom}(G, g)$ of a simple compact Lie group G with a left-invariant Riemannian metric g is contained in the group $L(G)R(G)$, the product of left and right translations. Hence G is a normal subgroup in $\mathrm{Isom}(G, g)$, which is locally isomorphic to the group $G \times K$, where K is a closed subgroup of G, with action $(a, b)(c) = acb^{-1}$, where $a, c \in G$ and $b \in K$.

In [3], Alekseevski and Nikonorov showed that, if we choose G as the isometry group of the compact Lie group G with a left-invariant Riemannian metric, then we have the following Proposition.

Proposition 2.2 *([3] Proposition 8). A compact Lie group G with a left-invariant metric g is a g.o. space if and only if the corresponding Euclidean metric $(\ , \ )$ on the Lie algebra $\mathfrak{g}$ is bi-invariant.*

In [11], Nikonorov consider the isometry group of a compact simple Lie group G as $G \times K$, where K is a closed subgroup of G. Then he obtained the equivalent algebraic description of g.o. metrics g on compact simple Lie groups G as follows.

Theorem 2.3 *([11] Proposition 10). Let (G, g) be a compact simple Lie group with a left-invariant Riemannian metric. Then the following are equivalent:* (i) *(G, g) is a geodesic orbit manifold,* (ii) *there is a closed connected subgroup K of G such that, for any $X \in \mathfrak{g}$, there is $W \in \mathfrak{k}$ such that $([X + W, Y], X) = 0$ for every $Y \in \mathfrak{g}$ and* (iii) *$[A(X), X + W] = 0$, where $A : \mathfrak{g} \to \mathfrak{g}$ is the metric endomorphism for (G, g).*

Let B denote the negative of the Killing form of $\mathfrak{g}$, the Lie algebra of G. Then we have an inner product on $\mathfrak{g}$ given by

$$(\ , \ ) = A_0 B(\ , \ )|_{\mathfrak{k}_0} + x_1 B(\ , \ )|_{\mathfrak{k}_1} + \cdots + x_p B(\ , \ )|_{\mathfrak{k}_p} + y_1 B(\ , \ )|_{\mathfrak{m}_1} + \cdots + y_q B(\ , \ )|_{\mathfrak{m}_q}, \tag{2.1}$$

where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \cdots \mathfrak{k}_p$ is the decomposition of $\mathfrak{k}$ into non-isomorphic simple ideals and center, $\mathfrak{m}$ is the B-orthogonal complement of $\mathfrak{k}$ and $\mathfrak{m} = \mathfrak{m}_1 \oplus \cdots \oplus \mathfrak{m}_q$ is the decomposition of $\mathfrak{m}$ into irreducible and mutually inequivalent $Ad(K)$-modules.

D'Atri and Ziller [8] have investigated naturally reductive metrics among the left-invariant metrics on compact Lie groups, and have given a complete classification in the case of simple Lie groups. The following is a description of naturally reductive left-invariant metrics on a compact simple Lie group (Theorem 2.4).

Theorem 2.4 *([8] Theorem 1, Theorem 3). Under the notations above, a left-invariant metric on G of the form*

$$(\, , \,) = xB|_{\mathfrak{m}} + A_0|_{\mathfrak{k}_0} + u_1 B|_{\mathfrak{k}_1} + \cdots + u_p B|_{\mathfrak{k}_p}, (x, u_1, \cdots, u_p \in \mathbb{R}^+) \tag{2.2}$$

is naturally reductive with respect to $G \times K$, where $G \times K$ acts on G by $(g,k)y = gyk^{-1}$ and where A_0 is an arbitrary metric on $\mathfrak{k}_0$. Conversely, if a left-invariant metric $(\, , \,)$ on a compact simple Lie group G is naturally reductive, then there exists a closed subgroup K of G such that the metric $(\, , \,)$ is given by the form (2.2).

We have the following corollary.

Corollary 2.5. *Let g of the form (2.1) be a non-naturally reductive g.o. metric on compact Lie group G and let $\tilde{g}$ be the restriction of g on $\mathfrak{m}$, denote the corresponding metric endomorphism by A and $\tilde{A}$, respectively. Then $(M = G/K, \tilde{g})$ is a g.o. metric on M not homothetic to the standard metric.*

Proof. Since g is a g.o. metric on G, then by Theorem 2.3 we have that, for any $X \in \mathfrak{m}$, there exists $W \in \mathfrak{k}$ such that

$$[W + X, A(X)] = [W + X, \tilde{A}(X)] = 0 \in \mathfrak{k};$$

by Theorem 2.1, $(M = G/K, \tilde{g})$ is a g.o. space. From Theorem 2.4, we know that $\tilde{g}$ is not homothetic to the standard metric, because g is non-naturally reductive. □

Next, we describe some basic facts about flag manifolds.

Definition 2.6 *([15], or see [5]).* A *flag manifold* is a homogeneous space of the form $G/K = G/C(S)$, where G is a compact connected Lie group, S is a torus in G, and $C(S)$ is the centralizer of S in G.

Let $G/K = G/C(S)$ be a flag manifold, where G is a compact semisimple Lie group and S is a torus in G, here $C(S)$ denotes the centralizer of S in G. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of the Lie groups G and K respectively, and $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{k}^{\mathbb{C}}$ be the complexifications of $\mathfrak{g}$ and $\mathfrak{k}$, respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be a reductive decomposition with respect to B with $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$. Let H be a maximal torus containing S. Then this is a maximal torus in K. Let $\mathfrak{h}$ be the Lie algebra of H and $\mathfrak{h}^{\mathbb{C}}$ its complexification. Then $\mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Let R be a root system $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$ and $\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R} \mathfrak{g}^{\mathbb{C}}_{\alpha}$ be the root space decomposition.

Obviously, $\mathfrak{k}^{\mathbb{C}}$ contains $\mathfrak{h}^{\mathbb{C}}$, so there exists a subset R_K of R such that $\mathfrak{k}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} + \sum_{\alpha \in R_K} \mathfrak{g}^{\mathbb{C}}_{\alpha}$. We can choose Π and Π_K to be simple roots of R and R_K, respectively, such that $\Pi_K \subset \Pi$. Let $R_M = R \setminus R_K$, then we have $\mathfrak{m}^{\mathbb{C}} = \sum_{\alpha \in R_M} \mathfrak{g}^{\mathbb{C}}_{\alpha}$ and

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R_K} \mathfrak{g}^{\mathbb{C}}_{\alpha} \oplus \sum_{\alpha \in R_M} \mathfrak{g}^{\mathbb{C}}_{\alpha}.$$

We choose a Weyl basis $\{H_\alpha, E_\alpha | \alpha \in R\}$ in $\mathfrak{g}^{\mathbb{C}}$ with $B(E_\alpha, E_{-\alpha}) = 1$, $[E_\alpha, E_{-\alpha}] = H_\alpha$ and

$$[E_\alpha, E_\beta] = \begin{cases} 0 & \text{if } \alpha + \beta \notin R \text{ and } \alpha + \beta \neq 0 \\ N_{\alpha,\beta} E_{\alpha+\beta} & \text{if } \alpha + \beta \in R, \end{cases}$$

where $N_{\alpha,\beta} (\neq 0)$ is the structure constant with $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$ and $N_{\alpha,\beta} = -N_{\beta,\alpha}$. The following is a compact real form of $\mathfrak{g}^{\mathbb{C}}$:

$$\mathfrak{g}_\mu = \sum_{\alpha \in R^+} \mathbb{R}\sqrt{-1} H_\alpha \oplus \sum_{\alpha \in R^+} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha),$$

where R^+ is the positive root system of $\mathfrak{g}$ and $A_\alpha = E_\alpha - E_{-\alpha}$, $B_\alpha = \sqrt{-1}(E_\alpha + E_{-\alpha})$. Since any two compact real forms of $\mathfrak{g}^{\mathbb{C}}$ are conjugated, we can identify $\mathfrak{g}$ with $\mathfrak{g}_\mu$. If we set $R^+_M = R^+ \setminus R^+_K$, then we have

$$\mathfrak{k} = \sum_{\alpha \in R^+} \mathbb{R}\sqrt{-1} H_\alpha \oplus \sum_{\alpha \in R^+_K} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha) \text{ and } \mathfrak{m} = \sum_{\alpha \in R^+_M} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha).$$

The next lemma shows the bracket computation of $\mathfrak{g}$, which we will make much use of in the proof of our main theorem.

Lemma 2.7. *The Lie brackets among $\{A_\alpha = E_\alpha - E_{-\alpha}, B_\alpha = \sqrt{-1}(E_\alpha + E_{-\alpha}), \sqrt{-1} H_\beta \mid \alpha \in R^+, \beta \in \Pi\}$ of $\mathfrak{g}$ are given by*

$$\begin{aligned}
&[\sqrt{-1} H_\alpha, A_\beta] = \beta(H_\alpha) B_\beta, \quad [A_\alpha, A_\beta] = N_{\alpha,\beta} A_{\alpha+\beta} + N_{-\alpha,\beta} A_{\alpha-\beta} (\alpha \neq \beta), \\
&[\sqrt{-1} H_\alpha, B_\beta] = -\beta(H_\alpha) A_\beta, [B_\alpha, B_\beta] = -N_{\alpha,\beta} A_{\alpha+\beta} - N_{\alpha,-\beta} A_{\alpha-\beta} (\alpha \neq \beta), \\
&[A_\alpha, B_\alpha] = 2\sqrt{-1} H_\alpha, [A_\alpha, B_\beta] = N_{\alpha,\beta} B_{\alpha+\beta} + N_{\alpha,-\beta} B_{\alpha-\beta} (\alpha \neq \beta),
\end{aligned}$$

where $N_{\alpha,\beta}$ are the structural constants in Weyl basis.

In flag manifolds, the so-called $\mathfrak{t}$-roots play an very important role, which we now describe. These results are essentially due to Kostant in 1965; see [14, Theorem 8.13.3].

From now on we fix a system of simple roots $\Pi = \{\alpha_1, \cdots, \alpha_r, \phi_1, \cdots, \phi_k\}$ of R, so that $\Pi_K = \{\phi_1, \cdots, \phi_k\}$ is a basis of the root system R_K and $\Pi_M = \Pi \setminus \Pi_K = \{\alpha_1, \cdots, \alpha_r\}(r + k = l)$. Let $\{h_{\alpha_1}, \cdots, h_{\alpha_r}, h_{\phi_1}, \cdots, h_{\phi_k}\}$ be the fundamental weights. Let

$$\mathfrak{t} = \mathfrak{z}(\mathfrak{k}^{\mathbb{C}}) \cap \sqrt{-1}\mathfrak{h},$$

where $\mathfrak{z}(\mathfrak{k}^{\mathbb{C}})$ is the center of $\mathfrak{k}^{\mathbb{C}}$. Consider the restriction map $\pi : (\mathfrak{h}^{\mathbb{C}})^* \to \mathfrak{t}^*$ defined by $\pi(\alpha) = \alpha|_{\mathfrak{t}}$, and set $R_{\mathfrak{t}} = \pi(R) = \pi(R_M)$. $\mathfrak{t}$-roots are the elements of $R_{\mathfrak{t}}$. For an invariant ordering $R_M^+ = R^+ \setminus R_K^+$ in R_M, we set $R_{\mathfrak{t}}^+ = \pi(R_M^+)$ and $R_{\mathfrak{t}}^- = -R_{\mathfrak{t}}^+$. It is obvious that $R_{\mathfrak{t}}^- = \pi(R_M^-)$, thus the splitting $R_{\mathfrak{t}} = R_{\mathfrak{t}}^- \cup R_{\mathfrak{t}}^+$ defines an ordering in $R_{\mathfrak{t}}$. A $\mathfrak{t}$-root $\xi \in R_{\mathfrak{t}}^+$ (respectively $\xi \in R_{\mathfrak{t}}^-$) will be called positive (respectively negative). A $\mathfrak{t}$-root is called simple if it is not a sum of two positive $\mathfrak{t}$-roots.

Theorem 2.8. *([14, Theorem 8.13.3]; or see [4, Corollary 3.1]) There is one-to-one correspondence between $\mathfrak{t}$-roots and complex irreducible ad$(\mathfrak{k}^{\mathbb{C}})$-submodules $\mathfrak{m}_\xi$ of $\mathfrak{m}^{\mathbb{C}}$. This correspondence is given by*

$$R_{\mathfrak{t}} \ni \xi \leftrightarrow \mathfrak{m}_\xi = \sum_{\alpha \in R_M, \pi(\alpha) = \xi} \mathbb{C}E_\alpha,$$

Hence $\mathfrak{m}^{\mathbb{C}} = \sum_{\xi \in R_{\mathfrak{t}}} \mathfrak{m}_\xi$. Moreover, these submodules are non-equivalent ad$(\mathfrak{k}^{\mathbb{C}})$-modules.

Since the complex conjugation $\tau : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ with respect to the compact real form $\mathfrak{g}$ interchanges the root spaces, a decomposition of the real $ad(\mathfrak{k})$-module $\mathfrak{m} = (\mathfrak{m}^{\mathbb{C}})^\tau$ into real irreducible $ad(\mathfrak{k})$-submodule is given by

$$\mathfrak{m} = \sum_{\xi \in R_{\mathfrak{t}}^+} (\mathfrak{m}_\xi \oplus \mathfrak{m}_{-\xi})^\tau, \tag{2.3}$$

where V^τ denotes the set of fixed points of the complex conjugation τ in a vector subspace $V \subset \mathfrak{g}^{\mathbb{C}}$. If we set $R_{\mathfrak{t}}^+ = \{\xi_1, \cdots, \xi_s\}$, then according to (2.3) each real irreducible $ad(\mathfrak{k})$-submodule $\mathfrak{m}_i = (\mathfrak{m}_{\xi_i} \oplus \mathfrak{m}_{-\xi_i})^\tau (1 \le i \le s)$ corresponding to the positive $\mathfrak{t}$-roots ξ_i, is given by

$$\mathfrak{m}_i = \sum_{\alpha \in R_M^+, \pi(\alpha) = \xi_i} (\mathbb{R}A_\alpha + \mathbb{R}B_\alpha).$$

3. Main theorem and its proof

In this section, we will state and prove our main theorem.

Theorem 3.1. *All the g.o. metrics on compact simple Lie groups G of the form (2.1) arising from flag manifolds are naturally reductive.*

In [2], the authors investigated all g.o. metrics on flag manifolds on compact simple Lie groups and they proved that only $\mathrm{SO}(2l+1)/\mathrm{U}(l)(l \ge 2)$ and $\mathrm{Sp}(l)/\mathrm{U}(1)\mathrm{Sp}(l-1)(l \ge 3)$ can admit g.o. metrics not homothetic to the standard metrics. As a result of Corollary 2.5, we only need to consider whether there are non-naturally reductive g.o. metrics on $\mathrm{SO}(2l+1)(l \ge 2)$ and $\mathrm{Sp}(l)(l \ge 3)$ with the corresponding metric forms. For these two special flag manifolds, the metric for (2.1) can be simplified as follows:

$$(\ ,\ ) = B(\ ,\ )|_{\mathfrak{u}(1)} + uB(\ ,\ )|_{\mathfrak{k}_0} + xB(\ ,\ )|_{\mathfrak{m}_1} + yB(\ ,\ )|_{\mathfrak{m}_2}, \tag{3.1}$$

where $\mathfrak{u}(1)$ is a 1-dimensional center of $\mathfrak{k}$ and $\mathfrak{k}_0$ is a simple Lie algebra.

When we apply Theorem 2.3 to the metric form (3.1), we can immediately obtain the following equivalent description of g.o. metric of the following form.

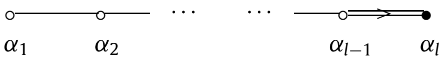
Theorem 3.2. *A compact simple Lie group G with the left-invariant metric induced by (3.1) is a geodesic orbit space if and only if, for any $T \in \mathfrak{u}(1)$, $H \in \mathfrak{k}_0$, $X_1 \in \mathfrak{m}_1$, $X_2 \in \mathfrak{m}_2$, there exists $K \in \mathfrak{k}$ such that the following three conditions hold:*

(1) $[H, K] = 0$;
(2) $[(x-1)T + (x-u)H + xK + (x-y)X_2, X_1] = 0$;
(3) $[(y-1)T + (y-u)H + yK, X_2] = 0$.

In the following, we will prove that all the g.o. metrics of the form (3.1) on $\mathrm{SO}(2l+1)(l \ge 2)$ and $\mathrm{Sp}(l)(l \ge 3)$ are naturally reductive for each case.

3.1. Case of SO(2l + 1)

The painted Dynkin diagram of this case is

B_l :


Hence we can give the basis for each of the four parts in the decomposition $\mathfrak{so}(2l+1) = \mathfrak{u}(1) \oplus \mathfrak{su}(l) \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$.
$\mathfrak{u}(1) = \text{span}_{\mathbb{R}}\{\sqrt{-1}H_{\alpha_l}\}$,
$\mathfrak{su}(l) = \text{span}_{\mathbb{R}}\{A_\alpha, B_\alpha, \sqrt{-1}H_\beta | \alpha = \alpha_p + \cdots + \alpha_k, 1 \le p \le k \le l-1; \beta = \alpha_p, 1 \le p \le l-1\}$,
$\mathfrak{m}_1 = \text{span}_{\mathbb{R}}\{A_\alpha, B_\alpha | \alpha = \alpha_k + \cdots + \alpha_{l-1} + \alpha_l, 1 \le k \le l\}$,
$\mathfrak{m}_2 = \text{span}_{\mathbb{R}}\{A_\alpha, B_\alpha | \alpha = \alpha_k + \cdots + 2\alpha_p + \cdots + 2\alpha_l, 1 \le k < p \le l\}$.
Then we choose $T = \sqrt{-1}H_{\alpha_l}$, $H = \sum_{i=1}^{l-1}\sqrt{-1}H_{\alpha_i}$, $X_1 = B_{\alpha_l}$, $X_2 = A_{\alpha_1+\cdots+\alpha_{l-1}+2\alpha_l}$, and we assume that the metric of the form (3.1) is a g.o. metric; by Theorem 3.2, there exists some $K \in \mathfrak{k}$ such that

(1) $[H, K] = 0$;
(2) $[(x-1)T + (x-u)H + xK + (x-y)X_2, X_1] = 0$;
(3) $[(y-1)T + (y-u)H + yK, X_2] = 0$.

From (2), we have $[(x-1)\sqrt{-1}H_{\alpha_l} + (x-u)\sum_{i=1}^{l-1}\sqrt{-1}H_{\alpha_i} + xK, B_{\alpha_l}] = (y-x)[A_{\alpha_1+\cdots+\alpha_{l-1}+2\alpha_l}, B_{\alpha_l}]$.
By Lemma 2.7, we have

$$[(x-1)\sqrt{-1}H_{\alpha_l} + (x-u)\sum_{i=1}^{l-1}\sqrt{-1}H_{\alpha_i} + xK, B_{\alpha_l}] = (y-x)N_{\alpha_1+\cdots+\alpha_{l-1}+2\alpha_l,-\alpha_l}A_{\alpha_1+\cdots+\alpha_{l-1}+\alpha_l}.$$

We next prove that there is no $A_{\alpha_1+\cdots+\alpha_{l-1}+\alpha_l}$-component in $[(x-1)\sqrt{-1}H_{\alpha_l} + (x-u)\sum_{i=1}^{l-1}\sqrt{-1}H_{\alpha_i} + xK, B_{\alpha_l}]$; in fact, we only need to show that K does not contain any $B_{\alpha_1+\cdots+\alpha_{l-1}}$-component by Lemma 2.7. If K contains a $B_{\alpha_1+\cdots+\alpha_{l-1}}$-component, then

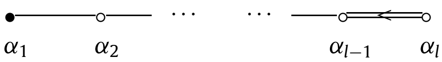
$$[\sum_{i=1}^{l-1}\sqrt{-1}H_{\alpha_i}, B_{\alpha_1+\cdots+\alpha_{l-1}}] = -\sum_{i=1}^{l-1}(\alpha_1 + \cdots + \alpha_{l-1})(H_{\alpha_i})A_{\alpha_1+\cdots+\alpha_{l-1}} \tag{3.2}$$

$$= -\sum_{i=1}^{l-1} < \alpha_1 + \cdots + \alpha_{l-1}, \alpha_i > A_{\alpha_1+\cdots+\alpha_{l-1}} \tag{3.3}$$

From the Cartan matrix of B_l, we know $[\sum_{i=1}^{l-1}\sqrt{-1}H_{\alpha_i}, B_{\alpha_1+\cdots+\alpha_{l-1}}] \ne 0$, which is a contradiction to (1) above. As a result, there is no $A_{\alpha_1+\cdots+\alpha_{l-1}+\alpha_l}$-component in $[(x-1)\sqrt{-1}H_{\alpha_l} + (x-u)\sum_{i=1}^{l-1}\sqrt{-1}H_{\alpha_i} + xK, B_{\alpha_l}]$. Hence, $x = y$. By Theorem 2.4, geodesic orbit metrics on SO(2l + 1) of the form (3.1) are naturally reductive with respect to SO(2l + 1) × U(l).

3.2. Case of Sp(l)

The painted Dynkin diagram of this case is

C_l :


The basis of each part of the decomposition $\mathfrak{sp}(l) = \mathfrak{u}(1) \oplus \mathfrak{sp}(l-1) \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$ are as follows:
$\mathfrak{u}(1) = \text{span}_{\mathbb{R}}\{\sqrt{-1}H_{\alpha_1}\}$,
$\mathfrak{sp}(l-1) = \text{span}_{\mathbb{R}}\{A_\alpha, B_\alpha, \sqrt{-1}H_\beta | \beta = \alpha_i (2 \le i \le l); \alpha = \alpha_p + \cdots + \alpha_k (2 \le p \le k \le l)$ or $\alpha = \alpha_p + \alpha_{p+1} + \cdots + 2\alpha_k + \cdots + 2\alpha_{l-1} + \alpha_l (2 \le p \le k \le l-1)\}$,
$\mathfrak{m}_1 = \text{span}_{\mathbb{R}}\{A_\alpha, B_\alpha | \alpha = \alpha_1 + \cdots + \alpha_k (1 \le k \le l)$ or $\alpha = \alpha_1 + \alpha_2 + \cdots + 2\alpha_p + \cdots + 2\alpha_{l-1} + \alpha_l (2 \le p \le l-1)\}$,
$\mathfrak{m}_2 = \text{span}_{\mathbb{R}}\{A_{2\alpha_1+\cdots+2\alpha_{l-1}+\alpha_l}, B_{2\alpha_1+\cdots+2\alpha_{l-1}+\alpha_l}\}$.
We assume the metric of the form (3.1) on Sp(l) is a geodesic orbit metric, then for $T = \sqrt{-1}H_{\alpha_1}$, $H = \sum_{i=2}^{l}\sqrt{-1}H_{\alpha_i}$, $X_1 = B_{\alpha_1}$, $X_2 = A_{2\alpha_1+\cdots+2\alpha_{l-1}+\alpha_l}$, by Theorem 3.2, there exists some $K \in \mathfrak{k}$ such that

(1) $[H, K] = 0$;

(2) $[(x-1)T+(x-u)H+xK+(x-y)X_2, X_1]=0$;
(3) $[(y-1)T+(y-u)H+yK, X_2]=0$.

From (2) above, we have $[(x-1)\sqrt{-1}H_{\alpha_1}+(x-u)\sum_{i=2}^{l}\sqrt{-1}H_{\alpha_i}+xK, B_{\alpha_1}]=(y-x)[A_{2\alpha_1+\cdots+2\alpha_{l-1}+\alpha_l}, B_{\alpha_1}]$.

By Lemma 2.7, we have

$$[(x-1)\sqrt{-1}H_{\alpha_1}+(x-u)\sum_{i=2}^{l}\sqrt{-1}H_{\alpha_i}+xK, B_{\alpha_1}]=(y-x)N_{2\alpha_1+\cdots+2\alpha_{l-1}+\alpha_l,-\alpha_1}A_{\alpha_1+2\alpha_2+\cdots+2\alpha_{l-1}+\alpha_l}.$$

We next prove that there is no $A_{\alpha_1+2\alpha_2+\cdots+2\alpha_{l-1}+\alpha_l}$-component in $[(x-1)\sqrt{-1}H_{\alpha_1}+(x-u)\sum_{i=2}^{l}\sqrt{-1}H_{\alpha_i}+xK, B_{\alpha_1}]$, in fact, we only need to show K does not contain any $B_{2\alpha_2+\cdots+2\alpha_{l-1}+\alpha_l}$-component by Lemma 2.7. If K contains a $B_{2\alpha_2+\cdots+2\alpha_{l-1}+\alpha_l}$-component, then

$$[\sum_{i=2}^{l}\sqrt{-1}H_{\alpha_i}, B_{2\alpha_2+\cdots+2\alpha_{l-1}+\alpha_l}]=-\sum_{i=2}^{l}(2\alpha_2+\cdots+2\alpha_{l-1}+\alpha_l)(H_{\alpha_i})A_{2\alpha_2+\cdots+2\alpha_{l-1}+\alpha_l} \tag{3.4}$$

$$=-\sum_{i=2}^{l}<2\alpha_2+\cdots+2\alpha_{l-1}+\alpha_l, \alpha_i>A_{2\alpha_2+\cdots+2\alpha_{l-1}+\alpha_l} \tag{3.5}$$

From the Cartan matrix of C_l, we know that $[\sum_{i=2}^{l}\sqrt{-1}H_{\alpha_i}, B_{2\alpha_2+\cdots+2\alpha_{l-1}+\alpha_l}]\neq 0$. This contradicts (1) above, so $[(x-1)\sqrt{-1}H_{\alpha_1}+(x-u)\sum_{i=2}^{l}\sqrt{-1}H_{\alpha_i}+xK, B_{\alpha_1}]$ has no $A_{\alpha_1+2\alpha_2+\cdots+2\alpha_{l-1}+\alpha_l}$-component. Hence, $x=y$. By Theorem 2.4, geodesic orbit metrics on $\mathrm{Sp}(l)$ of the form (3.1) are naturally reductive with respect to $\mathrm{Sp}(l)\times(\mathrm{U}(1)\times\mathrm{Sp}(l-1))$.

That completes the proof of Theorem 3.1.

Remark 3.3. For the details of the relationship between painted Dynkin diagrams and flag manifolds, see [15] (for the viewpoint of complex groups and manifolds), [1] and [5] (for the viewpoint of compact groups and manifolds).

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