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Solutions of a class of multiplicatively advanced differential equations

Solutions d'une classe d'équations différentielles multiplicativement avancées

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A R T I C L E   I N F O

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A B S T R A C T

The multiplicatively advanced differential equations (MADEs) of form $f^{(n)}(t) = \alpha f(\beta t)$ with $\alpha \neq 0$, $\beta > 1$ are studied along with a class of their solutions of type $f_{\mu,\lambda}(t)$ defined on $[0, \infty)$. For $\lambda \in \mathbb{Q}^+$, $\mu \in \mathbb{R}$, the solutions $f_{\mu,\lambda}(t)$ are extended to $(-\infty, \infty)$ in a non-unique manner to obtain Schwartz wavelet solutions $F_{\mu,\lambda}(t)$ of the original MADE, with all moments of $F_{\mu,\lambda}(t)$ vanishing. Examples are studied in detail. The Fourier transform of each $F_{\mu,\lambda}(t)$ is computed and, in a number of examples, is related to the Jacobi theta function. Additional conditions sufficient for the uniqueness of certain MADE initial value problems are given. Conditions for decay and non-decay at $-\infty$ are obtained. Decay rates at $\pm\infty$ in terms of familiar functions are established.

R É S U M É

Des équations différentielles multiplicativement avancées (MADE) de la forme $f^{(n)}(t) = \alpha f(\beta t)$ avec $\alpha \neq 0$, $\beta > 1$ sont étudiées dans le cadre des solutions de type $f_{\mu,\lambda}(t)$ définies sur $[0, \infty)$. Pour $\lambda \in \mathbb{Q}^+$, $\mu \in \mathbb{R}$, les solutions $f_{\mu,\lambda}(t)$ sont prolongées sur $(-\infty, \infty)$ d'une manière non unique pour obtenir des solutions ondelettes dans l'espace de Schwartz $F_{\mu,\lambda}(t)$ de l'originale MADE, avec tous les moments de $F_{\mu,\lambda}(t)$ nuls. Des exemples sont étudiés en détail. La transformée de Fourier de chaque $F_{\mu,\lambda}(t)$ est calculée et, dans un certain nombre d'exemples, est liée à la fonction thêta de Jacobi. Des conditions supplémentaires suffisantes pour l'unicité de la solution de certaines MADE avec condition initiale sont données. Les conditions de décroissance et de non-décroissance à $-\infty$ sont obtenues. Les taux de décroissance à $\pm\infty$ en termes de fonctions familières sont établis.

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1. Introduction: multiplicatively advanced differential equations (MADEs)

This article is a study of homogeneous multiplicatively advanced differential equations (MADEs) of the form

$$f^{(n)}(t) = \alpha f(\beta t) \quad \text{or equivalently} \quad f^{(n)}(t) - \alpha f(\beta t) = 0, \tag{1}$$

where $\alpha \neq 0$ and $\beta > 1$. Note that the argument βt in the second term of (1) is multiplicatively advanced by the advancing parameter $\beta > 1$, making (1) a MADE. We shall approach this study through the examination of a new class of functions $f_{\mu,\lambda}(t)$ given in the following definition.

Definition 1.1. Let $q > 1$, $\mu \in \mathbb{R}$, and $\lambda > 0$. Then for $t \geq 0$, the function $f_{\mu,\lambda}(t)$ is given by

$$f_{\mu,\lambda}(t) \equiv \sum_{m=-\infty}^{\infty} (-1)^m \frac{e^{-q^m t}}{q^{m(m-\mu)/\lambda}} . \tag{2}$$

From (2), observe that $|f_{\mu,\lambda}(t)| \leq \sum_{m=-\infty}^{\infty} 1/q^{m(m-\mu)/\lambda} < \infty$. So $f_{\mu,\lambda}(t)$ is bounded and converging uniformly on $t \in [0,\infty)$. For λ rational, the $f_{\mu,\lambda}(t)$ satisfy the MADE (18) below, which by choice of parameters involved can be shown to be equivalent to (1). This equivalence is shown in the Remark 7 following Theorem 2.2 below. Note that if one complexifies the argument t to obtain the complex argument z in (2), then the above bound $|f_{\mu,\lambda}(z)| \leq \sum_{m=-\infty}^{\infty} 1/q^{m(m-\mu)/\lambda} < \infty$ still holds for z in the right half-plane $\mathcal{R}(z) \geq 0$. As the uniform limit of the analytic functions given by the truncated summations (for m ranging from $-N$ to N in (2) as $N \to \infty$), $f_{\mu,\lambda}(z)$ is analytic [28] on the open right half-plane $\mathcal{R}(z) > 0$. Thus $f_{\mu,\lambda}(t)$ is real analytic in t on $(0,\infty)$ and it is $\mathcal{C}^\infty$ in t on $[0,\infty)$. For these values of t, it is also real analytic in the parameters μ and λ, but only $\mathcal{C}^\infty$ in the parameter $q > 1$.

After obtaining the properties of $f_{\mu,\lambda}(t)$ on the positive half-line $[0,\infty)$ including the MADE that each solves, the $f_{\mu,\lambda}(t)$ are extended to $F_{\mu,\lambda}(t)$ globally defined and $\mathcal{C}^\infty$ on all of the real line. These $F_{\mu,\lambda}(t)$ also satisfy the original MADE satisfied by $f_{\mu,\lambda}(t)$. The extensions $F_{\mu,\lambda}(t)$ are shown to be decaying rapidly at $\pm\infty$ and are in fact Schwartz wavelet functions on $\mathbb{R}$. As decaying global functions, they are amenable to Fourier transform computations, which are obtained and seen to be related to the Jacobi theta function in a range of cases. Ultimately, this study expands the connection between global solutions of MADEs such as (1) with the harmonic analysis of Schwartz wavelets, which, in turn, can be connected with the special function theory of the Jacobi theta function. As a first such connection, we point out that the formal MacLaurin series for $f_{\mu,\lambda}(t)$ is given by

$$\sum_{n\geq 0} \frac{f^{(n)}_{\mu,\lambda}(0)}{n!} z^n = \sum_{n\geq 0} \frac{(-1)^n \theta(q^{2/\lambda}; -q^{(\mu+n\lambda-1)/\lambda})}{n!} z^n , \tag{3}$$

where $\theta(q;u)$ is the Jacobi theta function given by (22) below, and where equality in (3) follows from (12) and (28). As will be seen in general in the proof of Proposition 2.3 below, the formal MacLaurin series given by (3) has radius of convergence 0 when $f_{\mu,\lambda}(t)$ is not flat at $t = 0$. Hence, $f_{\mu,\lambda}(t)$ and its extension $F_{\mu,\lambda}(t)$ cannot be real analytic at $t = 0$. Thus, in this study we restrict $F_{\mu,\lambda}(t)$ to t on the real line in the $\mathcal{C}^\infty(\mathbb{R})$ case, as opposed to attempting to extend the $f_{\mu,\lambda}(z)$ analytically beyond the imaginary axis in the complex plane, which in many cases is problematic via the Remark 4 at the end of this section. In special cases, methods of extending exponential series beyond a natural boundary, such as the imaginary axis encountered in (2), are well studied, see for instance [5]. Also, restriction of $f_{\mu,\lambda}(z)$ to the imaginary axis $z = it$ yields an almost periodic function of t, as per p. 289 of [1], see also [2], [3].

While the MADE (1) may at first appear counter-intuitive, its solutions for special values of μ and λ are generating a number of interesting applications. These special case applications include: modeling tsunami waves [25]; modeling rogue waves [25]; obtaining Schwartz functions ${}_qCos(t)$ and ${}_qSin(t)$ which well-approximate $\cos(t)$ and $\sin(t)$, respectively, on compact sets as $q \to 1^+$ [24], as illustrated in Fig. 1; obtaining smooth Schwartz approximations of the Haar wavelet [27]; obtaining Schwartz approximations of truncated Legendre polynomials [26], [27]; and obtaining Schwartz approximations of spherical Bessel functions of the first kind [27]. The majority of these solutions also turn out to be Schwartz wavelets generating wavelet frames for $\mathcal{L}^2(\mathbb{R})$, and in turn these solutions comprise a rich set within each $\mathcal{L}^p(\mathbb{R})$ space and have good decay and localization while satisfying perturbations of classic differential equations (see Remark 8 after Theorem 2.2). The solutions of (1) will also provide further interesting applications to physics. Each of the solutions described in the applications above relate to special function theory in the sense that all of them have Fourier transforms that can be expressed in terms of the Jacobi theta function (see (22) below). A pattern is emerging that clarifies the relation of solutions of MADEs such as (1) to: wavelets and wavelet frames, special function theory, approximation theory, self-similarity, and physical applications.

Thus the MADE (1) and the functions (2) deserve study in their own right. We note that Definition 1.1 is motivated by and generalizes: (i) the results in [22], where the mother wavelet $K(t) = f_{-1,2}(t)$ was shown to satisfy the MADE $K'(t) = K(qt)$

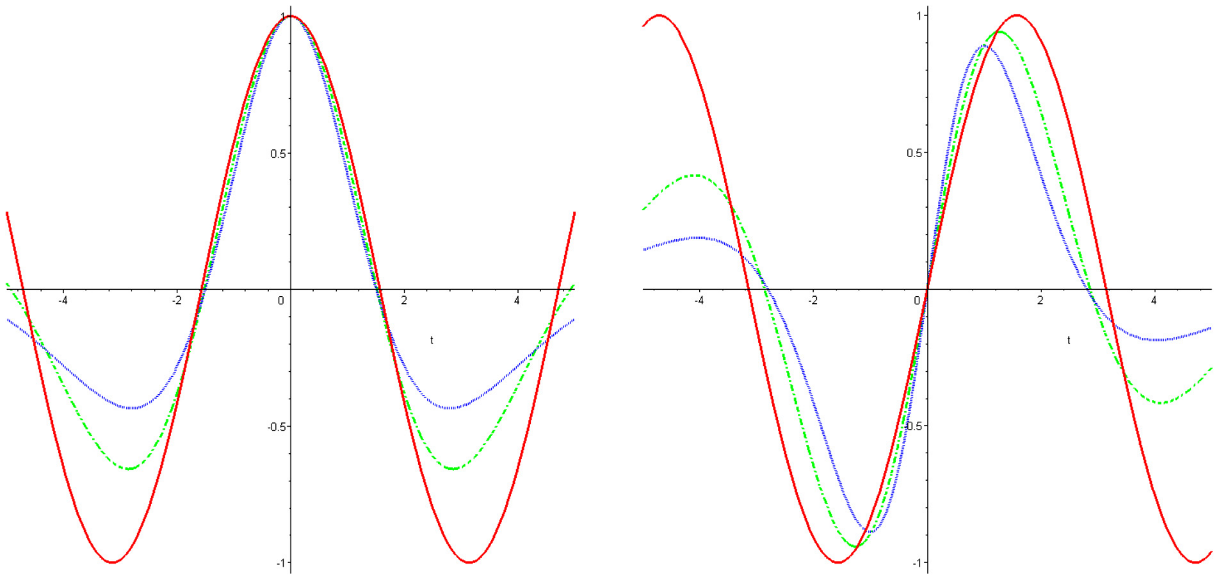

Fig. 1. Left: $y = \cos(x)$ (solid red) approached by $y = {}_qCos(t)$ for: $q = 1.5$ (dotted blue), $q = 1.2$ (dashed green). **Right:** $y = \sin(x)$ (solid red) approached by $y = {}_qSin(t)$ for: $q = 1.5$ (dotted blue), $q = 1.2$ (dashed green).

with $K(t)$ related to the Jacobi theta function, and (ii) the results in [24], where the mother wavelets ${}_qCos(t)$ (which is $f_{0,1}(t)$ normalized by the scale factor $1/f_{0,1}(0)$) and ${}_qSin(t)$ (which is $f_{1,1}(t)$ also scaled by $1/f_{0,1}(0)$) were studied. In this second setting one has the MADEs ${}_qCos''(t) = -q\ {}_qCos(qt)$ and ${}_qSin''(t) = -q^2\ {}_qSin(qt)$, which are perturbations of the harmonic oscillator $f''(t) = -f(t)$ parameterized by the perturbation/advancing parameter $q > 1$. As q approaches 1 from above, these second-order MADE perturbations approach the classical harmonic oscillator, and their L^2 solutions ${}_qCos(t)$ and ${}_qSin(t)$ converge uniformly to $cos(t)$ and $sin(t)$ on compact sets of $\mathbb{R}$, as shown in [24] and illustrated in Fig. 1. Thus, the current study of the $f_{\mu,\lambda}(t)$ can be seen as a more comprehensive, but comparable, study of global solutions of more general MADEs which are perturbations of classical ODE's. Furthermore, this study lays the ground work for understanding the convergence of these Schwartz MADE solutions to their classical analogues, generalizing the convergence seen in Fig. 1.

To place things in historical context, observe that the functions $f_{\mu,\lambda}(t)$ given in (2) are Dirichlet-like series. Classical general Dirichlet series take the form $\sum_{m=0}^{\infty} a_m \exp(-\lambda_m t)$ where it is assumed that $m \geq 0$ and the spectrum consisting of the λ_m is increasing and unbounded, see [9]. However, in equation (2) above, the m fall in $\mathbb{Z}$ giving a double-sidedness to the series (as $m \to -\infty$ or as $m \to \infty$); also the spectrum $\lambda_m = q^m$ is geometric in nature, increasing in m, and becoming unbounded as m approaches ∞. In contrast to the case of classical Dirichlet series, the spectrum $\lambda_m = q^m$ accumulates to 0 as m approaches $-\infty$.

The use of Dirichlet-like series has previously occurred in the study of other multiplicatively advanced/delayed functional differential equations. For instance, in [30] Zhang shows that, for $0 < q < 1$, a Dirichlet-like series of form $F(\mu; q, x) = C(\mu, q) \sum_{m \geq 0} A(\mu, q, m) e^{-q^m x}$ given in (2.1) of [30] [with geometric decreasing spectrum $\lambda_m = q^m$ accumulating to 0] satisfies the multiplicatively delayed functional differential pantograph equation (0.2) of [30], namely $y'(x) = q^{\mu} y(qx) - y(x)$. Also, for $0 < q < 1$, in Lemma 22 and 24 of [12], a family of multiplicatively delayed q-difference partial differential equations is studied with the aid of Dirichlet-like series. In their case, the spectrum $\lambda_m = Dq^{dm}$ is geometric, decreasing, and accumulating to 0. They also utilize a decay estimate at infinity similar in spirit to Proposition 8.1 below (which under reciprocation of the argument yields a flatness condition at 0). In Lemma 3 of [15] for $q > 1$ a related Dirichlet like series with decreasing geometric spectrum accumulating to 0 is used to obtain a decay estimate that is also similar to Proposition 8.1 below in the study of multiplicatively advanced partial differential equations. Decay rates similar to Proposition 8.1 were obtained in [20], and also appear in [23–26]. In equation (4) and Lemma 9 of [17] and in Lemma 30 of [13] other Dirichlet-like series with decreasing non-geometric spectra of form $\lambda_m = 1/(m+1)^{\alpha}$ accumulating to 0 occur, where they are used to obtain key estimates. Related and interesting references also include [16], [11], [14], [18].

In [7], conditions are given for unique solutions possessing exponential decay. However, though the MADE (1) in the case $n = 1$ relates to the advanced differential equation under study in Dung's paper, [7], it does not meet a key assumption required in [7] to obtain uniqueness and exponential decay. This is discussed here in detail at the end of Section 8.1. Instead, and in contrast to [7], we obtain generic decay rates at $\pm\infty$ of type $K_1|t|^{-K_2 \ln|t| + K_3}$ via Proposition 8.1 below, and we encounter non-uniqueness.

This study proceeds as follows. For $\lambda \in \mathbb{Q}^+$ and $\mu \in \mathbb{R}$, the multiplicatively advanced differential equations (MADEs) satisfied by the $f_{\mu,\lambda}(t)$ are determined. A key extension problem is solved, namely, for $\lambda \in \mathbb{Q}^+$ extension of $f_{\mu,\lambda}(t)$ to the whole real line is accomplished, obtaining decaying extensions $F_{\mu,\lambda}(t)$ and in special cases, non-decaying bounded solutions. The $F_{\mu,\lambda}(t)$ are shown to be Schwartz wavelets with all moments vanishing (see [4], [19] for related discussion).

Explicit examples are constructed, which illustrate our process and which show that the assumptions required to obtain extension of $f_{\mu,\lambda}(t)$ to $(-\infty, 0]$ are generic in nature. Fourier transforms are computed and related to the Jacobi theta function. Non-uniqueness of such extensions as well as of solutions to initial value problems (IVPs) with initial values given at $t = 0$ is demonstrated. Additional conditions sufficient for uniqueness are obtained. Detailed proofs of decay rates at $\pm\infty$ and of relations to Jacobi theta functions are given.

Before proceeding further, we provide a few contextual remarks.

Remark 1. Note that a solution of (1) by $f(t)$ will allow for a solution of the related MADE

$$g^{(n)}(t) = \alpha g(\beta t + \delta) \,, \tag{4}$$

by letting

$$g(t) = f\left(t + \frac{\delta}{\beta - 1}\right) . \tag{5}$$

To see this, first observe that

$$g\left(t - \frac{\delta}{\beta - 1}\right) = f(t) \,, \tag{6}$$

from which one has

$$g^{(n)}(t) = f^{(n)}\left(t + \frac{\delta}{\beta - 1}\right) = \alpha f\left(\beta t + \beta\frac{\delta}{\beta - 1}\right) = \alpha g\left(\beta t + \beta\frac{\delta}{\beta - 1} - \frac{\delta}{\beta - 1}\right) = \alpha g(\beta t + \delta) \,.$$

Remark 2. We mention that an inhomogeneous version of the MADE in (1), of form $f^{(n)}(t) - \alpha f(\beta t) = h(t)$ for a given $h(t)$, can be solved with a higher order homogeneous MADE via annihilator techniques [8]. That is, if A_h denotes the annihilator of $h(t)$, one sees that a solution of the inhomogeneous MADE falls among the solutions of the related higher-order homogeneous MADE $A_h[f^{(n)}(t) - \alpha f(\beta t)] = A_h[h(t)] = 0$. As a special example, if $h(t) = f_{-1,2}(t)$ (as in Example 1 in Section 5, or as in Section 7 equation (155) below), then for fixed $q > 1$, with $\alpha = -1/q, \beta = q$ in (1), one sees that $A_h = D_t - \hat{A}_q$ is the annihilator of h, where D_t denotes differentiation with respect to t and $\hat{A}_q$ denotes the advancing operator that multiplies the argument of a function by q, (see either: (13) with $L = 1 = k$ and $\mu = -1$, or (102) with $\mu = -1$ and $\lambda = 2$ below). The inhomogeneous MADE $f^{(1)}(t) + (1/q)f(qt) = h(t)$ has solutions that fall among those of $[D_t - \hat{A}_q][D_t + (1/q)\hat{A}_q]f(t) = [D_t^2 - (1/q)\hat{A}_{q^2}]f(t) = 0$, which is of type (1).

Remark 3. Note that the substitution $t = \mathrm{e}^u$ and the associated function $G(u) = f(\mathrm{e}^u)$ converts the MADE (1) to an additively advanced differential equation on $u \in (-\infty, \infty)$ with a u-solution that is in general not a wavelet. However, the $t = \mathrm{e}^u$ substitution, ignores the available extension of the solution of the original MADE (1) to $t \in (-\infty, 0]$, which yields global wavelet solutions of (1) on all of $t \in (-\infty, \infty)$ in the general setting. For instance, in the case that $\alpha = 1$ and $\beta = q > 1$ in (1), one has $f^{(1)}(t) = f(qt)$. Letting $t = \mathrm{e}^u$ and $G(u) = f(\mathrm{e}^u)$, the resulting differential equation becomes $G^{(1)}(u) = \mathrm{e}^u G(u + \ln(q))$ for $-\infty < u < \infty$ (corresponding only to the case that $t > 0$ and yielding a differential equation with exponential coefficients). The solution $G(u)$ is not in general a wavelet (in the standard Lebesgue measure of $\mathbb{R}$). However, the solutions to the original MADE $f^{(1)}(t) = f(qt)$ for t in all of $\mathbb{R}$ are seen to be wavelets in general, as seen in Section 4 below, and they have physical applications, see for instance [25]. This is strong supporting evidence that the original MADE version of the differential equation, namely (1), is the natural version to study. The study of such solutions of (1), along with their extensions to the negative real numbers, is a main focal point of this work.

Remark 4. If one replaces e^{-t} by z in (2), sets $q > 1$ to be an integer, and restricts the index m to lie in $\mathbb{N}_0$, then the resulting series is $\sum_{m=0}^{\infty}(-1)^m z^{q^m}/q^{m(m-\mu)/\lambda}$. This series in z has radius of convergence equal to 1. Furthermore, the spectral gaps are $q^{m+1}/q^m = q > 1$, and by the Ostrowski–Hadamard theorem (see [10]) the series cannot be analytically extended beyond the boundary, namely the unit circle. Thus this series in z is lacunary, and extension of the original complexified series (2) beyond the imaginary axis is problematic.

Remark 5. We mention that one can also study multiplicatively advanced fractional differential equations. Let $\mathcal{F}[f(t)](\omega)$ denote the Fourier transform of $f(t)$. For $\nu > 0$, take the ν-th fractional derivative of $f(t)$ be

$$f^{(\nu)}(t) \equiv \mathcal{F}^{-1}[(\mathrm{i}\omega)^{\nu}\mathcal{F}[f(t)](\omega)](t),$$

for appropriate branch of $(\mathrm{i}\omega)^{\nu}$, as is done for instance in [29]. Then for $q > 1$ and for $\theta(q; u)$ the Jacobi theta function given in (22), let $g(\omega) \equiv [1/(\omega\theta(q^{\nu}; \omega^{\nu}))]$ for $\omega > 0$ and $g(\omega) \equiv 0$ for $\omega \le 0$. We show in Remark 9 in Section 2.1 that $f(t) = \mathcal{F}^{-1}[g(\omega)](t)$ satisfies the multiplicatively advanced fractional differential equation $f^{(\nu)}(t) = \mathrm{i}^{\nu} f(qt)$, analogous to (1).

2. Solutions $f_{\mu,\lambda}(t)$ of the MADE $f^{(n)}(t)=\alpha f(\beta t)$ and their properties

The main goal of this section is to obtain the MADE satisfied by $f_{\mu,\lambda}(t)$ for $\lambda\in\mathbb{Q}^+$, namely (18). This is accomplished in Theorem 2.2. The equivalence of the MADE (1) to the MADE (18) is established. In addition, related properties of the $f_{\mu,\lambda}(t)$ are discussed. We begin with two reduction formulas.

Proposition 2.1. *Let $q>1$, and $\lambda\in\mathbb{R}^+$. The following reduction formulas hold. For $\mu_1,\mu_2\in\mathbb{R}$ with $\mu_2-\mu_1=2L\in 2\mathbb{Z}$*

$$f_{\mu_2,\lambda}(t)=(-1)^L q^{L(L+\mu_1)/\lambda} f_{\mu_1,\lambda}(q^L t)\ . \tag{7}$$

For $\mu_1,\mu_2\in\mathbb{R}$ with $\mu_2-\mu_1=2L+1$

$$f_{\mu_2,\lambda}(t)=(-1)^L q^{L(L+\mu_1+1)/\lambda} f_{\mu_1+1,\lambda}(q^L t)\ . \tag{8}$$

Proof. For $\mu_1,\mu_2\in\mathbb{R}$ with $\mu_2-\mu_1=2L\in 2\mathbb{Z}$

$$f_{\mu_2,\lambda}(t)=\sum_{m=-\infty}^{\infty}(-1)^m\frac{\mathrm{e}^{-q^m t}}{q^{m(m-\mu_2)/\lambda}}=\sum_{m=-\infty}^{\infty}(-1)^m\frac{\mathrm{e}^{-q^m t}}{q^{m(m-2L-\mu_1)/\lambda}}$$

$$=\sum_{M=-\infty}^{\infty}(-1)^{M+L}\frac{\mathrm{e}^{-q^{M+L}t}}{q^{(M+L)(M-L-\mu_1)/\lambda}} \tag{9}$$

$$=\sum_{M=-\infty}^{\infty}(-1)^{M+L}\frac{\mathrm{e}^{-q^{M}(q^L t)}}{q^{(-L^2-L\mu_1)/\lambda}q^{M(M-\mu_1)/\lambda}}\ , \tag{10}$$

where the reindexing $m=M+L$ occurs in (9), yielding (10) which is equivalent to (7). The odd case $\mu_2-\mu_1=2L+1$ in (8) follows immediately if one replaces μ_1 by μ_1+1 in (7). □

Setting $\mu_2=2L$ and $\mu_1=0$ in (7), and $\mu_2=2L+1$ and $\mu_1=0$ in (8), respectively, yields:

Corollary 2.1. *Let $q>1$, and $\lambda\in\mathbb{R}^+$. For any $L\in\mathbb{Z}$,*

$$f_{2L,\lambda}(t)=(-1)^L q^{L^2/\lambda} f_{0,\lambda}(q^L t)\quad\text{and}\quad f_{2L+1,\lambda}(t)=(-1)^L q^{L(L+1)/\lambda} f_{1,\lambda}(q^L t)\ .$$

The derivative $f'_{\mu,\lambda}(t)$ is computed from (2) as

$$f'_{\mu,\lambda}(t)=\frac{\mathrm{d}f_{\mu,\lambda}(t)}{\mathrm{d}t}=\sum_{m=-\infty}^{\infty}(-1)^m\frac{\mathrm{e}^{-q^m t}(-q^m)}{q^{m(m-\mu)/\lambda}}$$

$$=-\sum_{m=-\infty}^{\infty}(-1)^m\frac{\mathrm{e}^{-q^m t}}{q^{m(m-\mu-\lambda)/\lambda}}=-f_{\mu+\lambda,\lambda}(t)\ , \tag{11}$$

from which one concludes that, for $n\geq 0$, the higher derivatives satisfy

$$f^{(n)}_{\mu,\lambda}(t)=(-1)^n\sum_{m=-\infty}^{\infty}(-1)^m\frac{\mathrm{e}^{-q^m t}}{q^{m(m-\mu-n\lambda)/\lambda}}=(-1)^n f_{\mu+n\lambda,\lambda}(t)\ . \tag{12}$$

Next, for $t\geq 0$, we examine the behavior of $f_{\mu,\lambda}(t)$ for rational values of λ.

Theorem 2.2. *For $\lambda=2L/k$ with $L,k\in\mathbb{Z}^+$ one has*

$$f^{(k)}_{\mu,\lambda}(t)=(-1)^{k+L}q^{L(L+\mu)/\lambda}f_{\mu,\lambda}(q^L t)\ . \tag{13}$$

For $\lambda=(2L+1)/k$ with $L\in\mathbb{Z}^+_0$, $k\in\mathbb{Z}^+$ one has

$$f^{(k)}_{\mu,\lambda}(t)=(-1)^{k+L}q^{L(L+\mu+1)/\lambda}f_{\mu+1,\lambda}(q^L t)\ , \tag{14}$$

and furthermore, for $\lambda=(2L+1)/k$, one has

$$f^{(2k)}_{\mu,\lambda}(t)=-q^{(2L+1)(2L+1+\mu)/\lambda}f_{\mu,\lambda}(q^{2L+1}t)\ . \tag{15}$$

Define the advancing power A to be the power of q in the coefficient of t on the right hand side of each of (13) and (15) as follows

$$A \equiv \begin{cases} L & \text{if} \quad \lambda = (2L)/k \\ 2L+1 & \text{if} \quad \lambda = (2L+1)/k \,, \end{cases} \tag{16}$$

and denote the order R of the MADE in (13) and (15) by

$$R \equiv \begin{cases} k & \text{if} \quad \lambda = (2L)/k \\ 2k & \text{if} \quad \lambda = (2L+1)/k \,. \end{cases} \tag{17}$$

Then (13) and (15) can be unified as the following MADE

$$f^{(R)}_{\mu,\lambda}(t) = (-1)^{R+A} q^{A(A+\mu)/\lambda} f_{\mu,\lambda}(q^A t) \,, \tag{18}$$

where A and R are as in (16) and (17), respectively, with $\lambda = 2A/R$.

Proof. To prove (13), observe that $k\lambda = k(2L/k) = 2L$. Applying (12) by setting $n = k$ yields

$$f^{(k)}_{\mu,\lambda}(t) = (-1)^k f_{\mu+k\lambda,\lambda}(t) = (-1)^k f_{\mu+2L,\lambda}(t) = (-1)^k(-1)^L q^{L(L+\mu)/\lambda} f_{\mu,\lambda}(q^L t) \,, \tag{19}$$

where the last equality in (19) follows from the reduction formula (7). To prove (14), observe that $k\lambda = k([2L+1]/k) = 2L+1$. Applying (12) by setting $n = k$ yields

$$f^{(k)}_{\mu,\lambda}(t) = (-1)^k f_{\mu+k\lambda,\lambda}(t) = (-1)^k f_{\mu+2L+1,\lambda}(t) = (-1)^k(-1)^L q^{L(L+\mu+1)/\lambda} f_{\mu+1,\lambda}(q^L t) \,, \tag{20}$$

where the last equality in (20) follows from the reduction formula (8). To prove (15), observe that $2k\lambda = 2k([2L+1]/k) = 2(2L+1)$. Applying (12) by setting $n = 2k$ yields

$$f^{(2k)}_{\mu,\lambda}(t) = (-1)^{2k} f_{\mu+2k\lambda,\lambda}(t) = f_{\mu+2(2L+1),\lambda}(t) = (-1)^{2L+1} q^{(2L+1)(2L+1+\mu)/\lambda} f_{\mu,\lambda}(q^{2L+1} t) \,, \tag{21}$$

where the last equality in (21) follows from the reduction formula (7).

Now (18) follows immediately from (13) and (15) via (16) and (17). □

Remark 6. The significance of (13) and (15) is that the functions $f_{\mu,\lambda}(t)$ all satisfy multiplicatively advanced differential equations (MADEs) in that the variable t in the argument of $f_{\mu,\lambda}$ in the right hand side of (13), respectively (15), has been multiplicatively advanced by the scaling $q^L > 1$, respectively $q^{2L+1} > 1$. And thus one arrives at the MADE (18), where the right hand side is multiplicatively advanced by $q^A > 1$, with $A = L$ or $A = 2L+1$ respectively. Hence we have the terminology of A as the advancing power and R as the order of (18).

Remark 7. Observe that the MADE (18) is equivalent to the MADE (1) via the following process. First, choose R in (18) to be n in (1). Second, choose A in (18) so that the sign of $(-1)^{R+A}$ in (18) equals the sign of α in (1). Third choose q in (18) so that q^A in (18) equals β in (1), that is set $q = \beta^{1/A}$. Fourth, set λ in (18) to be $2A/R$. Finally, choose μ in (18) so that $q^{A(A+\mu)/\lambda}$ in (18) equals $|\alpha|$ in (1), that is, take $\mu = -A + [2\ln(|\alpha)|]/[R\ln(q)]$. Thus we have recovered the MADE (1) from the MADE (18).

Remark 8. As the parameter $q \to 1^+$, the MADE (18) for fixed R and A can be interpreted as a perturbation of the standard differential equation $f^{(R)}(t) = (-1)^{R+A} f(t)$, and the solutions of (18) can be interpreted as perturbations of the solutions of $f^{(R)}(t) = (-1)^{R+A} f(t)$ in an appropriate neighborhood of $t = 0$.

2.1. The Jacobi theta function and its relation to $f_{\mu,\lambda}(0)$

Next, recall that for $q > 1$ the Jacobi theta function is given by

$$\theta(q;u) = \sum_{n=-\infty}^{\infty} \frac{u^n}{q^{n(n-1)/2}} = \mu_q \prod_{n=0}^{\infty} \left(1 + \frac{u}{q^n}\right)\left(1 + \frac{1}{uq^{n+1}}\right) \,, \tag{22}$$

where

$$\mu_q = \prod_{n=0}^{\infty} \left(1 - \frac{1}{q^{n+1}}\right) . \tag{23}$$

One property of the Jacobi theta function to be used later is that for all $p \in \mathbb{Z}$

$$\theta(q; q^p u) = q^{p(p+1)/2} u^p \theta(q; u) \ . \tag{24}$$

The Jacobi theta function plays a major role in this study, especially in expressing initial conditions, invertibility criteria, and residues in Fourier transform computations.

From the product formula in (22), one sees that

$$\theta(q; u) = 0 \iff u = -q^p \text{ for some } p \in \mathbb{Z} \ . \tag{25}$$

From the summation formula in (22) one also sees the following useful computational lemma.

Lemma 2.3. *For each $q > 1$, $\lambda > 0$, $\mu \in \mathbb{R}$, and $a \in \mathbb{R}$, one has*

$$\sum_{n=-\infty}^{\infty} \frac{a^n}{q^{n(n-\mu)/\lambda}} = \theta(q^{2/\lambda}; aq^{(\mu-1)/\lambda}) \ . \tag{26}$$

Proof. Consider

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{a^n}{q^{n(n-\mu)/\lambda}} &= \sum_{n=-\infty}^{\infty} \frac{a^n (q^{\mu/\lambda})^n}{q^{n(n-0)/\lambda}} \frac{q^{-n/\lambda}}{q^{-n/\lambda}} = \sum_{n=-\infty}^{\infty} \frac{(aq^{(\mu-1)/\lambda})^n}{q^{n(n-1)/\lambda}} \\ &= \sum_{n=-\infty}^{\infty} \frac{(aq^{(\mu-1)/\lambda})^n}{(q^{2/\lambda})^{n(n-1)/2}} = \theta(q^{2/\lambda}; aq^{(\mu-1)/\lambda}) \ , \end{aligned} \tag{27}$$

where the second equality in (27) follows from the summation formula in (22). □

From Lemma 2.3 one concludes the following.

Lemma 2.4. *For each $q > 1$, $\lambda > 0$, $\mu \in \mathbb{R}$ one has*

$$f_{\mu,\lambda}(0) = \theta(q^{2/\lambda}; -q^{(\mu-1)/\lambda}) \ . \tag{28}$$

Proof. From (2) one sees that

$$f_{\mu,\lambda}(0) = \sum_{n=-\infty}^{\infty} (-1)^n \frac{e^{-q^n \cdot 0}}{q^{n(n-\mu)/\lambda}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{q^{n(n-\mu)/\lambda}} = \theta(q^{2/\lambda}; (-1)q^{(\mu-1)/\lambda}) \ ,$$

where the last equality holds from (26). This proves the Lemma. □

From Lemma 2.4 in conjunction with (25), one concludes that

$$\begin{aligned} & f_{\mu,\lambda}(0) = 0 \\ \iff & \theta(q^{2/\lambda}; -q^{(\mu-1)/\lambda}) = 0 \\ \iff & \exists \ p \in \mathbb{Z} \text{ with} - q^{(\mu-1)/\lambda} = -q^{2p/\lambda} \\ \iff & \mu = 2p + 1 \text{ is an odd integer.} \end{aligned} \tag{29}$$

From the vanishing criterion (29) one obtains the following flatness criterion.

Proposition 2.2. *$f_{\mu,\lambda}$ is flat at 0 provided any one of the following equivalent conditions holds:*

$$f_{\mu,\lambda} \text{ is flat at } 0$$

$$\iff \forall\, k \in \mathbb{N}_0 \ \text{ one has } \ f^{(k)}_{\mu,\lambda}(0) = (-1)^k f_{\mu+k\lambda,\lambda}(0) = 0 \tag{30}$$

$$\iff f_{\mu,\lambda}(0) = 0 \ \text{ and } \ f'_{\mu,\lambda}(0) = 0 \tag{31}$$

$$\iff \exists\, n \in \mathbb{N}_0 \text{ with } f^{(n)}_{\mu,\lambda}(0) = 0 \ \text{ and } \ f^{(n+1)}_{\mu,\lambda}(0) = 0 \tag{32}$$

$$\iff \mu \text{ is an odd integer and } \lambda \text{ is an even integer} \ . \tag{33}$$

Proof. The equivalence in (30) is by definition. The forward implications in (30) $\Rightarrow$ (31) and (31) $\Rightarrow$ (32) are automatic. To show that (32) $\Rightarrow$ (33), assume $f^{(n)}_{\mu,\lambda}(0) = (-1)^n f_{\mu+n\lambda,\lambda}(0) = 0$ and $f^{(n+1)}_{\mu,\lambda}(0) = (-1)^{n+1} f_{\mu+(n+1)\lambda,\lambda}(0) = 0$. From the vanishing criterion (29), we must have that both $\mu + n\lambda$ and $\mu + (n+1)\lambda$ are odd integers. Thus, as the difference of two odd integers, one has $(\mu + (n+1)\lambda) - (\mu + n\lambda) = \lambda$ is an even integer. Thus, as the difference of an odd and even integer, one has $(\mu + n\lambda) - n(\lambda) = \mu$ is an odd integer, giving the implication (32) $\Rightarrow$ (33). Finally, assume μ is odd and λ is even. Then for all $k \in \mathbb{N}_0$, one has $\mu + k\lambda$ is odd. Again, by the vanishing criterion (29), one has $f^{(k)}_{\mu,\lambda}(0) = (-1)^k f_{\mu+k\lambda,\lambda}(0) = 0$ for all $k \in \mathbb{N}_0$, and thus $f_{\mu,\lambda}$ is flat at 0, and (33) $\Rightarrow$ (30) is shown. □

Remark 9. We now show Remark 5 of Section 1 regarding multiplicatively advanced fractional differential equations. Let $\mathcal{F}[f(t)](\omega)$ denote the Fourier transform of $f(t)$ given by (98). For $\nu > 0$, take the ν-th fractional derivative of $f(t)$ to be $f^{(\nu)}(t) \equiv \mathcal{F}^{-1}[(\mathrm{i}\omega)^\nu \mathcal{F}[f(t)](\omega)](t)$, for appropriate branch of $(\mathrm{i}\omega)^\nu$. By direct computation, one sees in general that $\mathcal{F}[f(qt)](\omega) = (1/q)\mathcal{F}[f(t)](\omega/q)$. Then for $q > 1$, let $g(\omega) \equiv [1/(\omega\theta(q^\nu;\omega^\nu))]$ for $\omega > 0$ and $g(\omega) \equiv 0$ for $\omega \le 0$. Then for $\omega > 0$

$$\begin{aligned}(\mathrm{i}\omega)^\nu g(\omega) &= (\mathrm{i}\omega)^\nu[1/(\omega\theta(q^\nu;\omega^\nu))] \;=\; \mathrm{i}^\nu[1/(\omega\omega^{-\nu}\theta(q^\nu;\omega^\nu))]\\ &= \mathrm{i}^\nu[1/(\omega(q^\nu)^{(-1)(-1+1)/2}\omega^{-\nu}\theta(q^\nu;\omega^\nu))]\\ &= \mathrm{i}^\nu[1/(\omega(q^\nu)^{(-1)(-1+1)/2}(\omega^\nu)^{-1}\theta(q^\nu;\omega^\nu))]\\ &= \mathrm{i}^\nu[1/(\omega\theta(q^\nu;(\omega/q)^\nu))] \;=\; \mathrm{i}^\nu(1/q)[1/([\omega/q]\theta(q^\nu;(\omega/q)^\nu))] \;=\; \mathrm{i}^\nu(1/q)g(\omega/q)\,,\end{aligned} \tag{34}$$

where (24) justifies the first equality in (34). When $\omega \le 0$ one has $(\mathrm{i}\omega)^\nu g(\omega) = \mathrm{i}^\nu(1/q)g(\omega/q)$ trivially, by vanishing of $g(\omega)$ in this setting. Thus for all $\omega \in \mathbb{R}$ one has $(\mathrm{i}\omega)^\nu g(\omega) = \mathrm{i}^\nu(1/q)g(\omega/q)$. Taking inverse Fourier transforms of each side of this last equality, and setting $f(t) = \mathcal{F}^{-1}[g(\omega)](t)$, gives that $f^{(\nu)}(t) = \mathrm{i}^\nu f(qt)$.

2.2. Non-analyticity of extensions of $f_{\mu,\lambda}(t)$ at $t = 0$

Relying on Proposition 2.2 along with (7) and (12), one sees that neither $f_{\mu,\lambda}(t)$ nor any of its extensions can be analytic at $t = 0$, as is seen in the next proposition.

Proposition 2.3. *For $\lambda \in \mathbb{Q}^+$, the function $f_{\mu,\lambda}(t)$ (and any extension of $f_{\mu,\lambda}(t)$ to $(-\infty, 0)$ with matching derivatives) is not analytic at $t = 0$.*

Proof. With A as in (16) and R as in (17), one has $\lambda = 2A/R$. For each $M \in \mathbb{N}_0$ one has

$$f^{(j+MR)}_{\mu,\lambda}(t) = (-1)^{j+MR} f_{\mu+[j+MR]\lambda,\lambda}(t) = (-1)^j(-1)^{MR} f_{\mu+j\lambda+M2A,\lambda}(t) \tag{35}$$

$$= (-1)^j(-1)^{MR}(-1)^{MA}q^{MA(MA+\mu+j\lambda)/\lambda} f_{\mu+j\lambda,\lambda}(q^{MA}t)\,, \tag{36}$$

where the first equality in (35) follows from (12), and the second equality in (35) follows from the fact that $R\lambda = 2A$. Equality in (36) follows from the reduction formula (7). Setting $t = 0$ above gives

$$\begin{aligned}f^{(j+MR)}_{\mu,\lambda}(0) &= (-1)^j f_{\mu+j\lambda,\lambda}(0)(-1)^{MR}(-1)^{MA}q^{MA(MA+\mu+j\lambda)/\lambda}\\ &= f^{(j)}_{\mu,\lambda}(0)(-1)^{M(R+A)}q^{MA(MA+\mu+j\lambda)/\lambda}\,,\end{aligned} \tag{37}$$

where the equality in (37) follows from (12).

Now if $f_{\mu,\lambda}(t)$ is flat at $t = 0$, then $f_{\mu,\lambda}(t)$ cannot be analytic at 0 in that $f_{\mu,\lambda}(t)$ would be the identically 0 function, as all of its derivatives are 0. However, such a flat $f_{\mu,\lambda}(t)$ cannot be identically zero (as is shown Corollary 6.4 in Section 6 below). So we assume that $f_{\mu,\lambda}(t)$ is not flat.

In the non-flat case, if $f_{\mu,\lambda}(t)$ were to be analytic at $t = 0$, one would have that

$$f_{\mu,\lambda}(z) = \sum_{n\ge 0} \frac{f^{(n)}_{\mu,\lambda}(0)}{n!} z^n \tag{38}$$

converges with radius of convergence $R_1 > 0$. By Proposition 2.2, Equation (31), one has that $f^{(j)}_{\mu,\lambda}(0) \ne 0$ either for $j = 0$ or for $j = 1$. Fix one such $j = 0$ or $j = 1$ with $f^{(j)}_{\mu,\lambda}(0) \ne 0$, and note that from (38), one would obtain that

$$\sum_{M\ge 0} \frac{f^{(j+MR)}_{\mu,\lambda}(0)}{(j+MR)!} z^{j+MR} \tag{39}$$

converges, with radius of convergence $R_2 \geq R_1 > 0$. Dividing by z^j and setting $u = z^R$ gives that

$$\sum_{M\geq 0} a_M u^M \quad \text{with} \quad a_M = \frac{f^{(j+MR)}_{\mu,\lambda}(0)}{(j+MR)!} \tag{40}$$

converges, with u radius of convergence $R_u = R_2^R > 0$. However, the ratio test gives

$$\begin{aligned}\left|\frac{a_{M+1}}{a_M}\right| &= \left|\frac{f^{(j+[M+1]R)}_{\mu,\lambda}(0)}{(j+[M+1]R)!} \Big/ \frac{f^{(j+MR)}_{\mu,\lambda}(0)}{(j+MR)!}\right| \\ &= \left|\frac{f^{(j)}_{\mu,\lambda}(0)(-1)^{[M+1][R+A]}q^{[M+1]A([M+1]A+\mu+j\lambda)/\lambda}}{f^{(j)}_{\mu,\lambda}(0)(-1)^{M[R+A]}q^{MA(MA+\mu+j\lambda)/\lambda}}\right| \cdot \frac{(j+MR)!}{(j+[M+1]R)!} \qquad (41)\\ &= \frac{q^{2MA^2/\lambda}q^{A[\mu+j\lambda+A]/\lambda}}{(j+MR+1)(j+MR+2)\dots(j+MR+R)}, \qquad (42)\end{aligned}$$

where equality in (41) follows from (37). Now (42) implies that the ratio $|a_{M+1}/a_M|$ approaches infinity as $M \to \infty$, yielding a radius $R_u = 0$. This contradicts that $R_u = R_2^R > 0$ from above. Thus $f_{\mu,\lambda}(t)$ cannot be analytic at 0. □

3. Extending $f_{\mu,\lambda}(t)$ to $t < 0$ for rational $\lambda > 0$

In order to compute the Fourier transforms of $f_{\mu,\lambda}(t)$, we will need to obtain an extension of $f_{\mu,\lambda}(t)$ from $[0,\infty)$ to the whole real line. Such an extension is the goal of this section, and is accomplished in Theorem 3.2.

We first look for a class of potential extension functions satisfying the same MADE (18) as does $f_{\mu,\lambda}(t)$. Furthermore, the derivatives of the extension should match the derivatives of $f_{\mu,\lambda}(t)$ at $t = 0$. The extensions will be constructed from the following class of functions $h(c,t)$ as given immediately below.

Let $q > 1$ be fixed. For fixed $\mu \in \mathbb{R}$, $c \in \mathbb{C}^*$, and $\lambda > 0$, define the functions

$$h(c,t) \equiv h_{\mu,\lambda}(c,t) \equiv \sum_{n=-\infty}^{\infty} a_n \frac{\mathrm{e}^{-q^n ct}}{q^{n(n-\mu)/\lambda}}, \tag{43}$$

where, for now, the $a_n \in \mathbb{C}$ are arbitrary, with the two constraints that:

$$\sum_{n=-\infty}^{\infty} \frac{|a_n|}{q^{n(n-\mu)/\lambda}} \quad \text{converges, and} \tag{44}$$

$$\mathcal{R}(-ct) \leq 0 \quad \text{where } \mathcal{R}(z) \text{ denotes the real part of } z. \tag{45}$$

These two constraints give that $h(c,t)$ is bounded and converges absolutely for t in the appropriate half-line: $(-\infty,0]$ when $\mathcal{R}(c) \leq 0$; or $[0,\infty)$ when $\mathcal{R}(c) \geq 0$. For now, the coefficients a_n are arbitrary with (44)–(45) holding; however, later, in order to have (43) satisfy the MADE (18), certain of the coefficients (namely $a_0,\dots,a_{A-1}$) will be freely chosen from $\mathbb{R}$ and the remaining coefficients a_n will depend on both the freely chosen coefficients $a_0,\dots,a_{A-1}$ and on the argument c, which is selected from the arguments of a class of scaled roots of unity, as determined in (48) through (55) below.

Note also that, for t in the appropriate interval of convergence (namely $(-\infty,0]$ or $[0,\infty)$), one has

$$h^{(m)}(c,t) \equiv \frac{\mathrm{d}^m h}{\mathrm{d}t^m}(c,t) = (-c)^m \sum_{n=-\infty}^{\infty} a_n \frac{\mathrm{e}^{-q^n ct}(q^n)^m}{q^{n(n-\mu)/\lambda}} = (-c)^m \sum_{n=-\infty}^{\infty} a_n \frac{\mathrm{e}^{-q^n ct}}{q^{n(n-\mu-m\lambda)/\lambda}}, \tag{46}$$

when $m \geq 0$. When $m < 0$, we take the $|m|$-th anti-derivative of $h(c,t)$ to be

$$h^{(m)}(c,t) = (-c)^m \sum_{n=-\infty}^{\infty} a_n \frac{\mathrm{e}^{-q^n ct}(q^n)^m}{q^{n(n-\mu)/\lambda}} = (-c)^m \sum_{n=-\infty}^{\infty} a_n \frac{\mathrm{e}^{-q^n ct}}{q^{n(n-\mu-m\lambda)/\lambda}}, \tag{47}$$

as in (85)–(87), where any polynomial term $p_m(t)$ of degree $|m|-1$ due to integration is set equal to 0 in (47). One reason for this is that the anti-derivatives $h^{(m)}(c,t)$ approach 0 at $\pm\infty$ precisely when $p_m(t) \equiv 0$, and when $\mathcal{R}(-ct) < 0$ as discussed in Section 8.

Let $q > 1$, $\mu \in \mathbb{R}$, $c \in \mathbb{C}^*$, and $k \in \mathbb{N}$ be fixed. We now let $\lambda > 0$ be rational with either; i) $\lambda = 2L/k$ with $L \in \mathbb{N}$, or ii) $\lambda = (2L+1)/k$ with $L \in \mathbb{N}_0$. Recall from the definition (16) and (17) that: $A = L$ and $R = k$ in the even numerator case $\lambda = 2L/k$; and $A = 2L+1$ and $R = 2k$ in the odd numerator case $\lambda = (2L+1)/k$. Then $\lambda = 2A/R$ in both cases. Let $h(c,t) = h_{\mu,\lambda}(c,t)$ be as in (43) above.

In order for $h(c,t)$ to satisfy the same MADE as $f_{\mu,\lambda}(t)$, namely (18), one must have that

$$h^{(R)}(c,t) = (-1)^{R+A} q^{A(A+\mu)/\lambda} h(c, q^A t) = (-1)^{R+A} q^{A(A+\mu)/\lambda} \sum_{n=-\infty}^{\infty} a_n \frac{e^{-q^n c(q^A t)}}{q^{n(n-\mu)/\lambda}}, \tag{48}$$

where the first equality in (48) follows from the hypothesis that (18) is satisfied, and the second equality in (48) follows from (43).

Observe that $R\lambda = 2A$, (that is in the even numerator case $R\lambda = k(2L/k) = 2L = 2A$, and in the odd numerator case $R\lambda = (2k)([2L+1]/k) = 2[2L+1] = 2A$). Thus one has

$$h^{(R)}(c,t) = (-c)^R \sum_{n=-\infty}^{\infty} a_n \frac{e^{-q^n ct}}{q^{n(n-\mu-R\lambda)/\lambda}} = (-c)^R \sum_{n=-\infty}^{\infty} a_n \frac{e^{-q^n ct}}{q^{n(n-\mu-2A)/\lambda}} \tag{49}$$

$$= (-1)^R c^R \sum_{N=-\infty}^{\infty} a_{N+A} \frac{e^{-q^{N+A} ct}}{q^{(N+A)(N-A-\mu)/\lambda}} \tag{50}$$

$$= (-1)^R c^R \sum_{N=-\infty}^{\infty} a_{N+A} \frac{e^{-q^{N+A} ct}}{q^{-(A^2+A\mu)/\lambda} q^{N(N-\mu)/\lambda}}$$

$$= (-1)^{R+A} q^{A(A+\mu)/\lambda} \sum_{N=-\infty}^{\infty} (-1)^A c^R a_{N+A} \frac{e^{-q^N c(q^A t)}}{q^{N(N-\mu)/\lambda}}, \tag{51}$$

where the R-th derivative in (49) is obtained from (46), the substitution $R\lambda = 2A$ gives the second equality in (49), and the re-indexing $n = N + A$ occurs in (50). One can indeed equate (51) with the right most expression in (48) if $a_N = (-1)^A c^R a_{N+A}$, for all $N \in \mathbb{Z}$. Thus we have that $h(c,t)$ satisfies the same MADE (18) as $f_{\mu,\lambda}(t)$ if

$$a_{N+A} = \frac{(-1)^A}{c^R} a_N \quad \forall\, N \in \mathbb{Z}, \tag{52}$$

which is equivalent to

$$a_{N+MA} = \left[\frac{(-1)^A}{c^R}\right]^M a_N \quad \forall\, N, M \in \mathbb{Z}. \tag{53}$$

We will work with real coefficients a_N, which occurs precisely when c^R and $a_0, \ldots, a_{A-1}$ are real. Letting $c = \gamma e^{i\theta}$ with $\gamma > 0$, one has that $c^R = \gamma^R e^{iR\theta}$ is real, or, equivalently, $e^{iR\theta} = \pm 1$. Thus $e^{i\theta}$ is an R-th root of ± 1, and $c = \gamma\omega^p$ with $\omega = e^{2\pi i/R}$, or $c = \gamma e^{i\pi/R}\omega^p$, for some $p = 0, \ldots, R-1$. In this setting of real coefficients, (53) becomes

$$a_{N+MA} = \left[\frac{(-1)^A}{(-1)^\ell \gamma^R}\right]^M a_N \ \ \forall\, N \in \mathbb{Z} \ \text{ when } c = \gamma e^{i\pi\ell/R}\omega^p \text{ with } \ell = 0, 1. \tag{54}$$

Thus, when (54) holds,

$$h(c,t) = \sum_{n=-\infty}^{\infty} a_n \frac{e^{-q^n ct}}{q^{n(n-\mu)/\lambda}} = \sum_{j=0}^{A-1} \sum_{M=-\infty}^{\infty} a_{MA+j} \frac{e^{-q^{MA+j} ct}}{q^{(MA+j)(MA+j-\mu)/\lambda}}$$

$$= \sum_{j=0}^{A-1} \sum_{M=-\infty}^{\infty} \left[\frac{(-1)^{A-\ell}}{\gamma^R}\right]^M a_j \frac{e^{-q^{MA+j}\gamma e^{i\pi\ell/R}\omega^p t}}{q^{(MA+j)(MA+j-\mu)/\lambda}}, \tag{55}$$

which is parametrized by $a_j \in \mathbb{R}$ for $j = 0, \ldots, A-1$, $\gamma \in \mathbb{R}^+$, $\ell = 0, 1$, and $p = 0, \ldots, R-1$, where the parameters ℓ and p are chosen for convergence on the appropriate interval $(-\infty, 0]$ or $[0, \infty)$. Thus a parameter space for the $h(c,t)$ as in (55) with real coefficients satisfying the MADE (18) is $\mathbb{R}^A \times \mathbb{R}^+ \times \mathbb{Z}_2 \times \mathbb{Z}_R$. In (55), we have first that (45) holds by choice of ℓ and p. Secondly, (44) holds by bounding (55) as follows:

$$|h(c,t)| \le \sum_{j=0}^{A-1} |a_j| \sum_{M=-\infty}^{\infty} \left[\frac{1}{\gamma^R}\right]^M \frac{1}{q^{(MA+j)(MA+j-\mu)/\lambda}}$$

$$\le \sum_{j=0}^{A-1} |a_j| q^{(-j^2+\mu j)/\lambda} \sum_{M=-\infty}^{\infty} \frac{1}{q^{MA(MA+2j-\mu+\lambda[R/A]\log_q \gamma)/\lambda}} < \infty. \tag{56}$$

Complexifying t to obtain z, the bound (56) also holds on the modulus of $h(c,z)$ for the half-plane $\mathcal{R}(-cz) \le 0$. As the uniform limit of the analytic functions given by the truncated sums, $h(c,z)$ is analytic on the open half-plane $\mathcal{R}(-cz) < 0$, as in [28]. Thus, for $t = \mathcal{R}(z)$, $h(c,t)$ is real analytic in t on the appropriate ray, say $(-\infty, 0)$, contained in $\mathcal{R}(-cz) < 0$ and $\mathcal{C}^\infty$ in t on $(-\infty, 0]$. For these values of t, it is also real analytic in the parameter $\mu \in (-\infty, \infty)$, however it is only $\mathcal{C}^\infty$ in the parameter $q > 1$.

Since the MADE in (18) is linear, we shall look for extensions $h(t)$ of $f_{\mu,\lambda}(t)$ of form

$$h(t) = \sum_{r=0}^{R-1} b_r h(c_r, t) = \sum_{r=0}^{R-1} b_r \left[\sum_{j=0}^{A-1} \sum_{M=-\infty}^{\infty} \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R} \right]^M a_{j,r} \frac{\mathrm{e}^{-q^{MA+j}\gamma_r \mathrm{e}^{\mathrm{i}\pi\ell_r/R}\omega^{p_r} t}}{q^{(MA+j)(MA+j-\mu)/\lambda}} \right] \tag{57}$$

with each $h(c_r, t)$ defined on $(-\infty, 0]$. From the discussion in the previous paragraph, $h(z)$ will be analytic on the open wedge emanating from 0 given by the intersection of the half-planes $\mathcal{R}(-c_r z) < 0$ for $r = 0, \ldots, R-1$. Thus, for $(-\infty, 0)$ contained in this wedge, $h(t)$ is: real analytic in t on $(-\infty, 0)$, $\mathcal{C}^\infty$ in t on $(-\infty, 0]$; real analytic in the parameter $\mu \in (-\infty, \infty)$; and $\mathcal{C}^\infty$ in the parameter $q > 1$. If $\mathcal{R}(-c_r t) = 0$ for even one r, then $h(t)$ is only $\mathcal{C}^\infty$ in t on $(-\infty, 0]$ but still real analytic in μ on $(-\infty, \infty)$. From (55)–(57) one observes that $h(t)$ is bounded and converging uniformly on $(-\infty, 0]$. Because the b_r values can be absorbed into the $a_{j,r}$ the parameter spaces of such $h(t)$ consists of sheets of $[\mathbb{R}^A \times \mathbb{R}^+]^R$, depending on a choice of values in $\mathbb{Z}_2 \times \mathbb{Z}_R$ that allows for extension.

Next, we insure the first 0 through $R-1$ derivatives of $h(t)$ match those of $f_{\mu,\lambda}(t)$ at 0. That is, for $h(t)$ as in (57), we find values of b_r so that

$$f^{(m)}_{\mu,\lambda}(0) = h^{(m)}(0) = \sum_{r=0}^{R-1} b_r h^{(m)}(c_r, 0)\,, \tag{58}$$

for $m = 0, 1, \ldots, R-1$. From the MADE (18), we shall see shortly that (58) is sufficient to guarantee that all derivatives and anti-derivatives of $f_{\mu,\lambda}(t)$ and $h(t)$ (in the sense of (85)–(87) below) will match at 0. In matrix form, (58) becomes

$$\begin{bmatrix} f_{\mu,\lambda}(0) \\ f^{(1)}_{\mu,\lambda}(0) \\ \vdots \\ f^{(m)}_{\mu,\lambda}(0) \\ \vdots \\ f^{(R-1)}_{\mu,\lambda}(0) \end{bmatrix} = \begin{bmatrix} h(c_0,0) & h(c_1,0) & \cdots & h(c_r,0) & \cdots & h(c_{R-1},0) \\ h^{(1)}(c_0,0) & h^{(1)}(c_1,0) & \cdots & h^{(1)}(c_r,0) & \cdots & h^{(1)}(c_{R-1},0) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ h^{(m)}(c_0,0) & h^{(m)}(c_1,0) & \cdots & h^{(m)}(c_r,0) & \cdots & h^{(m)}(c_{R-1},0) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ h^{(R-1)}(c_0,0) & h^{(R-1)}(c_1,0) & \cdots & h^{(R-1)}(c_r,0) & \cdots & h^{(R-1)}(c_{R-1},0) \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_r \\ \vdots \\ b_{R-1} \end{bmatrix} \tag{59}$$

which can be solved uniquely for the various b_m when the square matrix in equation (59) has non-zero determinant. Observe that the $R \times R$ matrix in (59), which we denote by H, has (m,r) entry $H_{m,r} = h^{(m)}(c_r, 0)$, and thus H is the Wronskian matrix of the functions $h(c_0,t), h(c_1,t), \ldots, h(c_{R-1},t)$ at $t = 0$. Letting $\boldsymbol{F}$ be the column matrix with m-th entry $F_m = f^{(m)}_{\mu,\lambda}(0)$ and $\boldsymbol{B}$ be the column matrix with r-th entry $B_r = b_r$, equation (59) is succinctly rewritten as

$$\boldsymbol{F} = H \cdot \boldsymbol{B}\,. \tag{60}$$

One can reduce the invertibility of H and subsequent solution for $\boldsymbol{B}$ to statements about theta functions, by relying on (12) and (28), and by expressing the entries $H_{m,r}$ of H in terms of theta function values. To effect this for the $H_{m,r}$, we begin by proving the next lemma which is a useful extension of (26).

Lemma 3.1. *For $q > 1$, $\mu \in \mathbb{R}$, $\lambda \in \mathbb{R}^+$, $A \in \mathbb{N}$, and $j \in \mathbb{Z}$, one has that for all $a \in \mathbb{R}$:*

$$\sum_{M=-\infty}^{\infty} \frac{a^M}{q^{(MA+j)(MA+j-\mu-m\lambda)/\lambda}} = q^{-j(j-\mu-m\lambda)/\lambda}\theta(q^{2A^2/\lambda}; aq^{A[\mu+m\lambda-2j-A]/\lambda})\,. \tag{61}$$

Proof. Set $Q = q^{A^2}$, and observe

$$\begin{aligned} \sum_{M=-\infty}^{\infty} \frac{a^M}{q^{(MA+j)(MA+j-\mu-m\lambda)/\lambda}} &= q^{-j(j-\mu-m\lambda)/\lambda} \sum_{M=-\infty}^{\infty} \frac{a^M}{q^{A^2 M[M+2j/A-\mu/A-m\lambda/A]/\lambda}} \\ &= q^{-j(j-\mu-m\lambda)/\lambda} \sum_{M=-\infty}^{\infty} \frac{a^M}{Q^{M[M+2j/A-\mu/A-m\lambda/A]/\lambda}} && (62) \\ &= q^{-j(j-\mu-m\lambda)/\lambda}\theta(Q^{2/\lambda}; aQ^{[\mu/A+m\lambda/A-2j/A-1]/\lambda}) && (63) \\ &= q^{-j(j-\mu-m\lambda)/\lambda}\theta(q^{2A^2/\lambda}; aq^{A[\mu+m\lambda-2j-A]/\lambda})\,, && (64) \end{aligned}$$

where: q^{A^2} was replaced by Q (62); Lemma 2.3 was used to move from (62) to (63); and Q was replaced by q^{A^2} in (64). The lemma is now proven. □

Now, with $h(c_r, t)$ as in (55), one computes

$$h^{(m)}(c_r,t) \equiv \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\sum_{j=0}^{A-1}\sum_{M=-\infty}^{\infty}\left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right]^M a_{j,r}\,\frac{\mathrm{e}^{-q^{MA+j}\gamma_r \mathrm{e}^{\mathrm{i}\pi\ell_r/R}\omega^{p_r}t}}{q^{(MA+j)(MA+j-\mu)/\lambda}}\right]$$

$$= (-\gamma_r \mathrm{e}^{\mathrm{i}\pi\ell_r/R}\omega^{p_r})^m\left[\sum_{j=0}^{A-1} a_{j,r}\sum_{M=-\infty}^{\infty}\left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right]^M \frac{\mathrm{e}^{-q^{MA+j}\gamma_r \mathrm{e}^{\mathrm{i}\pi\ell_r/R}\omega^{p_r}t}}{q^{(MA+j)(MA+j-\mu-m\lambda)/\lambda}}\right], \tag{65}$$

which when evaluated at $t = 0$ gives

$$h^{(m)}(c_r,0) = (-\gamma_r \mathrm{e}^{\mathrm{i}\pi\ell_r/R}\omega^{p_r})^m\left[\sum_{j=0}^{A-1} a_{j,r}\sum_{M=-\infty}^{\infty}\left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right]^M \frac{1}{q^{(MA+j)(MA+j-\mu-m\lambda)/\lambda}}\right]$$

$$= (-\gamma_r \mathrm{e}^{\mathrm{i}\pi\ell_r/R}\omega^{p_r})^m \sum_{j=0}^{A-1} a_{j,r} q^{-j(j-\mu-m\lambda)/\lambda}\,\theta\left(q^{2A^2/\lambda};\left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right] q^{A[\mu+m\lambda-2j-A]/\lambda}\right), \tag{66}$$

where equality in (66) follows from (61).

Relying on (12), (28), and (66), one sees that equation (59) (equivalently (60)) takes the form

$$\boldsymbol{F} = \begin{bmatrix}\vdots\\ f^{(m)}_{\mu,\lambda}(0)\\ \vdots\end{bmatrix} = \begin{bmatrix}\vdots\\ (-1)^m f_{\mu+m\lambda,\lambda}(0)\\ \vdots\end{bmatrix} = \begin{bmatrix}\vdots\\ (-1)^m\theta(q^{2/\lambda}; -q^{(\mu+m\lambda-1)/\lambda})\\ \vdots\end{bmatrix} \tag{67}$$

$$= \begin{bmatrix}\vdots & \vdots & \vdots\\ \cdots & h^{(m)}(c_r,0) & \cdots\\ \vdots & \vdots & \vdots\end{bmatrix}\begin{bmatrix}\vdots\\ b_r\\ \vdots\end{bmatrix} \tag{68}$$

$$= \begin{bmatrix}\vdots & \vdots & \vdots\\ \cdots & (-\gamma_r \mathrm{e}^{\mathrm{i}\pi\ell_r/R}\omega^{p_r})^m \sum_{j=0}^{A-1} a_{j,r} q^{-j(j-\mu-m\lambda)/\lambda}\theta\left(q^{2A^2/\lambda};\left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right]q^{A[\mu+m\lambda-2j-A]/\lambda}\right) & \cdots\\ \vdots & \vdots & \vdots\end{bmatrix}\begin{bmatrix}\vdots\\ b_r\\ \vdots\end{bmatrix}$$

$$= H \cdot \boldsymbol{B}\,, \tag{69}$$

where the (m, r) entry $h^{(m)}(c_r, 0)$ in (68) is given by the theta values in (66). Thus (67)–(69) is in effect a statement about theta function values. Furthermore, the invertibility of H is expected generically via the rich choice of parameters $\gamma_r, \omega^{p_r}, \ell_r$, and $a_{j,r}$ giving the entries $h^{(m)}(c_r, 0)$ of H, as specified in (66).

In summary, we have reached the following theorem.

Theorem 3.2. *For $t \geq 0$, the functions*

$$f_{\mu,\lambda}(t) = \sum_{m=-\infty}^{\infty}(-1)^m \frac{\mathrm{e}^{-q^m t}}{q^{m(m-\mu)/\lambda}}$$

with $q > 1$, $\mu \in \mathbb{R}$, and $\lambda = 2L/k$ or $\lambda = (2L+1)/k$ with $L, k \in \mathbb{N}$ can be extended to all of $\mathbb{R}$ in the following manner. Let $A = L$ if $\lambda = 2L/k$, and let $A = 2L+1$ if $\lambda = (2L+1)/k$. Also, let $R = k$ if $\lambda = 2L/k$, and let $R = 2k$ if $\lambda = (2L+1)/k$. For $r = 0, \ldots, R-1$, choose $(\gamma_r, \ell_r, p_r) \in \mathbb{R}^+ \times \mathbb{Z}_2 \times \mathbb{Z}_R$, with $c_r \equiv \gamma_r \mathrm{e}^{\mathrm{i}\pi\ell_r/R}\omega^{p_r}$ where $\omega = \mathrm{e}^{2\pi\mathrm{i}/R}$ and real part $\mathcal{R}(\mathrm{e}^{\mathrm{i}\pi\ell_r/R}\omega^{p_r}) \leq 0$. For $j = 0, \ldots, A-1$ choose any $a_{j,r} \in \mathbb{R}$ with at least one $a_{j,r} \neq 0$, and set

$$h(c_r,t) = \sum_{j=0}^{A-1}\sum_{M=-\infty}^{\infty}\left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right]^M a_{j,r}\,\frac{\exp[-q^{MA+j}\cdot\gamma_r\cdot \mathrm{e}^{\mathrm{i}\pi\ell_r/R}\cdot\omega^{p_r}\cdot t]}{q^{(MA+j)(MA+j-\mu)/\lambda}}. \tag{70}$$

Let $\boldsymbol{F}$ *be the* $R \times 1$ *column vector with m-th entry* F_m *given by*

$$F_m = (-1)^m \theta(q^{2/\lambda}; -q^{(\mu+m\lambda-1)/\lambda}),$$

as in (67)*. And let* H *be the* $R \times R$ *matrix with* (m,r) *entry given by*

$$H_{m,r} = (-\gamma_r e^{i\pi\ell_r/R}\omega^{p_r})^m \sum_{j=0}^{A-1} a_{j,r} q^{-j(j-\mu-m\lambda)/\lambda} \theta\left(q^{2A^2/\lambda}; \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right] q^{A[\mu+m\lambda-2j-A]/\lambda}\right), \tag{71}$$

as in (66) *and* (68)*–*(69)*.*

Then if $\det H \neq 0$*, there exists an* $R \times 1$ *column vector* $\boldsymbol{B} = H^{-1}\boldsymbol{F}$*, as in* (69)*, with r-th entry* b_r *so that* $f_{\mu,\lambda}(t)$ *can be extended to* $h(t)$ *for* $t \leq 0$ *where*

$$h(t) = \sum_{r=0}^{R-1} b_r h(c_r, t) = \sum_{r=0}^{R-1} b_r \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{M=-\infty}^{\infty} \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right]^M \frac{\exp[-q^{MA+j}\cdot\gamma_r\cdot e^{i\pi\ell_r/R}\cdot\omega^{p_r}\cdot t]}{q^{(MA+j)(MA+j-\mu)/\lambda}}\right]. \tag{72}$$

Furthermore, the $h(c_r,t)$ *on* $(-\infty,0]$*, and* $h(t)$ *satisfies the same MADE* (18) *as* $f_{\mu,\lambda}(t)$*, namely*

$$h^{(R)}(t) = (-1)^{R+A} q^{A(A+\mu)/\lambda} h(q^A t)\ . \tag{73}$$

Moreover, at $t=0$*, the extension* $h(t)$ *has the same derivatives (and anti-derivatives) of all orders as does* $f_{\mu,\lambda}(t)$*. That is*

$$f_{\mu,\lambda}^{(m)}(0) = h^{(m)}(0) \quad \forall\, m \in \mathbb{Z}\ , \tag{74}$$

where for $m<0$ *the anti-derivatives* $f_{\mu,\lambda}^{(m)}(t)$ *and* $h^{(m)}(t)$ *are given by* (85)*–*(87) *below. Since each entry* $f_{\mu,\lambda}^{(m)}(0)$ *of* $\boldsymbol{F}$ *is real valued, and since the entries* $H_{m,r}$ *of* H *may be complex valued, the coefficients* b_r *may be complex in general. However, the real part* $\mathcal{R}(h(t))$ *of* $h(t)$ *will be a real extension of* $f_{\mu,\lambda}(t)$ *to* $t \leq 0$ *satisfying* (73) *and* (74)*. Thus* $f_{\mu,\lambda}(t)$ *can be extended to a bounded real valued function on* $\mathbb{R}$ *satisfying the MADE* (73) *with initial conditions* (74)*.*

The parameter space $\mathcal{P}_{\mu,\lambda}$ *available for selection of the* $h(c_r,t)$ *for* $r=0,\ldots,R-1$ *consists of a union of open subsets of the connected components of*

$$[\mathbb{P}^{A-1}(\mathbb{R}) \times \mathbb{R}^+ \times \mathbb{Z}_2 \times \mathbb{Z}_R]^R \tag{75}$$

with real multiples of $\boldsymbol{a}_r = (a_{0,r}, a_{1,r}, \ldots, a_{(A-1),r})$ *giving the parameter* $[\boldsymbol{a}_r] \in \mathbb{P}^{A-1}(\mathbb{R})$*, and with* $c_r = \gamma_r e^{i\pi\ell_r/R}\omega^{p_r}$ *giving the parameters* $\gamma_r \in \mathbb{R}^+$ *and* $(\ell_r, p_r) \in \mathbb{Z}_2 \times \mathbb{Z}_R$*. These must satisfy that* $\mathcal{R}(e^{i\pi\ell_r/R}\omega^{p_r}) \leq 0$*, which determines the connected component. The open set condition comes from the condition that* $\det H \neq 0$ *for the associated matrix* H *whose entries* $H_{m,r}$ *are given by* (71)*. Thus* $\mathcal{P}_{\mu,\lambda}$ *is an open manifold with each connected component having dimension* RA*, with* $\mathcal{P}_{\mu,\lambda} \subset [\mathbb{P}^{A-1}(\mathbb{R}) \times \mathbb{R}^+ \times \mathbb{Z}_2 \times \mathbb{Z}_R]^R$*.*

Proof. From (48)–(54) one has that the MADE (73) holds on $t \leq 0$ for each $h(c_r,t)$ for the choices $a_{j,r}, \gamma_r, \ell_r, p_r$. Thus (73) holds for any linear combination of the $h(c_r,t)$, including for $h(t)$ as in (72) and for its real part $\mathcal{R}(h(t))$. Note (73) is the same MADE (18) holding for $f_{\mu,\lambda}(t)$ on $t \geq 0$. From (55)–(57) the function $h(t)$ is bounded, and from the remarks following (2) the function $f_{\mu,\lambda}(t)$ is bounded. We conclude the extension is bounded on $\mathbb{R}$. From (12) and (28), one has that the initial condition is given by each of the column vectors in (67). From (66)–(69), one has that the matrix H as in (71) is the Wronskian matrix of the chosen $h(c_0,t),\ldots,h(c_{R-1},t)$ at $t=0$. Thus by hypothesis, $\det H \neq 0$ gives that H is invertible and that the $h(c_0,t),\ldots,h(c_{R-1},t)$ are linearly independent on $(-\infty,0]$. For each fixed r, from the choices $a_{j,r}$ with $j=0,\ldots,A-1$, a scalar may be pulled and subsumed into the b_r, thus for each r the $a_{j,r}$ modulo a scalar is real projective space $\mathbb{P}^{A-1}(\mathbb{R})$, with $\gamma_r \in \mathbb{R}^+$ and $(\ell_r,p_r) \in \mathbb{Z}_2 \times \mathbb{Z}_R$ satisfying $\mathcal{R}(e^{i\pi\ell_r/R}\omega^{p_r}) \leq 0$ to give each $h(c_r,t)$ defined on $(-\infty,0]$. This gives the parameter space $\mathcal{P}_{\mu,\lambda}$ in (75).

From (58)–(59), one has that (74) holds for $m=0,\ldots,R-1$. It remains to show that (74) holds for $m \geq R$ (and later for $m<0$). Recall $R\lambda = 2A$. From (65), with $m=s+nR$ and $0 \leq s \leq R-1$, one sees

$$h^{(s+nR)}(c_r,t) = (-\gamma_r e^{i\pi\ell_r/R}\omega^{p_r})^{s+nR} \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{M=-\infty}^{\infty} \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right]^M \frac{e^{-q^{MA+j}\gamma_r e^{i\pi\ell_r/R}\omega^{p_r}t}}{q^{(MA+j)(MA+j-\mu-E)/\lambda}}\right] \tag{76}$$

where $E=(s+nR)\lambda = s\lambda + n2A$, by relying on the fact that $R\lambda = 2A$. Using this latter expression for E, and re-indexing with $M=N+n$ gives

$$
\begin{aligned}
h^{(s+nR)}(c_r,t) &= (-\gamma_r e^{i\pi\ell_r/R}\omega^{p_r})^s(-\gamma_r)^{nR}(e^{i\pi\ell_r})^n \\
&\qquad \cdot\left[\sum_{j=0}^{A-1} a_{j,r}\sum_{N=-\infty}^{\infty}\left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right]^{N+n}\frac{e^{-q^{NA+nA+j}\gamma_r e^{i\pi\ell_r/R}\omega^{p_r}t}}{q^{(NA+nA+j)(NA-nA+j-\mu-s\lambda)/\lambda}}\right] \\
&= (-\gamma_r e^{i\pi\ell_r/R}\omega^{p_r})^s(-\gamma_r)^{nR}[(-1)^{\ell_r}]^n\left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right]^n \\
&\qquad \cdot\left[\sum_{j=0}^{A-1} a_{j,r}\sum_{N=-\infty}^{\infty}\left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right]^{N}\frac{e^{-q^{NA+nA+j}\gamma_r e^{i\pi\ell_r/R}\omega^{p_r}t}}{q^{-nA[nA+\mu+s\lambda]/\lambda}q^{(NA+j)(NA+j-\mu-s\lambda)/\lambda}}\right] \\
&= (-1)^{n[R+A]}q^{nA[nA+\mu+s\lambda]/\lambda}(-\gamma_r e^{i\pi\ell_r/R}\omega^{p_r})^s \\
&\qquad \cdot\left[\sum_{j=0}^{A-1} a_{j,r}\sum_{N=-\infty}^{\infty}\left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right]^{N}\frac{e^{-q^{NA+j}\gamma_r e^{i\pi\ell_r/R}\omega^{p_r}(q^{nA}t)}}{q^{(NA+j)(NA+j-\mu-s\lambda)/\lambda}}\right] \\
&= (-1)^{n[R+A]}q^{nA[nA+\mu+s\lambda]/\lambda}\, h^{(s)}(c_r,q^{nA}t)\,,
\end{aligned} \tag{77}
$$

where the last equality follows from (65).

This compares with the $(s+nR)$-th derivative of $f_{\mu,\lambda}(t)$, with $0\le s\le R-1$, as follows. From (12) along with the fact that $R\lambda=2A$, one has

$$
\begin{aligned}
f^{(s+nR)}_{\mu,\lambda}(t) &= (-1)^{s+nR}f_{\mu+(s+nR)\lambda,\lambda}(t) \;=\; (-1)^{s+nR}f_{\mu+s\lambda+n2A,\lambda}(t) \\
&= (-1)^{s+nR}\sum_{M=-\infty}^{\infty}(-1)^M\frac{e^{-q^M t}}{q^{M(M-\mu-s\lambda-2nA)/\lambda}}\,.
\end{aligned}
$$

Re-indexing with $M=N+nA$ gives

$$
\begin{aligned}
f^{(s+nR)}_{\mu,\lambda}(t) &= (-1)^{s+nR}\sum_{N=-\infty}^{\infty}(-1)^{N+nA}\frac{e^{-q^{N+nA}t}}{q^{(N+nA)(N-nA-\mu-s\lambda)/\lambda}} \\
&= (-1)^{nR+nA}(-1)^s\sum_{N=-\infty}^{\infty}(-1)^N\frac{e^{-q^N(q^{nA}t)}}{q^{-nA(nA+\mu+s\lambda)/\lambda}q^{N(N-\mu-s\lambda)/\lambda}} \\
&= (-1)^{n[R+A]}q^{nA(nA+\mu+s\lambda)/\lambda}(-1)^s\sum_{N=-\infty}^{\infty}(-1)^N\frac{e^{-q^N(q^{nA}t)}}{q^{N(N-\mu-s\lambda)/\lambda}} \\
&= (-1)^{n[R+A]}q^{nA(nA+\mu+s\lambda)/\lambda}(-1)^s f_{\mu+s\lambda,\lambda}(q^{nA}t) && (78)\\
&= (-1)^{n[R+A]}q^{nA(nA+\mu+s\lambda)/\lambda}f^{(s)}_{\mu,\lambda}(q^{nA}t)\,, && (79)
\end{aligned}
$$

where equality in (78) follows from (2), and equality in (79) follows from (12).

Now with $h(t)$ as in (72) and $0\le s\le R-1$, one has

$$
\begin{aligned}
h^{(s+nR)}(0) &= \sum_{r=0}^{R-1} b_r h^{(s+nR)}(c_r,0) && (80)\\
&= \sum_{r=0}^{R-1} b_r(-1)^{n[R+A]}q^{nA[nA+\mu+s\lambda]/\lambda}h^{(s)}(c_r,0) && (81)\\
&= (-1)^{n[R+A]}q^{nA[nA+\mu+s\lambda]/\lambda}\sum_{r=0}^{R-1} b_r h^{(s)}(c_r,0) && (82)\\
&= (-1)^{n[R+A]}q^{nA[nA+\mu+s\lambda]/\lambda}f^{(s)}_{\mu,\lambda}(0) && (83)\\
&= f^{(s+nR)}_{\mu,\lambda}(0)\,, && (84)
\end{aligned}
$$

where equality in (81) follows from (77), the equality in (83) follows from (67)–(69), and the equality in (84) follows from (79). Thus the derivatives of $h(t)$ of all positive and 0 orders match those of $f_{\mu,\lambda}(t)$ at $t=0$ as claimed.

In the case of anti-derivatives (or derivatives of negative order), define for $s < 0$

$$h^{(s)}(c_r, t) \equiv (-c_r)^s \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{M=-\infty}^{\infty} \left[\frac{(-1)^{A-\ell_r}}{|c_r|^R} \right]^M \frac{e^{-q^{MA+j} c_r t}}{q^{(MA+j)(MA+j-\mu-s\lambda)/\lambda}} \right]$$
$$\equiv (-\gamma_r e^{i\pi\ell_r/R} \omega^{p_r})^s \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{M=-\infty}^{\infty} \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R} \right]^M \frac{e^{-q^{MA+j} \gamma_r e^{i\pi\ell_r/R} \omega^{p_r} t}}{q^{(MA+j)(MA+j-\mu-s\lambda)/\lambda}} \right], \tag{85}$$

and

$$f_{\mu,\lambda}^{(s)}(t) \equiv (-1)^s f_{\mu+s\lambda,\lambda}(t) = (-1)^s \sum_{M=-\infty}^{\infty} (-1)^M \frac{e^{-q^M t}}{q^{M(M-\mu-s\lambda)/\lambda}}, \tag{86}$$

and

$$h^{(s)}(t) \equiv \sum_{r=0}^{R-1} b_r h^{(s)}(c_r, t), \tag{87}$$

which by construction are the $|s|$-th anti-derivatives of $h(c_r, t)$, $f_{\mu,\lambda}(t)$, and $h(t)$, respectively, with ALL constants of integration set equal to 0. Now let n be the positive integer such that $s + nR$ satisfies $0 \le s + nR \le R - 1$, and repeat the computations (76) through (79) verbatim. By previous work, one has that $h^{(s+nR)}(0) = f_{\mu,\lambda}^{(s+nR)}(0)$ because $0 \le s+nR \le R-1$. Thus, in the current $s < 0$ setting, equations (80)–(84) still hold. Dividing (82)–(83) by $(-1)^{n[R+A]} q^{nA[nA+\mu+s\lambda]/\lambda}$, one concludes that for $s < 0$

$$h^{(s)}(0) = \sum_{r=0}^{R-1} b_r h^{(s)}(c_r, 0) = f_{\mu,\lambda}^{(s)}(0).$$

Thus all the anti-derivatives (or derivatives of negative order) of $h(t)$ and of $f_{\mu,\lambda}(t)$ match at 0 as well, and the theorem is proven. □

Remark 10. The assumption that each $\mathcal{R}(c_r) = \mathcal{R}(e^{i\pi\ell_r/R}\omega^{p_r}) \le 0$ in Theorem 3.2 is sufficient to gain extension while meeting initial conditions and solving the relevant MADE. However, to also gain decay at $-\infty$, it will be necessary to assume the strict inequality that $\mathcal{R}(c_r) < 0$, as seen in Propositions 8.1 and 8.2 below. Therefore, this strict inequality will be a blanket assumption throughout the remainder of the paper, unless otherwise explicitly indicated.

Remark 11. Theorem 3.2 starts with a function $f_{\mu,\lambda}(t)$ defined on $[0, \infty)$ and solving the MADE (18). This function extends non-uniquely to a family of $\mathcal{C}^\infty$ solutions of (18) defined on all of $\mathbb{R}$ by assuming the extension parameters c_r in the $h(c_r, t)$ satisfy the requirement that $\mathcal{R}(-c_r t) \le 0$ for $t \le 0$. By a parallel argument, it is of course possible to start with a function defined on $(-\infty, 0]$ satisfying the MADE (18) and extending non-uniquely to a family of $\mathcal{C}^\infty$ solutions of (18) defined on all of $\mathbb{R}$ by assuming the extension parameters c_r in $h(c_r, t)$ satisfy the requirement that $\mathcal{R}(-c_r t) \le 0$ for $t \ge 0$. We do not pursue the details here, but do illustrate a case of this forward non-uniqueness in Section 7. Note also that, by taking the difference of any two distinct extensions to $\mathbb{R}$ of the $f_{\mu,\lambda}(t)$ defined on $[0, \infty)$ that both solve the MADE (18), one has a solution to (18) that is flat at the origin, vanishing on $[0, \infty)$, and non-identically 0 on $(-\infty, 0)$.

4. The extensions $F_{\mu,\lambda}(t)$ as Schwartz wavelets with vanishing moments

It is now convenient to make the following definition.

Definition 4.1. Let $q > 1$, and let $\mu \in \mathbb{R}$, $\lambda \in \mathbb{Q}^+$, and $s \in \mathbb{Z}$ be fixed. Let $\mathcal{R}h(t)$ be an extension of $f_{\mu,\lambda}(t)$ to the negative real line, associated with the parameters $[\boldsymbol{a}_r]$, γ_r, ℓ_r, p_r for $0 \le r \le R-1$ in $\mathcal{P} \subset [\mathbb{P}^{A-1}(\mathbb{R}) \times \mathbb{R}^+ \times \mathbb{Z}_2 \times \mathbb{Z}_R]^R$ as in Theorem 3.2, with the further assumption that each $\mathcal{R}(c_r) < 0$ for $c_r = \gamma_r e^{i\pi\ell_r/R}\omega^{p_r}$. We denote such a global extension by

$$F_{\mu,\lambda}^{(s)}(t) \equiv \begin{cases} f_{\mu,\lambda}^{(s)}(t) & \text{if } t \ge 0 \\ (\mathcal{R}h)^{(s)}(t) & \text{if } t < 0 \end{cases} \tag{88}$$
$$\equiv F_{\mu,\lambda}^{(s)}([\boldsymbol{a}_0], \gamma_0, \ell_0, p_0; \ldots; [\boldsymbol{a}_{R-1}], \gamma_{R-1}, \ell_{R-1}, p_{R-1}; t). \tag{89}$$

Here A and R are as in (16) and (17), respectively, with $\lambda = 2A/R$.

Each $F^{(s)}_{\mu,\lambda}(t)$ is real analytic in t on $(-\infty,0)\cup(0,\infty)$, but not at $t=0$ as a consequence of Proposition 2.3. It is $\mathcal{C}^\infty$ on $\mathbb{R}$. Note that (89) explicitly indicates the dependence of $(\mathcal{R}h)^{(s)}(t)$ (and therefore of $F^{(s)}_{\mu,\lambda}(t)$) on all parameters, whereas (88) suppresses the expression of this dependence for the sake of conciseness. Also, by Proposition 8.1, the assumption that each c_r has negative real part forces very rapid decay at $\pm\infty$ of $F_{\mu,\lambda}(t)$ and its derivatives of all orders.

A series of results on the properties of the $F^{(s)}_{\mu,\lambda}(t)$ now follows. First, note from Theorem 3.2 that the family

$$\left\{F^{(s)}_{\mu,\lambda}(t)|s\in\mathbb{Z}\right\} \tag{90}$$

is a family of functions in $\mathcal{C}^\infty(\mathbb{R})$ with $D_t F^{(s)}_{\mu,\lambda}(t)=F^{(s+1)}_{\mu,\lambda}(t)$ for each $s\in\mathbb{Z}$. Furthermore, each $F^{(s)}_{\mu,\lambda}(t)$ satisfies the MADE

$$\frac{\mathrm{d}^R}{\mathrm{d}t^R}\left[F^{(s)}_{\mu,\lambda}(t)\right]=(-1)^{R+A}q^{A(A+\mu+s\lambda)/\lambda}F^{(s)}_{\mu,\lambda}(q^A t)\ , \tag{91}$$

as can be seen from (77) and (79). Here A and R are as in (16) and (17), respectively, with $\lambda=2A/R$.

Next one sees that all moments of $F^{(s)}_{\mu,\lambda}(t)$ vanish.

Proposition 4.1. *For $n\in\mathbb{N}_0$, the n-th moment of $F^{(s)}_{\mu,\lambda}(t)$ vanishes. That is,*

$$M_n[F^{(s)}_{\mu,\lambda}(t)]\equiv\int_{-\infty}^{\infty}t^n F^{(s)}_{\mu,\lambda}(t)\,\mathrm{d}t=0 \tag{92}$$

for each non-negative integer n.

Proof. Observe that for an integer $k\geq 1$ one has

$$\int_{-\infty}^{\infty}t^k F^{(s)}_{\mu,\lambda}(t)\,\mathrm{d}t=t^k F^{(s-1)}_{\mu,\lambda}(t)\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\left(t^k\right)' F^{(s-1)}_{\mu,\lambda}(t)\,\mathrm{d}t=-k\int_{-\infty}^{\infty}t^{k-1}F^{(s-1)}_{\mu,\lambda}(t)\,\mathrm{d}t\ , \tag{93}$$

where the vanishing of $t^k F^{(s-1)}_{\mu,\lambda}(t)\Big|_{-\infty}^{\infty}$ follows from (179) in Corollary 8.1 below. Thus, by iterating (93) n times, we have

$$\int_{-\infty}^{\infty}t^n F^{(s)}_{\mu,\lambda}(t)\,\mathrm{d}t=(-1)^n n!\int_{-\infty}^{\infty}F^{(s-n)}_{\mu,\lambda}(t)\,\mathrm{d}t=(-1)^n n!\ F^{(s-n-1)}_{\mu,\lambda}(t)\Big|_{-\infty}^{\infty}=0\ , \tag{94}$$

where the vanishing of $F^{(s-n-1)}_{\mu,\lambda}(t)$ at $\pm\infty$ is also given by (179) in Corollary 8.1 below. □

Each of the $F^{(s)}_{\mu,\lambda}(t)$ turns out to exhibit wavelet properties. Recall [6] that a function $\phi(t)$ is considered to be a wavelet if

$$\phi\in\mathcal{L}^1(\mathbb{R})\cap\mathcal{L}^2(\mathbb{R})\cap\mathcal{L}^\infty(\mathbb{R})\ , \tag{95}$$

$$\int_{-\infty}^{\infty}\phi(t)\,\mathrm{d}t=0\ , \tag{96}$$

$$\int_{-\infty}^{\infty}\frac{|\mathcal{F}[\phi(t)](\omega)|^2}{|\omega|}\,\mathrm{d}\omega<\infty\ , \tag{97}$$

where the Fourier transform is given by

$$\mathcal{F}[\phi(t)](\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\omega t}\phi(t)\,\mathrm{d}t\ . \tag{98}$$

Proposition 4.2. *Each $F^{(s)}_{\mu,\lambda}(t)$ in the family (90) is a Schwartz wavelet with $D_t F^{(s)}_{\mu,\lambda}(t)=F^{(s+1)}_{\mu,\lambda}(t)$.*

Proof. From the fact that each $F^{(s)}_{\mu,\lambda}(t)$ is $\mathcal{C}^\infty$ and from (179) in Corollary 8.1, one has that for all $p, n \in \mathbb{N}_0$

$$\lim_{t\to\pm\infty} \left| t^p D_t^n F^{(s)}_{\mu,\lambda}(t) \right| = 0\,. \tag{99}$$

Thus there exist constants $B_{p,n,s}$ with

$$\left| t^p D_t^n F^{(s)}_{\mu,\lambda}(t) \right| \le B_{p,n,s}\,. \tag{100}$$

We conclude that each $F^{(s)}_{\mu,\lambda}(t)$ is Schwartz, and thus each $F^{(s)}_{\mu,\lambda}(t)$ lies in each $\mathcal{L}^p(\mathbb{R})$. Hence wavelet condition (95) is met. From Proposition 4.1, all moments of $F^{(s)}_{\mu,\lambda}(t)$ vanish, including the case $M_0[F^{(s)}_{\mu,\lambda}(t)] = 0$, from which we observe that wavelet condition (96) is met. Finally, since $\mathcal{F}[tf(t)](\omega) = iD_\omega \mathcal{F}[f(t)](\omega)$, one concludes that for each $n \in \mathbb{N}_0$

$$(iD_\omega)^n \mathcal{F}[F^{(s)}_{\mu,\lambda}(t)](0) = \mathcal{F}[t^n F^{(s)}_{\mu,\lambda}(t)](0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\cdot 0\cdot t} t^n F^{(s)}_{\mu,\lambda}(t)\,\mathrm{d}t = \frac{1}{\sqrt{2\pi}} M_n[F^{(s)}_{\mu,\lambda}(t)] = 0\,. \tag{101}$$

Hence $\mathcal{F}[F^{(s)}_{\mu,\lambda}(t)](\omega)$ vanishes to infinite order at $\omega = 0$. This high order of vanishing at $\omega = 0$ coupled with the fact that $F^{(s)}_{\mu,\lambda}(t)$, and hence $\mathcal{F}[F^{(s)}_{\mu,\lambda}(t)](\omega)$, is Schwartz gives that (97) holds. Thus each $F^{(s)}_{\mu,\lambda}(t)$ is a Schwartz wavelet. □

5. Examples and genericity

The purpose of this section is two-fold. One goal is to observe via example that the hypothesis that $\det H \neq 0$ in Theorem 3.2 is a generic condition in the choice of parameters given in $\mathcal{P}_{\mu,\lambda}$ in Theorem 3.2. Another objective is to explicitly obtain extensions of certain $f_{\mu,\lambda}(t)$ to $(-\infty, 0]$.

Example 1. In this example, we let $\lambda = 2L/k = 2$ with $L = k = 1 = R = A$. In this setting we have $f_{\mu,2}(t) = \sum_{m=-\infty}^{\infty} (-1)^m \mathrm{e}^{-q^m t}/q^{m(m-\mu)/2}$, which from (13) or (18) satisfies the MADE

$$f'_{\mu,2}(t) = q^{(\mu+1)/2} f_{\mu,2}(qt)\,. \tag{102}$$

Here the 1×1 column matrix $\boldsymbol{F}$, as obtained from (67), is given by

$$\boldsymbol{F} = \left[f_{\mu,2}(0) \right] = \left[\theta(q; -q^{(\mu-1)/2}) \right]. \tag{103}$$

In the notation of Theorem 3.2 immediately preceding (70), observe that the $k = 1 = R$ root of unity is $\omega = \mathrm{e}^{2\pi\mathrm{i}/R} = 1$, so we must choose $\ell_0 = 1$ resulting in $\mathcal{R}(\mathrm{e}^{\mathrm{i}\pi\ell_0}\omega) < 0$ in order to have $h(c_0, t)$ defined on $t \le 0$. Thus, recalling $A = 1$ in this example, one has $\mathrm{e}^{\mathrm{i}\pi\ell_0} = -1$ and, from (70),

$$h(c_0, t) = a_{0,0} \sum_{M=-\infty}^{\infty} \left[\frac{1}{\gamma_0} \right]^M \frac{\mathrm{e}^{q^M \gamma_0 t}}{q^{M(M-\mu)/2}}\,, \tag{104}$$

where $\gamma_0 > 0$ and $c_0 = \gamma_0 \mathrm{e}^{\mathrm{i}\pi} \cdot \omega = -\gamma_0$. Note that the form of (104) gives that (102) holds for $h(c_0, t)$, by (48)–(54). Thus the MADE (102) also holds for the extension $h(t)$ given below in (108), which follows from (73). From (71), the $k \times k = 1 \times 1$ matrix H has sole entry of the form

$$H_{0,0} = a_{0,0} h(c_0, 0) = a_{0,0} \sum_{M=-\infty}^{\infty} \left[\frac{1}{\gamma_0} \right]^M \frac{1}{q^{M(M-\mu)/2}} = a_{0,0} \theta\left(q; \left[\frac{1}{\gamma_0} \right] q^{(\mu-1)/2} \right)\,, \tag{105}$$

where equality in (105) follows from (26). The 1×1 column $\boldsymbol{B}$ has entry b_0 to be determined by $\boldsymbol{F} = H \cdot \boldsymbol{B}$, from the required initial conditions $\boldsymbol{F} = H \cdot \boldsymbol{B}$ that the first $k - 1 = 1 - 1 = 0$ derivatives match at $t = 0$, as in (67)–(69). In the current case, from (103) and (105), $\boldsymbol{F} = H \cdot \boldsymbol{B}$ becomes

$$\theta(q; -q^{(\mu-1)/2}) = \left[a_{0,0} \theta\left(q; \left[\frac{1}{\gamma_0} \right] q^{(\mu-1)/2} \right) \right] b_0\,, \tag{106}$$

and, by (25), one has $\theta\left(q; \left[\frac{1}{\gamma_0} \right] q^{(\mu-1)/2} \right) \neq 0$. Thus, since $a_{0,0} \neq 0$, one sees that the expression

$$\det H = a_{0,0} \theta\left(q; \left[\frac{1}{\gamma_0} \right] q^{(\mu-1)/2} \right)$$

never vanishes for any choice of the parameters $a_{0,0} \neq 0$ and $\gamma_0 > 0$. Thus we see that the assumption that $\det H \neq 0$ in Theorem 3.2 is generic here in Example 1. From (106), one then has that

$$\frac{\theta(q; -q^{(\mu-1)/2})}{\theta\left(q; \left[\frac{1}{\gamma_0}\right] q^{(\mu-1)/2}\right)} = a_{0,0} b_0 \,. \tag{107}$$

From (107) and from (72), the extension $h(t)$ of $f_{\mu,2}(t)$ to $t \le 0$ becomes the real-valued function

$$\begin{aligned} h(t) = b_0 a_{0,0} h(c_0, t) &= b_0 a_{0,0} \sum_{M=-\infty}^{\infty} \left[\frac{1}{\gamma_0}\right]^M \frac{\mathrm{e}^{q^M \gamma_0 t}}{q^{M(M-\mu)/2}} \\ &= \frac{\theta(q; -q^{(\mu-1)/2})}{\theta\left(q; \left[\frac{1}{\gamma_0}\right] q^{(\mu-1)/2}\right)} \sum_{M=-\infty}^{\infty} \left[\frac{1}{\gamma_0}\right]^M \frac{\mathrm{e}^{q^M \gamma_0 t}}{q^{M(M-\mu)/2}} \,, \end{aligned} \tag{108}$$

which is real analytic for $t < 0$ and $\mathcal{C}^\infty$ for $t \le 0$. Now $\mu = 2p + 1$ is an odd integer if and only if the numerator $\theta(q; -q^{(\mu-1)/2})$ vanishes, and in this case the only extension of the form $h(t)$, as in (108), is the identically 0 extension obtained by taking $b_0 = 0$, since $a_{0,0} \neq 0$. Note, in this $\mu = 2p+1$ case, that, since $\lambda = 2$ is even with μ odd, one has $f_{\mu,2}$ is flat at 0 by (33). As a special example of this flat case, let $\mu = -1$ along with $\lambda = 2$ to obtain $f_{-1,2}(t) = K(t)$, where $K(t)$ is studied extensively in [22] and where $K(t)$ is applied in tsunami modeling in [25]. Thus Example 1 is presented here as a generalization of $K(t)$.

On the other hand, if μ is not an odd integer then (108) gives a family of solutions parametrized by $\gamma_0 > 0$. This family is q-periodic in γ_0 in the sense that for each $p \in \mathbb{Z}$ and each $\gamma_0 \in \mathbb{R}^+$

$$\begin{aligned} &\frac{\theta(q; -q^{(\mu-1)/2})}{\theta\left(q; \left[\frac{1}{q^p \gamma_0}\right] q^{(\mu-1)/2}\right)} \sum_{M=-\infty}^{\infty} \left[\frac{1}{q^p \gamma_0}\right]^M \frac{\mathrm{e}^{q^M (q^p \gamma_0) t}}{q^{M(M-\mu)/2}} \\ &= \frac{\theta(q; -q^{(\mu-1)/2})}{\theta\left(q; \left[\frac{1}{\gamma_0}\right] q^{(\mu-1)/2}\right)} \sum_{M=-\infty}^{\infty} \left[\frac{1}{\gamma_0}\right]^M \frac{\mathrm{e}^{q^M \gamma_0 t}}{q^{M(M-\mu)/2}} \,, \end{aligned} \tag{109}$$

but the family is not constant in γ_0. For each γ_0, as $q \to 1^+$ the extension $h(t)$ as given by (108) behaves as $h(t) \approx \theta(q; -q^{(\mu-1)/2}) \cdot \mathrm{e}^t$ and its normalization $h(t)/\theta(q; -q^{(\mu-1)/2})$ approaches e^t uniformly on $(-\infty, 0]$. So for small values of $q > 1$ it can be difficult to distinguish the graphs of $h(t)$ for γ_0 in the interval $[1, q]$. However, the family in (108) can be seen to be non-constant in γ_0 for large values of q. In particular, for two values of γ_0 generating different extension functions $h(t)$ in (108), these functions must be linearly independent, as follows. If one were a constant multiple (by C) of the other, by agreement of the two functions at $t = 0$ one has $C = 1$. However, this is a contradiction as the two functions were different. Note that this implies that the fundamental set, as in [8], has dimension greater than or equal to 2, which is greater than the order 1 of the MADE (102). Fig. 2 Left exhibits different such extensions (108) to the negative real line of $f_{\mu,2}(t)$ with μ not an odd integer for varying γ_0. In particular, we have explicit non-uniqueness of the extension $h(t)$. Finally, by Proposition 4.2, any extension $F_{\mu,2}(t)$ agreeing with $f_{\mu,2}(t)$ on $[0, \infty)$ and with $h(t)$ as in (108) on $(-\infty, 0]$ is a Schwartz wavelet.

Example 2. In this example, we let $\lambda = 2L/k = 1$ with $L = 1 = A$, $k = 2 = R$. In this setting we have $f_{\mu,1}(t) = \sum_{m=-\infty}^{\infty} (-1)^m \mathrm{e}^{-q^m t}/q^{m(m-\mu)/1}$, which, from (13) or (18), satisfies the MADE

$$f_{\mu,1}^{(2)}(t) = -q^{\mu+1} f_{\mu,1}(qt) \,. \tag{110}$$

Here the 2×1 column matrix $\boldsymbol{F}$, as obtained from (67), is given by

$$\boldsymbol{F} = \begin{bmatrix} f_{\mu,1}(0) \\ -f_{\mu+1,1}(0) \end{bmatrix} = \begin{bmatrix} \theta(q^2; -q^{\mu-1}) \\ -\theta(q^2; -q^{\mu}) \end{bmatrix}. \tag{111}$$

In the notation of Theorem 3.2 immediately preceding (70), observe that the $k = 2 = R$ roots of unity are $\mathrm{e}^{2\pi\mathrm{i}/R} = \omega = -1$, $\omega^0 = \omega^2 = 1$. From (70), with $c_r = \gamma_r \mathrm{e}^{\mathrm{i}\pi\ell_r/R} \omega^{p_r}$ for $r = 0, 1$, one has

$$h(c_0, t) = a_{0,0} \sum_{M=-\infty}^{\infty} \left[\frac{(-1)^{1-\ell_0}}{\gamma_0^2}\right]^M \frac{\mathrm{e}^{-q^M \gamma_0 \mathrm{e}^{\mathrm{i}\pi\ell_0/2} \omega^{p_0} t}}{q^{M(M-\mu)}} \tag{112}$$

and

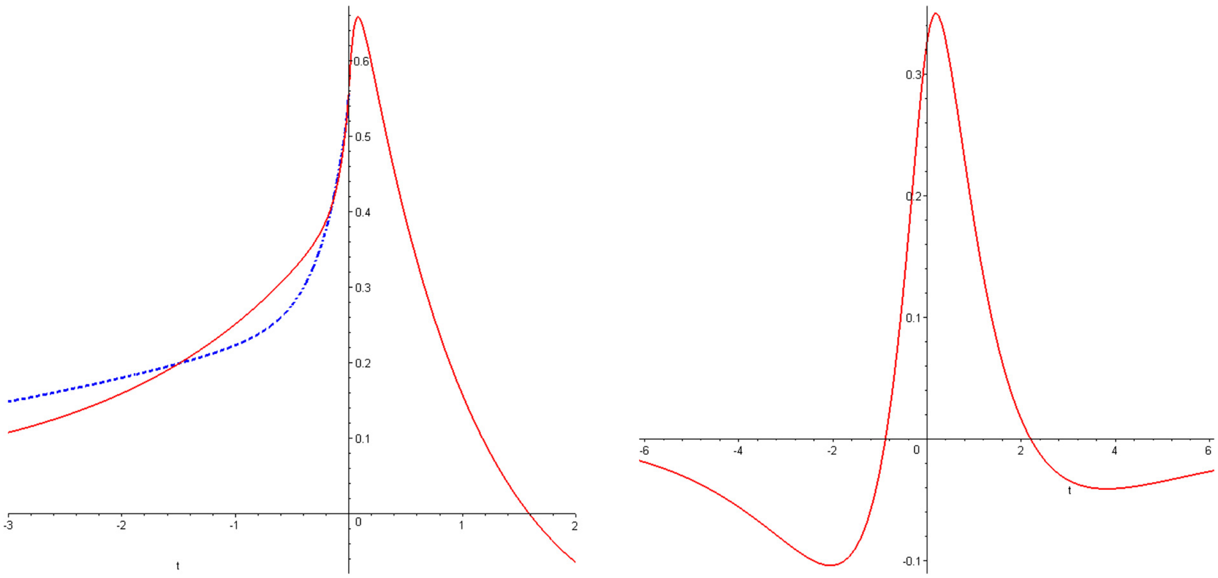

Fig. 2. Left: In Example 1 with $\lambda = 2$, $f_{\mu,2}(t)$ can be extended non-uniquely to the negative real axis solving the relevant MADE (102), as is illustrated here when $\mu = 0$, $q = 20$, with one extension (solid) having set $\gamma_0 = 0.2$ in equation (108) and with a second extension (dotted) having set $\gamma_0 = 0.5$ in equation (108). **Right:** In Example 2 Case A with $\lambda = 1$ (decaying in both terms), $f_{\mu,1}(t)$ is extended (non-uniquely) to the negative real axis solving the relevant MADE (110), as is illustrated here when $\mu = 0.3$, $q = 3$, with the values $\gamma_0 = 0.2$ and $\gamma_1 = 1.1$ in (124), where $\ell_0 = 0 = \ell_1$ and $p_0 = 1 = p_1$. Compare with the extensions in Fig. 3, Cases B and C, below, where $f_{0.3,1}(t)$ with $q = 3$ is also extended to $(-\infty, 0]$ in different manners.

$$h(c_1, t) = a_{0,1} \sum_{M=-\infty}^{\infty} \left[\frac{(-1)^{1-\ell_1}}{\gamma_1^2} \right]^M \frac{\mathrm{e}^{-q^M \gamma_1 \mathrm{e}^{\mathrm{i}\pi\ell_1/2} \omega^{p_1} t}}{q^{M(M-\mu)}} , \tag{113}$$

where $\gamma_0, \gamma_1 > 0$ and where, for $r = 0, 1$, the ℓ_r, p_r are chosen to satisfy $\mathcal{R}(\mathrm{e}^{\mathrm{i}\pi\ell_r/2}\omega^{p_r}) \leq 0$ in order to satisfy that $h(c_r, t)$ be defined on $t \leq 0$. Thus one has either that (i) $\ell_r = 0$ and $p_r = 1$ (the real and decaying case, as per Proposition 8.1 in Section 8.1), or (ii) $\ell_r = 1$ and $p_r = 0, 1$ (the complex case and NON-decaying case). We remark that in case (ii) one has that $\mathcal{R}(c_r) = 0$ and thus there need not be decay at $-\infty$, as per Remark 16 following Corollary 8.1 in Section 8.1 along with Proposition 8.2 in Section 8.2. We will include an analysis of case (ii) here for completeness. But to be guaranteed to work with decaying Schwartz functions one should confine oneself to case (i).

From (71), let H be the 2×2 matrix with (m, r) entry, for $0 \leq m, r \leq 1$, given by

$$H_{m,r} = (-\gamma_r \mathrm{e}^{\mathrm{i}\pi\ell_r/2} \omega^{p_r})^m a_{0,r} \theta \left(q^2; \left[\frac{(-1)^{1-\ell_r}}{\gamma_r^2} \right] q^{\mu+m-1} \right) . \tag{114}$$

That is, H is given by

$$\begin{bmatrix} a_{0,0}\theta \left(q^2; \left[\frac{(-1)^{1-\ell_0}}{\gamma_0^2} \right] q^{\mu-1} \right) & a_{0,1}\theta \left(q^2; \left[\frac{(-1)^{1-\ell_1}}{\gamma_1^2} \right] q^{\mu-1} \right) \\ (-\gamma_0 \mathrm{e}^{\mathrm{i}\pi\ell_0/2} \omega^{p_0}) a_{0,0}\theta \left(q^2; \left[\frac{(-1)^{1-\ell_0}}{\gamma_0^2} \right] q^{\mu} \right) & (-\gamma_1 \mathrm{e}^{\mathrm{i}\pi\ell_1/2} \omega^{p_1}) a_{0,1}\theta \left(q^2; \left[\frac{(-1)^{1-\ell_1}}{\gamma_1^2} \right] q^{\mu} \right) \end{bmatrix} . \tag{115}$$

We then define the related matrix $\tilde{H}$ as given by

$$\begin{bmatrix} \theta \left(q^2; \left[\frac{(-1)^{1-\ell_0}}{\gamma_0^2} \right] q^{\mu-1} \right) & \theta \left(q^2; \left[\frac{(-1)^{1-\ell_1}}{\gamma_1^2} \right] q^{\mu-1} \right) \\ (-\gamma_0 \mathrm{e}^{\mathrm{i}\pi\ell_0/2} \omega^{p_0}) \theta \left(q^2; \left[\frac{(-1)^{1-\ell_0}}{\gamma_0^2} \right] q^{\mu} \right) & (-\gamma_1 \mathrm{e}^{\mathrm{i}\pi\ell_1/2} \omega^{p_1}) \theta \left(q^2; \left[\frac{(-1)^{1-\ell_1}}{\gamma_1^2} \right] q^{\mu} \right) \end{bmatrix} . \tag{116}$$

Now from (111) and (114), as well as from (67)–(69), $\boldsymbol{F} = H \cdot \boldsymbol{B}$ becomes

$$\begin{bmatrix} \theta(q^2; -q^{\mu-1}) \\ -\theta(q^2; -q^{\mu}) \end{bmatrix} = H \begin{bmatrix} b_0 \\ b_1 \end{bmatrix} = \tilde{H} \begin{bmatrix} a_{0,0} b_0 \\ a_{0,1} b_1 \end{bmatrix} , \tag{117}$$

where H is the 2×2 matrix in (115) and $\tilde{H}$ is the 2×2 matrix in (116). To check invertibility of $\tilde{H}$, one computes

$$\det\tilde{H} = (-\gamma_1 e^{i\pi\ell_1/2}\omega^{p_1})\theta\left(q^2;\left[\frac{(-1)^{1-\ell_1}}{\gamma_1^2}\right]q^{\mu}\right)\theta\left(q^2;\left[\frac{(-1)^{1-\ell_0}}{\gamma_0^2}\right]q^{\mu-1}\right)$$
$$-(-\gamma_0 e^{i\pi\ell_0/2}\omega^{p_0})\theta\left(q^2;\left[\frac{(-1)^{1-\ell_0}}{\gamma_0^2}\right]q^{\mu}\right)\theta\left(q^2;\left[\frac{(-1)^{1-\ell_1}}{\gamma_1^2}\right]q^{\mu-1}\right). \tag{118}$$

We now have three cases, namely: Cases A, B, and C below.

Case A. (The decaying Schwartz case). If $\ell_0 = 0$ with $p_0 = 1$ in (112) as well as (114)–(118), and if $\ell_1 = 0$ with $p_1 = 1$ in (113) as well as (114)–(118), then from (118) $\det\tilde{H} = 0$ holds if and only if

$$\gamma_1\theta\left(q^2;\left[\frac{-1}{\gamma_1^2}\right]q^{\mu}\right)\theta\left(q^2;\left[\frac{-1}{\gamma_0^2}\right]q^{\mu-1}\right) - \gamma_0\theta\left(q^2;\left[\frac{-1}{\gamma_0^2}\right]q^{\mu}\right)\theta\left(q^2;\left[\frac{-1}{\gamma_1^2}\right]q^{\mu-1}\right) = 0 \tag{119}$$

if and only if

$$\frac{\gamma_1\theta\left(q^2;\left[\frac{-1}{\gamma_1^2}\right]q^{\mu}\right)}{\theta\left(q^2;\left[\frac{-1}{\gamma_1^2}\right]q^{\mu-1}\right)} = \frac{\gamma_0\theta\left(q^2;\left[\frac{-1}{\gamma_0^2}\right]q^{\mu}\right)}{\theta\left(q^2;\left[\frac{-1}{\gamma_0^2}\right]q^{\mu-1}\right)}. \tag{120}$$

Thus we examine

$$G(\gamma) \equiv \frac{\gamma\theta\left(q^2;\left[\frac{-1}{\gamma^2}\right]q^{\mu}\right)}{\theta\left(q^2;\left[\frac{-1}{\gamma^2}\right]q^{\mu-1}\right)} \tag{121}$$

for $\gamma > 0$. Note that for each $p \in \mathbb{Z}$ one has that $G(q^{p+\mu/2}) = 0$, that $G(q^{p+\mu/2-1/2})$ is unbounded, and that $G(\gamma) \neq 0$ for other values of $\gamma > 0$, all by (25). Hence G is not constant. Furthermore setting $z = q^{\mu}/\gamma^2$, equivalently $\gamma = q^{\mu/2}/\sqrt{z}$, one re-expresses (121) as

$$G(q^{\mu/2}/\sqrt{z}) = \frac{q^{\mu/2}}{\sqrt{z}}\frac{\theta\left(q^2;-z\right)}{\theta\left(q^2;-z/q\right)} \equiv \tilde{G}(z), \tag{122}$$

which can be extended to be analytic in z in the set U consisting of the right complex half-plane away from the poles at $z = q^{2p+1}$. Since G is non-constant, one has that $\tilde{G}$ is not constant. Thus, by the identity theorem, $\tilde{G}(z)$ can only equal a given constant at a discrete set of points without cluster point in U. As a consequence, for fixed γ_0, (120) holds only for a discrete set of γ_1 without cluster point in $\mathbb{R}^+ \cap U$. Thus, $\det\tilde{H} \neq 0$ for generic choice of $\gamma_0, \gamma_1 > 0$ and $\mu \in \mathbb{R}$ in Case A. For such choices of parameters, from (117), we have

$$\begin{bmatrix} a_{0,0}b_0 \\ a_{0,1}b_1 \end{bmatrix} = \begin{bmatrix} \theta\left(q^2;\left[\frac{-1}{\gamma_0^2}\right]q^{\mu-1}\right) & \theta\left(q^2;\left[\frac{-1}{\gamma_1^2}\right]q^{\mu-1}\right) \\ \gamma_0\theta\left(q^2;\left[\frac{-1}{\gamma_0^2}\right]q^{\mu}\right) & \gamma_1\theta\left(q^2;\left[\frac{-1}{\gamma_1^2}\right]q^{\mu}\right) \end{bmatrix}^{-1} \begin{bmatrix} \theta(q^2;-q^{\mu-1}) \\ -\theta(q^2;-q^{\mu}) \end{bmatrix} = \tilde{H}^{-1}\cdot\boldsymbol{F}. \tag{123}$$

From (112), (113), and with $a_{0,0}b_0, a_{0,1}b_1$ as in (123), one has that a desired extension $h(t)$ of $f_{\mu,1}(t)$ is given by

$$h(t) = a_{0,0}b_0\sum_{M=-\infty}^{\infty}\left[\frac{-1}{\gamma_0^2}\right]^M\frac{e^{q^M\gamma_0 t}}{q^{M(M-\mu)}} + a_{0,1}b_1\sum_{M=-\infty}^{\infty}\left[\frac{-1}{\gamma_1^2}\right]^M\frac{e^{q^M\gamma_1 t}}{q^{M(M-\mu)}}, \tag{124}$$

which is real analytic for $t < 0$ and $\mathcal{C}^{\infty}$ for $t \le 0$. By Proposition 4.2, the extension $F_{\mu,1}(t)$ agreeing with $f_{\mu,1}(t)$ on $[0,\infty)$ and with $h(t)$ as in (124) on $(-\infty,0]$ is a Schwartz wavelet. See Fig. 2 (right) where an example of Case A is graphed with $\mu = .3$, $q = 3$, $\gamma_0 = .2$ and $\gamma_1 = 1.1$.

Case B. (Non-decaying in one term). Let $\ell_0 = 0$ and $p_0 = 1$ in (112) as well as (114)–(118), and let $\ell_1 = 1$ and $p_1 = 0, 1$ in (113) as well as (114)–(118). Then from (118) $\det\tilde{H} = 0$ holds if and only if

$$\pm i\gamma_1\theta\left(q^2;\left[\frac{1}{\gamma_1^2}\right]q^{\mu}\right)\theta\left(q^2;\left[\frac{-1}{\gamma_0^2}\right]q^{\mu-1}\right) - \gamma_0\theta\left(q^2;\left[\frac{-1}{\gamma_0^2}\right]q^{\mu}\right)\theta\left(q^2;\left[\frac{1}{\gamma_1^2}\right]q^{\mu-1}\right) = 0. \tag{125}$$

By examining real and imaginary parts in (125), and recognizing that (by (25)) neither $\theta(q^2;q^{\mu}/\gamma_1^2)$ nor $\theta(q^2;q^{\mu-1}/\gamma_1^2)$ vanish, one sees that (125) holds if and only if

$$\theta\left(q^2;\left[\frac{-1}{\gamma_0^2}\right]q^{\mu-1}\right)=0=\theta\left(q^2;\left[\frac{-1}{\gamma_0^2}\right]q^{\mu}\right). \tag{126}$$

From (25), the vanishing in (126) occurs when there are $p,\hat{p}\in\mathbb{Z}$ with $q^{2p}=q^{\mu-1}/\gamma_0^2$ and $q^{2\hat{p}}=q^{\mu}/\gamma_0^2$. However, dividing the second equation by the first, one would have $q^{2[\hat{p}-p]}=q^1$. That is $\hat{p}-p=1/2$, a contradiction. Thus $\det\tilde{H}\neq 0$ for all choices of $\gamma_0,\gamma_1>0$ and $\mu\in\mathbb{R}$ in Case B.

From (117), for all choices of parameters γ_0,γ_1,μ in Case B we have

$$\begin{bmatrix} a_{0,0}b_0 \\ a_{0,1}b_1\end{bmatrix}=\begin{bmatrix}\theta\left(q^2;\left[\frac{-1}{\gamma_0^2}\right]q^{\mu-1}\right) & \theta\left(q^2;\left[\frac{1}{\gamma_1^2}\right]q^{\mu-1}\right)\\ \gamma_0\theta\left(q^2;\left[\frac{-1}{\gamma_0^2}\right]q^{\mu}\right) & \pm i\gamma_1\theta\left(q^2;\left[\frac{1}{\gamma_1^2}\right]q^{\mu}\right)\end{bmatrix}^{-1}\begin{bmatrix}\theta(q^2;-q^{\mu-1})\\ -\theta(q^2;-q^{\mu})\end{bmatrix}=\tilde{H}^{-1}\cdot\boldsymbol{F}. \tag{127}$$

From (112), (113), and with $a_{0,0}b_0$, $a_{0,1}b_1$ as in (127), one has that the desired extension $h(t)$ of $f_{\mu,1}(t)$ is given by

$$h(t)=a_{0,0}b_0\sum_{M=-\infty}^{\infty}\left[\frac{-1}{\gamma_0^2}\right]^M\frac{e^{q^M\gamma_0 t}}{q^{M(M-\mu)}}+a_{0,1}b_1\sum_{M=-\infty}^{\infty}\left[\frac{1}{\gamma_1^2}\right]^M\frac{e^{\pm iq^M\gamma_1 t}}{q^{M(M-\mu)}}, \tag{128}$$

where the sign of the $\pm$ in (128) matches the sign of the $\pm$ in both of the equations (127) and (125). The extension $h(t)$ in (128) is not real analytic in t, but it is $\mathcal{C}^\infty$ in t on $(-\infty,0]$. Observe that by Proposition 8.1 below, the first summation in (128) decays as t approaches $-\infty$, as does its real part. By Proposition 8.2, along with Remark 17 immediately following the proof of Proposition 8.2, one sees that the real part of the second summation in (128) does not decay for fixed real μ and rational $q>1$, and for all $\gamma_1>0$ and generic choice of $\gamma_0>0$. That is, in this setting, one only need to check the non-vanishing of (184) to establish non-decay of the second summation. From (128), this amounts to checking

$$0\neq\mathcal{R}\left(a_{0,1}b_1\sum_{M=-\infty}^{\infty}\left[\frac{1}{\gamma_1^2}\right]^M\frac{1}{q^{M(M-\mu)}}\right)=\mathcal{R}\left(a_{0,1}b_1\theta\left(q^2;\frac{1}{\gamma_1^2}q^{\mu-1}\right)\right)=\left[\mathcal{R}\left(a_{0,1}b_1\right)\right]\theta\left(q^2;\frac{1}{\gamma_1^2}q^{\mu-1}\right),$$

and since $\theta\left(q^2;q^{\mu-1}/\gamma_1^2\right)>0$ this is equivalent to checking that

$$0\neq\mathcal{R}\left(a_{0,1}b_1\right). \tag{129}$$

From equation (127), the real part of $a_{0,1}b_1$ is computed to be

$$\mathcal{R}(a_{0,1}b_1)=\frac{\theta\left(q^2;\left[\frac{1}{\gamma_1^2}\right]q^{\mu-1}\right)\cdot\gamma_0\left[\theta\left(q^2;\left[\frac{-1}{\gamma_0^2}\right]q^{\mu}\right)\right]}{\theta\left(q^2;\left[\frac{-1}{\gamma_0^2}\right]q^{\mu-1}\right)^2\gamma_1^2\theta\left(q^2;\left[\frac{1}{\gamma_1^2}\right]q^{\mu}\right)^2+\theta\left(q^2;\left[\frac{1}{\gamma_1^2}\right]q^{\mu-1}\right)^2\gamma_0^2\theta\left(q^2;\left[\frac{-1}{\gamma_0^2}\right]q^{\mu}\right)^2}$$
$$\cdot\left\{\gamma_0\theta\left(q^2;\left[\frac{-1}{\gamma_0^2}\right]q^{\mu}\right)\theta(q^2;-q^{\mu-1})+\theta\left(q^2;\left[\frac{-1}{\gamma_0^2}\right]q^{\mu-1}\right)\theta(q^2;-q^{\mu})\right\} \tag{130}$$

which is seen to vanish precisely when either the factor in square brackets vanishes, that is, when

$$\theta\left(q^2;\left[\frac{-1}{\gamma_0^2}\right]q^{\mu}\right)=0, \tag{131}$$

or when the factor in braces in (130) vanishes, that is, when

$$-\frac{\theta(q^2;-q^{\mu})}{\theta(q^2;-q^{\mu-1})}=\frac{\gamma_0\theta\left(q^2;\left[\frac{-1}{\gamma_0^2}\right]q^{\mu}\right)}{\theta\left(q^2;\left[\frac{-1}{\gamma_0^2}\right]q^{\mu-1}\right)}=G(\gamma_0), \tag{132}$$

where $G(\gamma)$ is given by (121) and has been seen to be non-constant in γ in Case A. This implies that, for fixed q and μ, condition (132) holds only for a discrete set of γ_0 without limit point in $\mathbb{R}^+$. Thus, for fixed q and μ, the vanishing of the factor in braces in (130) holds only for a discrete set of γ_0 without limit point in $\mathbb{R}^+$, a non-generic condition in γ_0. From (25), the vanishing of (131) occurs only when $-q^{\mu-1}/\gamma_0^2=q^{2p}$ for some integer p, which, for fixed q and μ, is a non-generic condition in γ_0. The above statements hold for all choices of γ_1 as the factor $\theta\left(q^2;q^{\mu-1}/\gamma_1^2\right)$ in equation (130) never vanishes, again by (25). Thus for rational $q>1$ and arbitrary μ both fixed, and for all choices of $\gamma_1>0$ and generic

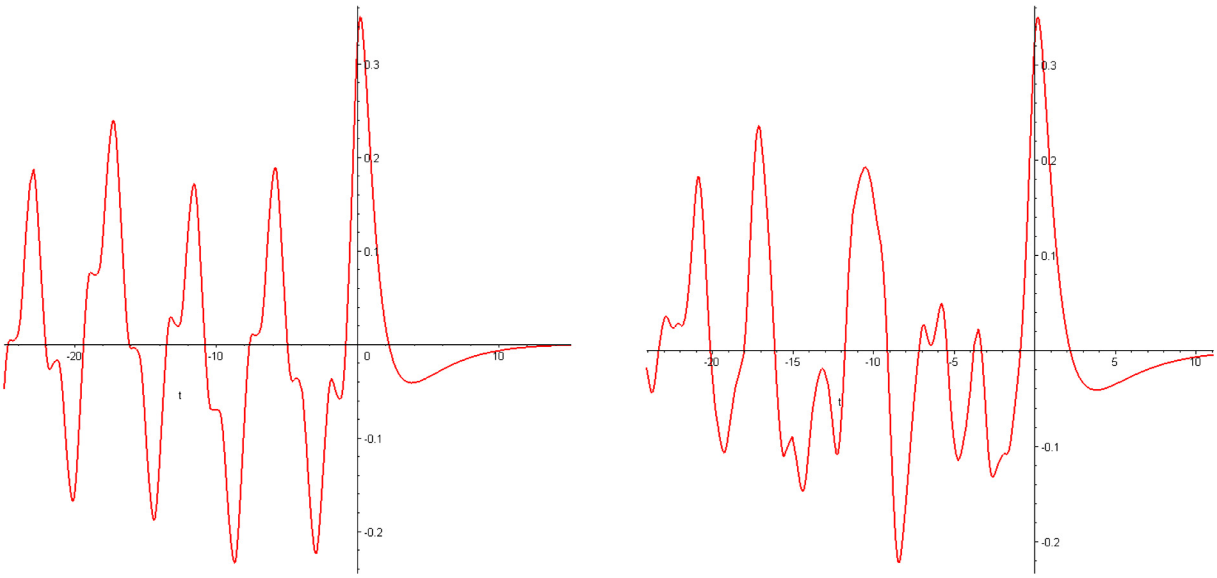

Fig. 3. Left: In Example 2 Case B with $\lambda = 1$ (decaying in one term and non-decaying in the second), $f_{\mu,1}(t)$ is extended (non-uniquely) to the negative real axis solving the relevant MADE (110), as is illustrated here when $\mu = 0.3$, $q = 3$, with the values $\gamma_0 = 0.2$ and $\gamma_1 = 1.1$ in (128), where $\ell_0 = 0$, $p_0 = 1$, $\ell_1 = 1$, $p_1 = 0$. Compare with Case A in Fig. 2 Right and Case C on the right, where $f_{0.3,1}(t)$ with $q = 3$ is also extended to $(-\infty, 0]$ in different manners. **Right:** In Example 2 Case C (i) with $\lambda = 1$ (non-decaying in both terms), $f_{\mu,1}(t)$ is extended (non-uniquely) to the negative real axis solving the relevant MADE (110), as is illustrated here when $\mu = 0.3$, $q = 3$, with the values $\gamma_0 = 0.2$ and $\gamma_1 = 1.1$, in (136), where $\ell_0 = 1$, $p_0 = 0$, $\ell_1 = 1$, $p_1 = 1$. Compare with Case A in Fig. 2 Right and Case B on the left, where $f_{0.3,1}(t)$ with $q = 3$ is also extended to $(-\infty, 0]$ in different manners.

choice of $\gamma_0 > 0$, the non-decay of the real part of the second summation in (128) is established. In this setting, taking the real part of $h(t)$ in (128) gives a real extension of $f_{\mu,1}(t)$ to $t < 0$, yielding an extension to $\mathbb{R}$ that is not among the $F_{\mu,\lambda}(t)$ of Definition 4.1 and that is non-decaying at $-\infty$. See Fig. 3 Left, where, for $\mu = .3$, $q = 3$, $\gamma_0 = .2$, and $\gamma_1 = 1.1$, we obtain $\mathcal{R}(a_{0,1}b_1) \approx .1359902757 \neq 0$, providing an example that is non-decaying in one term.

Case C. (Non-decaying in both terms). Let $\ell_0 = 1$ and $p_0 = 0, 1$ in (112) as well as in (114)–(118), and let $\ell_1 = 1$ and $p_1 = 0, 1$ in (113) as well as in (114)–(118). Then we have two subcases: (i) $p_0 \neq p_1$ where w.l.o.g. we assume $p_0 = 0$ and $p_1 = 1$, and (ii) $p_0 = p_1$ where both are 0 or both are 1.

In Case C (i), from (118) $\det \tilde{H} = 0$ holds if and only if

$$\mathrm{i}\gamma_1\theta\left(q^2;\left[\frac{1}{\gamma_1^2}\right]q^{\mu}\right)\theta\left(q^2;\left[\frac{1}{\gamma_0^2}\right]q^{\mu-1}\right)+\mathrm{i}\gamma_0\theta\left(q^2;\left[\frac{1}{\gamma_0^2}\right]q^{\mu}\right)\theta\left(q^2;\left[\frac{1}{\gamma_1^2}\right]q^{\mu-1}\right)=0 \tag{133}$$

if and only if

$$\frac{\gamma_1\theta\left(q^2;\left[\frac{1}{\gamma_1^2}\right]q^{\mu}\right)}{\theta\left(q^2;\left[\frac{1}{\gamma_1^2}\right]q^{\mu-1}\right)}=-\frac{\gamma_0\theta\left(q^2;\left[\frac{1}{\gamma_0^2}\right]q^{\mu}\right)}{\theta\left(q^2;\left[\frac{1}{\gamma_0^2}\right]q^{\mu-1}\right)}. \tag{134}$$

From (22), one sees that $\theta(Q;\omega) > 0$ for $\omega \in \mathbb{R}^+$, and since $\gamma_0, \gamma_1 > 0$, one sees that if (134) were to hold then a positive number would equal a negative number. Thus (134) cannot hold, with the consequence that $\det \tilde{H} \neq 0$ for all choices of parameters γ_0, γ_1, μ in Case C (i), and in this case one has from (117) that for all choices of parameters γ_0, γ_1, μ we have

$$\begin{bmatrix} a_{0,0}b_0 \\ a_{0,1}b_1 \end{bmatrix}=\begin{bmatrix} \theta\left(q^2;\left[\frac{1}{\gamma_0^2}\right]q^{\mu-1}\right) & \theta\left(q^2;\left[\frac{1}{\gamma_1^2}\right]q^{\mu-1}\right) \\ -\mathrm{i}\gamma_0\theta\left(q^2;\left[\frac{1}{\gamma_0^2}\right]q^{\mu}\right) & \mathrm{i}\gamma_1\theta\left(q^2;\left[\frac{1}{\gamma_1^2}\right]q^{\mu}\right) \end{bmatrix}^{-1}\begin{bmatrix} \theta(q^2;-q^{\mu-1}) \\ -\theta(q^2;-q^{\mu}) \end{bmatrix}=\tilde{H}^{-1}\cdot\boldsymbol{F}. \tag{135}$$

From (112), (113), and with $a_{0,0}b_0$, $a_{0,1}b_1$ as in (135), one has that the desired extension $h(t)$ of $f_{\mu,1}(t)$ is given by

$$h(t)=a_{0,0}b_0\sum_{M=-\infty}^{\infty}\left[\frac{1}{\gamma_0^2}\right]^M\frac{\mathrm{e}^{-\mathrm{i}q^M\gamma_0 t}}{q^{M(M-\mu)}}+a_{0,1}b_1\sum_{M=-\infty}^{\infty}\left[\frac{1}{\gamma_1^2}\right]^M\frac{\mathrm{e}^{\mathrm{i}q^M\gamma_1 t}}{q^{M(M-\mu)}}. \tag{136}$$

The extension $h(t)$ in (136) is not real analytic in t, but it is $\mathcal{C}^\infty$ in t on $(-\infty, 0]$. By Proposition 8.2, for μ not an odd integer, one observes non-decay of the real part of $h(t)$ in (136) for rational choice of q, γ_0, γ_1. That is, to check the non-vanishing of (184) in Proposition 8.2 one only need check in our setting that $\mathcal{R}(h(0)) \neq 0$. From (136) and (135), one observes $\mathcal{R}(h(0)) = \theta(q^2; -q^{\mu-1})$ and sees that for all choices of $\gamma_0, \gamma_1 > 0$ it never vanishes when μ is not an odd integer. Hence, by Proposition 8.2, with fixed μ not an odd integer, $\mathcal{R}(h(t))$ does not decay for fixed rational $q > 1$, for all rational choices of $\gamma_0, \gamma_1 > 0$. Taking the real part of such an $h(t)$ in (136) gives a real extension of $f_{\mu,1}(t)$ to $t < 0$, and yields an extension to all of $\mathbb{R}$ that does not fall among the $F_{\mu,\lambda}(t)$ of Definition 4.1. Furthermore, this extension does not decay at $-\infty$. See Fig. 3 Right for an illustration of such a non-decaying extension where $\mu = .3$ is not an odd integer, with $q = 3$, $\gamma_0 = .2$ and $\gamma_1 = 1.1$.

In Case C (ii), from (118) $\det \tilde{H} = 0$ holds if and only if

$$0 = (-1)^{p_0+1} \left[\mathrm{i}\gamma_1 \theta \left(q^2; \left[\frac{1}{\gamma_1^2} \right] q^\mu \right) \theta \left(q^2; \left[\frac{1}{\gamma_0^2} \right] q^{\mu-1} \right) - \mathrm{i}\gamma_0 \theta \left(q^2; \left[\frac{1}{\gamma_0^2} \right] q^\mu \right) \theta \left(q^2; \left[\frac{1}{\gamma_1^2} \right] q^{\mu-1} \right) \right] \tag{137}$$

if and only if

$$\frac{\gamma_1 \theta \left(q^2; \left[\frac{1}{\gamma_1^2} \right] q^\mu \right)}{\theta \left(q^2; \left[\frac{1}{\gamma_1^2} \right] q^{\mu-1} \right)} = \frac{\gamma_0 \theta \left(q^2; \left[\frac{1}{\gamma_0^2} \right] q^\mu \right)}{\theta \left(q^2; \left[\frac{1}{\gamma_0^2} \right] q^{\mu-1} \right)}. \tag{138}$$

Observe that

$$\frac{z \theta \left(q^2; \left[\frac{1}{z^2} \right] q^\mu \right)}{\theta \left(q^2; \left[\frac{1}{z^2} \right] q^{\mu-1} \right)} \tag{139}$$

is defined and analytic on the connected open region $\mathbb{C}^* \backslash D$, where $D = \left\{ \pm \mathrm{i} q^{\mu/2 - p - 1/2} \mid p \in \mathbb{Z} \right\}$ is the set of poles of order 1 of (139). For $z = \mathrm{i}\gamma$, the expression in (139) becomes $\mathrm{i}G(\gamma)$, where G is given by (121). From the remarks following (121), $G(\gamma)$ is not constant in γ. Thus (139) cannot be constant in $\mathrm{i}\gamma$, nor on any set in $\mathbb{C}^* \backslash D$ having a limit point, including any such subset of $\mathbb{R}^+$. Thus fixing γ_0 in (138) there is only a discrete set of γ_1 without limit point with equality in (138) holding. Thus for fixed γ_0 one has $\det \tilde{H} \neq 0$ for a generic choice of γ_1 in Case C (ii), and in this case one has from (117) that for such a generic choice of parameters γ_0, γ_1, μ

$$\begin{bmatrix} a_{0,0} b_0 \\ a_{0,1} b_1 \end{bmatrix} = \tilde{H}^{-1} \cdot \boldsymbol{F} \tag{140}$$

$$= \begin{bmatrix} \theta \left(q^2; \left[\frac{1}{\gamma_0^2} \right] q^{\mu-1} \right) & \theta \left(q^2; \left[\frac{1}{\gamma_1^2} \right] q^{\mu-1} \right) \\ -\mathrm{i}\gamma_0 (-1)^p \theta \left(q^2; \left[\frac{1}{\gamma_0^2} \right] q^\mu \right) & -\mathrm{i}\gamma_1 (-1)^p \theta \left(q^2; \left[\frac{1}{\gamma_1^2} \right] q^\mu \right) \end{bmatrix}^{-1} \begin{bmatrix} \theta(q^2; -q^{\mu-1}) \\ -\theta(q^2; -q^\mu) \end{bmatrix},$$

where p is the common value of $p_0 = p_1$. From (112), (113), and with $a_{0,0}b_0, a_{0,1}b_1$ as in (140), one has that the desired extension $h(t)$ of $f_{\mu,1}(t)$ is given by

$$h(t) = a_{0,0} b_0 \sum_{M=-\infty}^{\infty} \left[\frac{1}{\gamma_0^2} \right]^M \frac{\mathrm{e}^{-\mathrm{i} q^M \gamma_0 (-1)^p t}}{q^{M(M-\mu)}} + a_{0,1} b_1 \sum_{M=-\infty}^{\infty} \left[\frac{1}{\gamma_1^2} \right]^M \frac{\mathrm{e}^{-\mathrm{i} q^M \gamma_1 (-1)^p t}}{q^{M(M-\mu)}}, \tag{141}$$

where p is the common value of $p_0 = p_1$. The extension $h(t)$ in (141) is not real analytic in t, but it is $\mathcal{C}^\infty$ in t on $(-\infty, 0]$. By Proposition 8.2, for fixed μ not an odd integer, one observes non-decay of the real part of $h(t)$ in (141) for fixed rational $q > 1$, for each $\gamma_0 > 0$ and for each γ_1 lying in an open dense set of $(0, \infty)$ by observing that $\mathcal{R}(h(0)) = \theta(q^2; -q^{\mu-1}) \neq 0$ in these cases. Taking the real part of such an $h(t)$ in (141), with γ_0, γ_1 as above and rational, gives a real extension of $f_{\mu,1}(t)$ to $t < 0$, and yields an extension to all of $\mathbb{R}$ that does not fall among the $F_{\mu,\lambda}(t)$ of Definition 4.1. Furthermore, this extension need not decay at $-\infty$.

Remark 12. In all of the cases A through C in Example 2, one has that $\det \tilde{H} \neq 0$ generically. Thus in Theorem 3.2 one has that the assumption that $\det H \neq 0$ holds generically in that $\det H = a_{0,0} a_{0,1} \det \tilde{H}$. Again, the hypothesis that $\det H \neq 0$ holds is a non-restrictive one, and we have a rich set of parameters for extension.

Remark 13. The multitude of answers produced in Example 2 (with the $\lambda = 2L/k = 1$ and $L = 1 = A$, $k = 2 = R$) shows again that one does not have uniqueness of solutions to initial value problems for MADEs at $t = 0$. For instance, examine the graphs in Fig. 2 Right and Fig. 3 Left and Right showing three different types of extensions of $f_{\mu,1}(t)$ to $\mathbb{R}$ for $\mu = 0.3$ and $q = 3$. We do note that the initial value problem for MADEs, coupled with the stronger additional requirement that solutions must agree on an open interval containing $t = 0$, will indeed have uniqueness of its solution, as indicated in Corollary 7.2.

Remark 14. In Case A of Example 2, if one sets $\mu = 0$ with $\lambda = 1$ and $\gamma_0 = 1$, and chooses $\gamma_1 \neq 1$ to be generic, then one obtains that $a_{0,0}b_0 = 1$ and $a_{0,1}b_1 = 0$ in (123) and (124). The resulting extension $F_{0,1}(t)$ of $f_{0,1}(t)$ is an even function, which when normalized by dividing by $F_{0,1}(0)$ gives the ${}_qCos(t)$ introduced in [24]. Furthermore, in Case A, if one sets $\mu = 1$ with $\lambda = 1$ and $\gamma_0 = 1$ and chooses $\gamma_1 \neq 1$ to be generic, then one obtains that $a_{0,0}b_0 = 0$ and $a_{0,1}b_1 = 1$ in (123) and (124). The resulting extension $F_{1,1}(t)$ of $f_{1,1}(t)$ is an odd function, which, when normalized by dividing by $F_{0,1}(0)$ as well, gives the ${}_qSin(t)$ also introduced in [24]. From this perspective, we see both Example 2 and Example 1 as significant generalizations of functions previously studied in [22], [23], [24].

Example 3. In this short example, we illustrate a case where $\lambda = (2L+1)/k$ has an odd numerator. Namely, we set $L = 0$ and $k = 1$, to obtain $\lambda = (2 \cdot 0 + 1)/1 = 1$. In this setting, we have $A = 2L + 1 = 2 \cdot 0 + 1 = 1$ and $R = 2k = 2$, with (18) becoming

$$f^{(2)}_{\mu,1}(t) = (-1)^{2+1} q^{1(1+\mu)/1} f_{\mu,1}(q^1 t) = -q^{1+\mu} f_{\mu,1}(qt)\,. \tag{142}$$

Thus, (142) becomes (110), and we proceed verbatim as in Example 2.

6. Fourier transforms and Jacobi theta functions

In this section let $F_{\mu,\lambda}(t)$, as in Definition 4.1, be any extension of $f_{\mu,\lambda}(t)$ to $\mathbb{R}$ matching derivatives of all orders at $t = 0$ and satisfying the MADE (18). Such an $F_{\mu,\lambda}(t)$ is Schwartz, as was shown in Proposition 4.2.

6.1. The general case: computation of the Fourier transform of $F_{\mu,\lambda}(t)$

The computation of the Fourier transform of $F_{\mu,\lambda}(t)$ is divided into two calculations, as follows:

$$\begin{aligned}\mathcal{F}[F_{\mu,\lambda}(t)](x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}xt} F_{\mu,\lambda}(t)\,\mathrm{d}t \\ &= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \mathrm{e}^{-\mathrm{i}xt} f_{\mu,\lambda}(t)\,\mathrm{d}t \;+\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}xt} \mathcal{R}(h(t))\,\mathrm{d}t\,.\end{aligned}$$

The integral over $[0,\infty)$ and the integral over $(-\infty, 0]$ are computed in the following pair of propositions.

Proposition 6.1. *For $\mu \in \mathbb{R}$, $\lambda \in \mathbb{R}^+$, and $q > 1$, one has*

$$\int_0^{\infty} \mathrm{e}^{-\mathrm{i}xt} f_{\mu,\lambda}(t)\,\mathrm{d}t = \sum_{k=-\infty}^{\infty} \left[\frac{(-1)^k}{q^{k(k-\mu)/\lambda}} \frac{1}{(\mathrm{i}\,x + q^k)} \right]. \tag{143}$$

Proof. The computation proceeds directly. Note that

$$\begin{aligned}\int_0^{\infty} \mathrm{e}^{-\mathrm{i}xt} f_{\mu,\lambda}(t)\,\mathrm{d}t &= \int_0^{\infty} \mathrm{e}^{-\mathrm{i}xt} \sum_{k=-\infty}^{\infty} (-1)^k \frac{\mathrm{e}^{-q^k t}}{q^{k(k-\mu)/\lambda}}\,\mathrm{d}t \\ &= \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{q^{k(k-\mu)/\lambda}} \int_0^{\infty} \mathrm{e}^{\left(-\mathrm{i}x - q^k\right)t}\,\mathrm{d}t = \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{q^{k(k-\mu)/\lambda}} \left[0 - \frac{1}{(-\mathrm{i}\,x - q^k)} \right],\end{aligned}$$

giving (143). Here moving the integral past the summation is justified by absolute summability of $f_{\mu,\lambda}(t)$. □

Proposition 6.2. *For $\mu \in \mathbb{R}$, $\lambda \in \mathbb{Q}^+$, and $q > 1$, let $h(t)$ as in (72) be an extension of $f_{\mu,\lambda}(t)$ as in Theorem 3.2, where the $r, b_r, \gamma_r, \ell_r, j$, and p_r are as in Theorem 3.2, and $\lambda = 2A/R$ with A given by (16) and R given by (17). One has*

$$\int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}xt}\mathcal{R}(h(t))\,\mathrm{d}t$$
$$= \frac{1}{2}\sum_{r=0}^{R-1} b_r \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{M=-\infty}^{\infty} \left\{ \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right]^M \frac{1}{q^{(MA+j)(MA+j-\mu)/\lambda}} \frac{(-1)}{[\mathrm{i}x + q^{MA+j}\gamma_r \mathrm{e}^{\mathrm{i}\pi\ell_r/R}\omega^{p_r}]} \right\}\right]$$
$$+ \frac{1}{2}\sum_{r=0}^{R-1} \overline{b_r} \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{M=-\infty}^{\infty} \left\{ \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right]^M \frac{1}{q^{(MA+j)(MA+j-\mu)/\lambda}} \frac{(-1)}{[\mathrm{i}x + q^{MA+j}\gamma_r \mathrm{e}^{-\mathrm{i}\pi\ell_r/R}\omega^{R-p_r}]} \right\}\right]. \tag{144}$$

Proof. Given an extension $h(t)$ as in (72) of Theorem 3.2, one has $\mathcal{R}(h(t)) = [h(t) + \overline{h(t)}]/2$. Thus

$$\mathcal{R}(h(t)) = \frac{1}{2}\left[\sum_{r=0}^{R-1} b_r h(c_r, t) + \sum_{r=0}^{R-1} \overline{b_r h(c_r, t)}\right]$$
$$= \frac{1}{2}\sum_{r=0}^{R-1} b_r \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{M=-\infty}^{\infty} \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right]^M \frac{\mathrm{e}^{-q^{MA+j}\gamma_r \mathrm{e}^{\mathrm{i}\pi\ell_r/R}\omega^{p_r} t}}{q^{(MA+j)(MA+j-\mu)/\lambda}}\right]$$
$$+ \frac{1}{2}\sum_{r=0}^{R-1} \overline{b_r} \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{M=-\infty}^{\infty} \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right]^M \frac{\mathrm{e}^{-q^{MA+j}\gamma_r \mathrm{e}^{-\mathrm{i}\pi\ell_r/R}\omega^{R-p_r} t}}{q^{(MA+j)(MA+j-\mu)/\lambda}}\right],$$

from which one has

$$\int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}xt}\mathcal{R}(h(t))\,\mathrm{d}t$$
$$= \frac{1}{2}\sum_{r=0}^{R-1} b_r \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{M=-\infty}^{\infty} \left\{ \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right]^M \frac{1}{q^{(MA+j)(MA+j-\mu)/\lambda}} \int_{-\infty}^{0} \mathrm{e}^{[-\mathrm{i}x - q^{MA+j}\gamma_r \mathrm{e}^{\mathrm{i}\pi\ell_r/R}\omega^{p_r}]t}\mathrm{d}t \right\}\right]$$
$$+ \frac{1}{2}\sum_{r=0}^{R-1} \overline{b_r} \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{M=-\infty}^{\infty} \left\{ \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right]^M \frac{1}{q^{(MA+j)(MA+j-\mu)/\lambda}} \int_{-\infty}^{0} \mathrm{e}^{[-\mathrm{i}x - q^{MA+j}\gamma_r \mathrm{e}^{-\mathrm{i}\pi\ell_r/R}\omega^{R-p_r}]t}\mathrm{d}t \right\}\right]$$

giving (144) after evaluating the integrals. □

The previous propositions immediately give the following result on Fourier transforms.

Theorem 6.1. *For $\mu \in \mathbb{R}$, $\lambda \in \mathbb{Q}^+$, and $q > 1$, let $f_{\mu,\lambda}(t)$ be given by (2). Let $F_{\mu,\lambda}(t)$, as in Definition 4.1, be any extension of $f_{\mu,\lambda}(t)$ to $\mathbb{R}$ matching derivatives of all orders at $t = 0$ and satisfying the MADE (18). Then the Fourier transform $\mathcal{F}[F_{\mu,\lambda}(t)](x)$ is given by*

$$\mathcal{F}[F_{\mu,\lambda}(t)](x) \tag{145}$$
$$= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \left[\frac{(-1)^k}{q^{k(k-\mu)/\lambda}} \frac{1}{(\mathrm{i}x + q^k)}\right] \tag{146}$$
$$+ \frac{1}{2\sqrt{2\pi}}\sum_{r=0}^{R-1} b_r \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{M=-\infty}^{\infty} \left\{ \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right]^M \frac{1}{q^{(MA+j)(MA+j-\mu)/\lambda}} \frac{(-1)}{[\mathrm{i}x + q^{MA+j}\gamma_r \mathrm{e}^{\mathrm{i}\pi\ell_r/R}\omega^{p_r}]} \right\}\right]$$
$$+ \frac{1}{2\sqrt{2\pi}}\sum_{r=0}^{R-1} \overline{b_r} \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{M=-\infty}^{\infty} \left\{ \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right]^M \frac{1}{q^{(MA+j)(MA+j-\mu)/\lambda}} \frac{(-1)}{[\mathrm{i}x + q^{MA+j}\gamma_r \mathrm{e}^{-\mathrm{i}\pi\ell_r/R}\omega^{R-p_r}]} \right\}\right]$$

where the r, b_r, γ_r, ℓ_r, j, $a_{j,r}$ and p_r are as in Theorem 3.2, and A and R are given by (16) respectively (17) with $\lambda = 2A/R$. Finally, $\mathcal{F}[F_{\mu,\lambda}(t)](x)$ vanishes to infinite order at $x = 0$.

Proof. Equality in (146) follows directly from the remarks beginning this section along with Propositions 6.1 and 6.2. The vanishing to infinite order of $\mathcal{F}[F_{\mu,\lambda}(t)](x)$ at $x=0$ follows from the vanishing of all moments of $F_{\mu,\lambda}(t)$ as shown in Proposition 4.1. We remark that the third summation grouping in (146) is not the conjugation of the second summation grouping in (146) when $x \neq 0$. □

6.2. Special cases: relating the Fourier transform of $F_{\mu,\lambda}(t)$ to the Jacobi theta function

We next demonstrate a relation of the above Fourier transforms to the Jacobi theta function in two cases: the first case contains those $f_{\mu,\lambda}(t)$ that are flat at $t=0$ and are extended to be identically 0 on the negative reals; the second case contains those $f_{\mu,1}(t)$ that can be extended to the negative reals to give either even or odd functions. Similar relations should hold in the other remaining cases, in that we expect the expression for the Fourier transform in Theorem 6.1 to be expressible in terms of Jacobi theta functions. The general case involves a delicate contour integration over regions in corresponding Riemann surfaces. To avoid such issues, and for conciseness, we restrict ourselves here to the two simpler cases mentioned above. In these two cases, the full proofs are given in Section 9.

We proceed to the flat case. Recall by equation (33) of Proposition 2.2 that $f_{\mu,\lambda}(t)$ is flat if and only if $\mu = 2N+1$ is an odd integer and $\lambda = 2n$ is an even integer. In this flat setting the derivatives of all orders vanish at $t=0$, and by (59) (equivalently (60)) one obtains the column matrix $\boldsymbol{F}=0$ in (60), and thus the column matrix $\boldsymbol{B}=0$ in (60) yielding that each entry in $\boldsymbol{B}$, namely b_r, vanishes. Thus, in the flat case, the methods of Theorem 3.2 only give the identically zero extension on $(-\infty, 0]$. In this flat setting extended to be 0 on the negative reals, we refer to the extension as $F_{2N+1,2n}(t)$ and immediately obtain the following corollary to Theorem 6.1.

Corollary 6.2. *For $f_{2N+1,2n}(t)$ flat at $t=0$, extend $f_{2N+1,2n}(t)$ to be identically 0 on $\mathbb{R}^-$ to obtain $F_{2N+1,2n}(t)$. The Fourier transform of the extension $F_{2N+1,2n}(t)$ is expressible as:*

$$\mathcal{F}[F_{2N+1,2n}(t)](x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \left[\frac{(-1)^k}{q^{k(k-[2N+1])/(2n)}} \frac{1}{(\mathrm{i}x + q^k)} \right]. \tag{147}$$

Proof. Set $\mu = 2N+1$, $\lambda = 2n$, and $b_r = 0$ for $0 \le r \le R-1$ in the Fourier transform expression (146) of Theorem 6.1. □

Relying on Corollary 6.2, one can relate the Fourier transform in the flat setting to an expression involving the Jacobi theta function as given by (22). Thus we arrive at a main result of this work: relating the Fourier transforms in (147) to special function theory while generalizing the work in [22].

Theorem 6.3. *For $f_{2N+1,2n}(t)$ flat at $t=0$, extend $f_{2N+1,2n}(t)$ to be identically 0 on $\mathbb{R}^-$ to obtain $F_{2N+1,2n}(t)$. The Fourier transform of the extension $F_{2N+1,2n}(t)$ is expressible in terms of the Jacobi theta function θ via the following finite sum*

$$\mathcal{F}[F_{2N+1,2n}(t)](x) = \frac{(-1)^N}{\sqrt{2\pi}} \mu^3_{q^{1/n}} q^{N(N+1)/(2n)} \frac{1}{\mathrm{i}x} \left[\frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{\theta(q^{1/n}; z_j(x)/q^{(N+1)/n})} \right] \tag{148}$$

where for $0 \le j \le n-1$

$$z_j(x) = -|x|^{1/n} \mathrm{e}^{3\pi \mathrm{i}/[2n]} \mathrm{e}^{\mathrm{i}[\arg(x)]/n} \mathrm{e}^{\mathrm{i}2\pi j/n} = -|x|^{1/n} \mathrm{e}^{3\pi \mathrm{i}/[2n]} \mathrm{e}^{\mathrm{i}[\arg(x)]/n} \omega^j \tag{149}$$

where $\omega = \mathrm{e}^{\mathrm{i}2\pi/n}$. Equivalently, the $z_j(x) \equiv z_j$ are the n distinct solutions of $(-z_j)^n = -\mathrm{i}x$.

The proof of Theorem 6.3 is given as a series of propositions in Section 9 below.

We now are in a position to obtain the following corollary, which is useful in substantiating the claim in the proof of Proposition 2.3 that $f_{\mu,\lambda}(t)$ cannot be identically 0. First, if $f_{\mu,\lambda}(t)$ is not flat at $t=0$ then it cannot be the identically zero function. However, if $f_{\mu,\lambda}(t)$ is flat at $t=0$, then by (33) one has $\mu = 2N+1$ is an odd integer, and $\lambda = 2n$ is an even positive integer. In this flat case, the extension $F_{2N+1,2n}(t)$ of $f_{2N+1,2n}(t)$ is defined to be zero for negative t. Now $F_{2N+1,2n}(t)$ cannot identically vanish, because its Fourier transform cannot vanish, as is seen in the next corollary.

Corollary 6.4. *$\mathcal{F}[F_{2N+1,2n}(t)](x)$ is not identically 0 in x. Hence $F_{2N+1,2n}(t)$ is not identically 0 in t.*

Proof. By (148) and (149) and the identity theorem, in order to see that $\mathcal{F}[F_{2N+1,2n}(t)](x)$ is not identically 0 in x it is sufficient to show that the function

$$\sum_{j=0}^{n-1} \frac{1}{\theta(Q\,;z\omega^j)} = \frac{\displaystyle\sum_{j=0}^{n-1}\left(\prod_{0\le k\ne j}^{n-1} \theta(Q\,;z\omega^k)\right)}{\displaystyle\prod_{j=0}^{n-1}\theta(Q\,;z\omega^j)} \tag{150}$$

is not identically 0 in z, where $Q = q^{1/n}$ and $\omega = \mathrm{e}^{2\pi\mathrm{i}/n}$. This in turn is equivalent to the numerator

$$\sum_{j=0}^{n-1}\left(\prod_{0\le k\ne j}^{n-1} \theta(Q\,;z\omega^k)\right) = \theta(Q\,;z\omega)\theta(Q\,;z\omega^2)\cdots\theta(Q\,;z\omega^{n-1}) + \theta(Q\,;z)\sum_{j=1}^{n-1}\left(\prod_{1\le k\ne j}^{n-1} \theta(Q\,;z\omega^j)\right) \tag{151}$$

not being identically 0 in z. Setting $z = -Q$ in (151) gives that $\theta(Q\,;-Q) = 0$ and

$$\sum_{j=0}^{n-1}\prod_{0\le k\ne j}^{n-1} \theta(Q\,;-Q\,\omega^k) = \theta(Q\,;-Q\,\omega)\theta(Q\,;-Q\,\omega^2)\cdots\theta(Q\,;-Q\,\omega^{n-1}) \ne 0 \tag{152}$$

where the non-vanishing is obtained from (25) along with the fact that $-Q\,\omega^j$ does not lie on the negative real axis for $j = 1, \ldots, n-1$. By the identity theorem, (151) does not vanish on any subset of $\mathbb{C}\backslash\{0\}$ having a limit point. One concludes that $\mathcal{F}[F_{2N+1,2n}(t)](x)$ is not identically 0 in x. It follows that $F_{2N+1,2n}(t)$ is not identically 0 in t as $\mathcal{F}$ is injective. Hence, $f_{2N+1,2n}(t)$ cannot be identically 0 in t either. □

We proceed next to the second case: studying those $f_{\mu,1}(t)$ with even/odd extensions, and computing their Fourier transforms. Again, this result illustrates a major point of this work – demonstrating the connection with special function theory and Jacobi theta functions, and recovering the Fourier transform results of [24] via an alternate method of contour integration.

Theorem 6.5. *The function $f_{\mu,1}(t)$ extends to be an even function precisely when $\mu = 2n$ is an even integer. Denote this extension $F_{2n,1}(t)$. On the other hand, $f_{\mu,1}(t)$ extends to be an odd function precisely when $\mu = 2n+1$ is an odd integer. Denote this extension $F_{2n+1,1}(t)$. Let N be any integer (either $2n$ or $2n+1$, respectively, above). The Fourier transform of $F_{N,1}(t)$ is given in terms of the Jacobi theta function θ as follows:*

$$\mathcal{F}[F_{N,1}(t)](x) \;=\; \frac{2\mu_{q^2}^3}{\sqrt{2\pi}}\,\frac{(-\mathrm{i}x)^N}{\theta(q^2;x^2)}\,. \tag{153}$$

The proof of Theorem 6.5 is given in Section 9 below.

7. Non-uniqueness of solutions of MADE IVPs at 0, and conditions sufficient for uniqueness

In this section, we will first demonstrate non-uniqueness of solutions to the MADEs under consideration, by examining the case $f_{\mu,\lambda}(t)$ with $\mu \in \mathbb{R}$ and $\lambda = 2 = 2L/k$ with $L = 1 = k = R = A$ as in (16) and (17), with $q > 1$ and $f_{\mu,2}(t)$ satisfying the MADE (18) of explicit form

$$f'_{\mu,2}(t) = q^{(1+\mu)/2} f_{\mu,2}(qt)\,, \tag{154}$$

which is given also as (102) in Example 1.

As a first example, let $\mu = -1$, that is $\mu = 2N+1$ with $N = -1$ and $\lambda = 2n$ with $n = 1$. One has that $f_{-1,2}(t)$ is flat at 0, extends to be identically 0 on the negative reals to give $F_{-1,2}(t)$, which satisfies the MADE

$$f'(t) = f(qt) \tag{155}$$

with $f(0) = 0$, as was seen in Example 1. Note that for any $C \in \mathbb{R}$, the scaled function $CF_{-1,2}(t)$ also satisfies (155) with vanishing initial condition $CF_{-1,2}(0) = 0$. And thus, due to linearity of (155), we have a 1-parameter family, parameterized by C, of solutions to the initial value problem (155) with $f(0) = 0$ that agree (and vanish) on $(-\infty, 0]$. This exhibits non-uniqueness of solutions to IVP MADEs in general.

As a second example, we let $\mu, C \in \mathbb{R}$ be our parameters and rely on $f_{\mu,2}(t)$, which satisfies (154) above, to construct the function

$$G_{\mu,2,C}(t) \equiv Cf_{\mu,2}(t/q^{(1+\mu)/2}) \tag{156}$$

for $t \ge 0$. Observe that

$$G'_{\mu,2,C}(t) = Cf'_{\mu,2}(t/q^{(1+\mu)/2}) \cdot (1/q^{(1+\mu)/2}) = Cq^{(1+\mu)/2} f_{\mu,2}(qt/q^{(1+\mu)/2}) \cdot (1/q^{(1+\mu)/2})$$
$$= Cf_{\mu,2}(qt/q^{(1+\mu)/2}) = G_{\mu,2,C}(qt) , \tag{157}$$

where the rightmost equation in (157) follows from (154). In this setting, if $\mu = 2N + 1$ with $N = -1$ we recover the first example of this section. If $\mu = 2N + 1$ is an odd integer, one also obtains flat solutions to (155) which also vanish at 0. If $\mu \neq 2N + 1$ is not an odd-integer, then $G_{\mu,2,C}(0) = Cf_{\mu,2}(0) \neq 0$, by (29). If one sets $C = (f_{\mu,2}(0))^{-1}$ for instance, then for each μ not an odd integer one has $G_{\mu,2,[1/f_{\mu,2}(0)]}(t)$ satisfies the MADE IVP (155) on $[0, \infty)$ with $f(0) = 1$.

As a third example, fix $\mu \neq 2N + 1$, and examine $G_{\mu,2,C_1}(t)$ as in (156), where $C_1 = [1/f_{\mu,2}(0)]$. Extend $f_{\mu,2}(t)$ to $F_{\mu,2}(t)$ via Theorem 3.2, and thus extend $G_{\mu,2,C_1}(t)$ to $C_1 F_{\mu,2}(t/q^{(1+\mu)/2})$ for $t < 0$. With this extended $G_{\mu,2,C_1}(t)$, define $f_{C_2}(t) = G_{\mu,2,C_1}(t) + C_2 F_{-1,2}(t)$ which satisfies (155) with $f_{C_2}(0) = 1$ for all $C_2 \in \mathbb{R}$, and with each function $f_{C_2}(t)$ in the family given by C_2 agreeing on $(-\infty, 0]$ but differing on $(0, \infty)$.

As illustrated in the above examples, one sees that in general it will not be enough to match derivatives at $t = 0$ to obtain uniqueness of solutions to initial value problems involving the MADEs under consideration in this study. Instead, one will also need to assume agreement of the solutions along an interval. In particular, if two solutions of a MADE agree in an open neighborhood of $t = 0$ then they agree on $\mathbb{R}$. As a canonical example, we examine uniqueness for the MADE $f'(t) = f(qt)$.

Proposition 7.1. *For $q > 1$, assume that*

$$f'(t) = f(qt) \tag{158}$$

and that there exists a $\hat{t} > 0$ with

$$f(t) = 0 \quad \text{for all} \quad t \in [\hat{t}, q\hat{t}] \tag{159}$$

Then $f(t) \equiv 0$ on the half line $[0, \infty)$.

Similarly, if $f(t)$ satisfies (158) and there exists a $\hat{t} < 0$ with $f(t) = 0$ for all $\hat{t}$ in the interval $[q\hat{t}, \hat{t}]$, then $f(t) \equiv 0$ on the half line $(-\infty, 0]$.

Proof. We prove the proposition in the case that $\hat{t} > 0$ and note that the proof of the case $\hat{t} < 0$ is similar. The vanishing of $f(t)$ on $[\hat{t}, q\hat{t}]$ holds by the hypothesis (159). For $t \in [\hat{t}/q, \hat{t}]$, one observes that $qt \in [\hat{t}, q\hat{t}]$ from which one has that

$$f(t) - f(\hat{t}/q) = \int_{\hat{t}/q}^{t} f'(u)\, du = \int_{\hat{t}/q}^{t} f(qu)\, du = \int_{\hat{t}}^{qt} f(v)\, dv/q = 0 , \tag{160}$$

where the substitution $v = qu$ was made in the next to last equality, and where the last equality was obtained from the hypothesis (159) on $[\hat{t}, q\hat{t}]$. Setting $t = \hat{t}$ in (160) gives that $0 = f(\hat{t}) - f(\hat{t}/q) = 0 - f(\hat{t}/q)$, from which one sees $0 = f(\hat{t}/q)$. Thus for all $t \in [\hat{t}/q, \hat{t}]$ equation (160) gives $f(t) = 0$. Repeating this argument successively, one has $f(t) = 0$ $\forall$ $t \in [\hat{t}/q^n, \hat{t}/q^{n-1}]$ for all $n \in \mathbb{N}$. It follows that $f(0) = 0$ by continuity.

From (158), one sees that

$$\int f(u)\, du = qf(u/q) + C , \text{ so we let } F(u) = qf(u/q) . \tag{161}$$

For $t \in [q\hat{t}, q^2\hat{t}]$, one has $t/q \in [\hat{t}, q\hat{t}]$, and thus

$$\int_{q\hat{t}}^{t} f(u)\, du = F(t) - F(q\hat{t}) = qf(t/q) - qf(\hat{t}) = 0 - 0 , \tag{162}$$

where the last equality follows from the hypothesis (159). From this, one has that for each sufficiently small $\epsilon > 0$

$$\int_{t}^{t+\epsilon} f(u)\, du = 0 \ \forall\, t \in [q\hat{t}, q^2\hat{t}) , \tag{163}$$

which implies that $f(t) \equiv 0$ on $[q\hat{t}, q^2\hat{t})$, and therefore on $[q\hat{t}, q^2\hat{t}]$ by continuity. Repeating the above argument successively implies that $f(t) = 0$ on $[q^n\hat{t}, q^{n+1}\hat{t}]$ for all $n \in \mathbb{N}$, and, together with the earlier computation, one sees that $f(t) \equiv 0$ on $[0, \infty)$, proving the proposition. □

From the preceding proposition, one obtains the following corollary.

Corollary 7.1. *For $q > 1$, assume that two functions $f_1(t)$ and $f_2(t)$ satisfy*

$$f_i'(t) = f_i(qt) \text{ for } i = 1, 2\,, \tag{164}$$

and that there exists a $\hat{t} > 0$ with

$$f_1(t) = f_2(t) \tag{165}$$

on the interval $[\hat{t}, q\hat{t}]$. Then $f_1(t) \equiv f_2(t)$ on the half line $[0, \infty)$.

Similarly, if $f_1(t)$, $f_2(t)$ satisfy (158) and there exists a $\hat{t} < 0$ with (165) holding on the interval $[q\hat{t}, \hat{t}]$, then $f_1(t) \equiv f_2(t)$ on the half line $(-\infty, 0]$.

Proof. Let $f(t) = f_1(t) - f_2(t)$ and apply Proposition 7.1. □

In a similar vein, one has the following uniqueness criterion.

Corollary 7.2. *For $q > 1$, assume that there is an interval $(-\epsilon, \epsilon)$ about $t = 0$ such that two functions $f_1(t)$ and $f_2(t)$ satisfy*

$$f_i'(t) = f_i(qt) \text{ for } i = 1, 2 \tag{166}$$

and agree

$$f_1(t) = f_2(t) \quad \text{on} \quad (-\epsilon, \epsilon)\,. \tag{167}$$

Then

$$f_1(t) = f_2(t) \quad \text{on} \quad (-\infty, \infty)\,. \tag{168}$$

Proof. Pick any $\hat{t} > 0$ such that $q\hat{t} < \epsilon$. Then both $[\hat{t}, q\hat{t}]$ and $[q(-\hat{t}), -\hat{t}]$ fall in the interval $(-\epsilon, \epsilon)$ of agreement. Applying Corollary 7.1 gives the result. □

Note that Proposition 7.1 through Corollary 7.2 hold for f, f_1 and f_2 in the $\mathcal{C}^\infty$ category (in which any solution of $f'(t) = f(qt)$ falls). We next examine solutions of (18) that fall in $\mathcal{C}^\infty(\mathbb{R})$ and are real analytic on $\mathbb{R}\backslash\{0\}$, comparing them with the solution $F_{\mu,\lambda}(t)$ given by Definition 4.1.

Proposition 7.2. *For $\lambda \in \mathbb{Q}^+$ with $\lambda = 2A/R$ and for $\mu \in \mathbb{R}$, let $F_{\mu,\lambda}(t)$ as given in Definition 4.1 be a solution of the MADE (18), where A, R are as in (16)–(17), respectively. Let f be any other solution of the given MADE (18) that is $\mathcal{C}^\infty$ on $\mathbb{R}$ and real analytic on $(-\infty, 0) \cup (0, \infty)$. Then if there is a subinterval (a, b) of $(0, \infty)$, respectively $(-\infty, 0)$, with $F_{\mu,\lambda}(t) = f(t)$ on (a, b), then $F_{\mu,\lambda}(t) = f(t)$ on $[0, \infty)$, respectively $(-\infty, 0]$. Furthermore, if there is an $\epsilon > 0$ with $F_{\mu,\lambda}(t) = f(t)$ on $(-\epsilon, \epsilon)$, then $F_{\mu,\lambda}(t) = f(t)$ on $(-\infty, \infty)$.*

Proof. From Definition 4.1, one has $F_{\mu,\lambda}(t) = f_{\mu,\lambda}(t)$ for $t \geq 0$, with $f_{\mu,\lambda}(t)$ given by (2). Complexifying (2), one has

$$f_{\mu,\lambda}(z) \equiv \sum_{m=-\infty}^{\infty} (-1)^m \frac{\mathrm{e}^{-q^m z}}{q^{m(m-\mu)/\lambda}} \tag{169}$$

for $\mathcal{R}(z) \geq 0$. On the open right half-plane $\mathcal{R}(z) > 0$, $f_{\mu,\lambda}(z)$ is the uniform limit of the (analytic) truncated sums (from $-N$ to N) in (169) as $N \to \infty$. As such, $f_{\mu,\lambda}(z)$ is analytic on $\mathcal{R}(z) > 0$ (and continuous on $\mathcal{R}(z) \geq 0$). From analyticity on $\mathcal{R}(z) > 0$, one concludes that $f_{\mu,\lambda}(z)$ restricts to the real analytic function $f_{\mu,\lambda}(t)$ on $t > 0$.

Again from Definition 4.1, one has $F_{\mu,\lambda}(t) = \mathcal{R}(h(t))$ for $t < 0$, with $h(t) = \sum_{r=0}^{R-1} b_r h(c_r, t)$ given by (72) with each $h(c_r, t)$ given by (70) where $\mathcal{R}(c_r) < 0$. Complexifying t one has each $h(c_r, z)$ is analytic on the open half-plane $\mathcal{R}(-c_r z) < 0$ (which contains the negative real axis by the requirement that $\mathcal{R}(c_r) < 0$). Thus $h(z) = \sum_{r=0}^{R-1} b_r h(c_r z)$ is analytic on the open wedge given by the intersection of the half-planes $\mathcal{R}(-c_r z) < 0$ for $r = 0, \ldots, R-1$. One concludes that $\mathcal{R}(h(t))$, as the real part of $h(z)$ restricted to the negative real axis, is real analytic on $(-\infty, 0)$.

By Theorem 3.2, $F_{\mu,\lambda}(t)$ is $\mathcal{C}^\infty$ on $\mathbb{R}$ and satisfies the MADE (18). By Proposition 2.3, $F_{\mu,\lambda}(t)$ is not analytic at $t = 0$. By the above remarks $F_{\mu,\lambda}(t)$ is real analytic on $(-\infty, 0) \cup (0, \infty)$. If $f(t)$ in $\mathcal{C}^\infty(\mathbb{R})$ is real analytic on $(-\infty, 0) \cup (0, \infty)$ and agrees with $F_{\mu,\lambda}(t)$ on an open interval (a, b) of $(0, \infty)$, respectively $(-\infty, 0)$, then by the identity theorem for real analytic functions one has agreement on all of $(0, \infty)$, respectively $(-\infty, 0)$, and by continuity on all of $[0, \infty)$, respectively $(-\infty, 0]$. If one has agreement on an open interval $(-\epsilon, \epsilon)$, then: one has agreement on $(0, \epsilon)$ thus on $[0, \infty)$; and similarly there is agreement on $(-\epsilon, 0)$ thus on $(-\infty, 0]$. The result now follows, in particular for f any solution of the MADE (18) falling in the category $\mathcal{C}^\infty(\mathbb{R})$ and real analytic on $(-\infty, 0) \cup (0, \infty)$. □

Remark 15. We point out that criterion for uniqueness of a solution of a distinct but analogous class of certain multiplicatively advanced/delayed differential equations has been given in the main theorem of [21]. This illustrates that an additional asymptotic condition at the origin implies uniqueness.

8. Proof of the decay rates at $\pm\infty$

This section will be divided into two subsections. The first will handle the decay rate of decaying solutions to the MADEs under study. These decaying solutions are the Schwartz wavelets discussed in Section 4, and they form the majority of the solutions, in general. The second subsection will be devoted to the minority of cases, those solutions of the MADEs under study that do not decay. Though these solutions do not decay, they do remain bounded.

8.1. Decaying solutions

The goal of this section is to obtain sharp estimates – in terms of more familiar functions – for the decay rates at $+\infty$ of the new functions $f_{\mu,\lambda}(t)$, along with the decay rates of their extensions $h(t)$ and $F_{\mu,\lambda}(t)$ at $-\infty$.

Proposition 8.1. *Let $q>1$, and $\mathcal{A}, a>0$ and $\mathcal{B}, b, c\in\mathbb{R}$ all be fixed parameters for the function of $u\ge 0$ given by the summation in (170). Then for any fixed $\alpha>0$ (as in (172), (174), and (175) below), there are constants K_j with $j=1,2,3$ depending on the parameters $q, a, b, c, \mathcal{A}, \mathcal{B}$, and α such that for u satisfying (172) one has a bound of form*

$$0<\sum_{k=-\infty}^{\infty}\frac{\mathrm{e}^{-uq^{\mathcal{A}k+\mathcal{B}}}}{q^{ak^2+bk+c}}\le K_1 u^{-K_2\ln(u)+K_3}\,. \tag{170}$$

Consequently, for each of the functions $f_1(t)\equiv t^p D_t^n\left(f^{(s)}_{\mu,\lambda}(t)\right)$, $f_2(t)\equiv t^p D_t^n\left(h^{(s)}(c_r,t)\right)$ and $f_3(t)\equiv t^p D_t^n\left(h^{(s)}(t)\right)$ and $f_4(t)\equiv t^p D_t^n\left((\mathcal{R}h)^{(s)}(t)\right)$ satisfying that all $\mathcal{R}(c_r)<0$, and $f_5(t)\equiv t^p D_t^n\left(F^{(s)}_{\mu,\lambda}(t)\right)$, there are constants $\hat{K}_j$ with $j=1,2,3$ depending on the parameters $q, a, b, c, \mathcal{A}, \mathcal{B}, \alpha, p, n, \mu, \lambda, s$, and i along with the allowable parameters in (75) for $\mathrm{i}\ge 2$ so that

$$|f_i(t)|\le \hat{K}_1|t|^{-\hat{K}_2\ln(|t|)+\hat{K}_3} \tag{171}$$

for $|t|$ sufficiently large.

Proof. For fixed $q>1$, fixed $\mathcal{A}, a>0$ and fixed $\mathcal{B}, b, c\in\mathbb{R}$, along with fixed $\alpha>0$ as in (172), (174), and (175) below, from Propositions 7, 8, and 9 of [24] we immediately have that for

$$u>\max\left\{\frac{1}{\mathcal{A}}q^{\mathcal{A}(b+1)/(2a)-\mathcal{B}}\,,\ \alpha\mathcal{A}^{-1}q^{-\mathcal{B}}\mathrm{e}^{-1-b/\alpha}\right\}>0\,, \tag{172}$$

one has

$$0<\sum_{k=-\infty}^{\infty}\frac{\mathrm{e}^{-uq^{\mathcal{A}k+\mathcal{B}}}}{q^{ak^2+bk+c}}\le\frac{\mathrm{e}^{b/\mathcal{A}}}{q^c}\cdot\mathrm{e}^{-a[L_1(u)]^2\ln(q)+[b\ln(q)-2a/\mathcal{A}][L_2(u)]}\left\{1+\sqrt{\frac{\pi}{a\ln(q)}}\right\}, \tag{173}$$

where

$$L_1(u)\equiv\frac{\ln(u)-\ln(\alpha)+1+\ln(\mathcal{A})+\mathcal{B}\ln(q)+b/\alpha}{2a/\alpha+\mathcal{A}\ln(q)} \tag{174}$$

and

$$L_2(u)\equiv\begin{cases}\dfrac{\ln(u)+\ln(\mathcal{A})+\mathcal{B}\ln(q)}{\mathcal{A}\ln(q)} & \text{if }\ [b\ln(q)-2a/\mathcal{A}]\ge 0\\[2ex] \dfrac{\ln(u)-\ln(\alpha)+1+\ln(\mathcal{A})+\mathcal{B}\ln(q)+b/\alpha}{2a/\alpha+\mathcal{A}\ln(q)} & \text{if }\ [b\ln(q)-2a/\mathcal{A}]<0\,.\end{cases} \tag{175}$$

By collecting powers of $\ln(u)$ in (173), each such set of parameters $q, a, b, c, \mathcal{A}, \mathcal{B}, \alpha$ give positive constants K_1, K_2 and $K_3\in\mathbb{R}$ yielding a bound of form (170).

Each of the functions $f_1(t)\equiv t^p D_t^n\left(f^{(s)}_{\mu,\lambda}(t)\right)$, $f_2(t)\equiv t^p D_t^n\left(h^{(s)}(c_r,t)\right)$, $f_3(t)\equiv t^p D_t^n\left(h^{(s)}(t)\right)$, $f_4(t)\equiv t^p D_t^n\left((\mathcal{R}h)^{(s)}(t)\right)$, and $f_5(t)\equiv t^p D_t^n\left(F^{(s)}_{\mu,\lambda}(t)\right)$ is bounded by finite linear combinations of infinite sums of form (170) with

$$u=\mathcal{R}(\hat{c})t>0 \tag{176}$$

sufficiently large, where: $\hat{c}=1$ for f_1; $\hat{c}=c_r$ for f_2 with c_r satisfying $\mathcal{R}(c_r)<0$ as in Definition 4.1; $\hat{c}=c_r$ for $0\le r\le R-1$ for f_3 and f_4 with c_r satisfying $\mathcal{R}(c_r)<0$ as in Definition 4.1. Such bounds hold on f_5 by cases f_1 and f_4.

By relying on the triangle inequality, maximizing over the K_1 and K_3 in (170), minimizing over the K_2 in (170), and then collecting powers of $\ln(|t|)$, one obtains that for each $f_i(t)$ with $i=1,\dots,5$ there are constants $\hat{K}_j$ with $j=1,2,3$ depending on the parameters $q,a,b,c,\mathcal{A},\mathcal{B},\alpha,p,n,\mu,\lambda,s$ and i with bounds of form (171) holding for $|t|$ sufficiently large. □

As a consequence of Proposition 8.1, one obtains the following corollary.

Corollary 8.1. *For all $p,n\in\mathbb{N}_0$, one has*

$$\lim_{t\to\infty}\left|t^pD_t^n\left(f_{\mu,\lambda}^{(s)}(t)\right)\right|=0,\qquad \lim_{t\to-\infty}\left|t^pD_t^n\left(h^{(s)}(c_r,t)\right)\right|=0, \tag{177}$$

$$\lim_{t\to-\infty}\left|t^pD_t^n\left(h^{(s)}(t)\right)\right|=0,\qquad \lim_{t\to-\infty}\left|t^pD_t^n\left((\mathcal{R}h)^{(s)}(t)\right)\right|=0, \tag{178}$$

$$\text{and}\qquad \lim_{t\to\pm\infty}\left|t^pD_t^n\left(F_{\mu,\lambda}^{(s)}(t)\right)\right|=0\,, \tag{179}$$

where each $\mathcal{R}(c_r)<0$.

Proof. The vanishing limits are an immediate consequence of (171). □

Remark 16. To obtain decay in Proposition 8.1 one must have $\mathcal{R}(\hat{c})\neq 0$ in (176). In the setting that $\mathcal{R}(\hat{c})=0$, one need not have decay. In the next subsection we give a criterion that guarantees non-decay when $\mathcal{R}(\hat{c})=0$.

Finally, in [7], N.T. Dung studies functions of the form

$$f'(t)+a(t)f(t+h(t))+b(t)f(t+r(t))=0\,, \tag{180}$$

which, under further key assumptions (in particular (2.4) in [7], given in the sentence containing (182) below), are seen to decay exponentially and have unique solutions. If one sets $n=1$ in (1) the following MADE is obtained:

$$f'(t)=\alpha f(\beta t)\,, \tag{181}$$

where $\alpha\neq 0$ and $\beta>1$. If one sets $a(t)=-\alpha, h(t)=(\beta-1)t$, and $b(t)=0=r(t)$ in (180) one recovers (181). In order to obtain uniqueness of the solution to (180) along with exponential decay, the following key assumption is made (assumption (2.4) in [7]), among other assumptions. Namely, that the supremum of the expression

$$E(t)\equiv\int_{t_0}^{t}\mathrm{e}^{-\int_s^t D(u)\,\mathrm{d}u}\left(|a(s)|\int_s^{s+h(s)}C(u)\,\mathrm{d}u+|b(s)|\int_s^{s+r(s)}C(u)\,\mathrm{d}u\right)\mathrm{d}s \tag{182}$$

on the interval $[t_0,\infty)$ is less than 1, where $C(t)=|a(t)|+|b(t)|$ and $D(t)=a(t)+b(t)$. However, if one substitutes $a(t)=-\alpha, h(t)=(\beta-1)t$, and $b(t)=0=r(t)$ in (182), one obtains

$$\begin{aligned}E(t)&=\int_{t_0}^{t}\mathrm{e}^{-\int_s^t(-\alpha)\,\mathrm{d}u}\left(|-\alpha|\int_s^{s+(\beta-1)s}(|-\alpha|+|0|)\,\mathrm{d}u\right)\mathrm{d}s\\ &=-\alpha(\beta-1)t-(\beta-1)+\alpha(\beta-1)t_0\mathrm{e}^{\alpha(t-t_0)}+(\beta-1)\mathrm{e}^{\alpha(t-t_0)}\,,\end{aligned} \tag{183}$$

which approaches ∞ as $t\to\infty$ independently of choice of α or β. Thus $E(t)$ does not have a supremum less than 1, and one cannot assume either uniqueness or exponential decay of the solution to (181).

8.2. Non-decaying bounded solutions

While Proposition 8.1 gives decay of $\mathcal{R}(h(t))$ as t approaches $-\infty$ when all $\mathcal{R}(c_r)<0$, there is no need for decay of $\mathcal{R}(h(t))$ as t approaches $-\infty$ if even one of the $\mathcal{R}(c_r)=0$. The purpose of the following proposition is to prove non-decay in this setting, under mild assumptions on the parameters.

Proposition 8.2. *Let $q = a/b > 1$ be rational with $a, b \in \mathbb{N}$. Let $h(t) = \sum_{r=0}^{R-1} b_r h(c_r, t)$ be an extension of $f_{\mu,\lambda}(t)$ to $(-\infty, 0]$ as in Theorem 3.2 with $c_r = \pm i\gamma_r$ satisfying $\mathcal{R}(c_r) = 0$ for $0 \le r \le \hat{r}$ and $\mathcal{R}(c_r) < 0$ for $\hat{r} + 1 \le r \le R - 1$, and with $\lambda = 2A/R \in \mathbb{Q}^+$. Furthermore, assume $\gamma_r = \alpha_r/\beta_r$ is rational with $\alpha_r, \beta_r \in \mathbb{N}$ for $0 \le r \le \hat{r}$. Then, under the assumption that*

$$\mathcal{R}\left(\sum_{r=0}^{\hat{r}} b_r h(c_r, 0)\right) = \mathcal{R}\left(\sum_{r=0}^{\hat{r}} b_r \sum_{j=0}^{A-1} a_{j,r} q^{-j(j-\mu)/\lambda} \theta\left(q^{2A^2/\lambda}; \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right] q^{A[\mu-2j-A]/\lambda}\right)\right) \neq 0\,, \tag{184}$$

given any $\epsilon > 0$, there exists a sequence $t_k \to -\infty$ such that

$$|\mathcal{R}(h(t_k))| > \left|\mathcal{R}\left(\sum_{r=0}^{\hat{r}} b_r h(c_r, 0)\right)\right| - \epsilon\,. \tag{185}$$

Hence $\mathcal{R}(h(t))$ does not decay as $t \to -\infty$.

Proof. Let $\epsilon > 0$ be given. Observe, by definition of $\hat{r}$, that $h(t)$ splits into a non-decaying part and a decaying part as follows:

$$h(t) = \sum_{r=0}^{\hat{r}} b_r h(c_r, t) + \sum_{r=\hat{r}+1}^{R-1} b_r h(c_r, t)\,, \tag{186}$$

where the second sum in (186) is decaying. That is, by Proposition 8.1 combined with the fact that $\mathcal{R}(c_r) < 0$ for $\hat{r} + 1 \le r \le R - 1$, there exists a $T \in -\mathbb{N}$ such that

$$\left|\sum_{r=\hat{r}+1}^{R-1} b_r h(c_r, t)\right| < \epsilon/3 \tag{187}$$

for $t \le T$. We proceed to show that the first sum in (186) does not decay (and therefore $h(t)$ does not decay). From (72), observe that this first term is of form

$$\sum_{r=0}^{\hat{r}} b_r h(c_r, t) = \sum_{r=0}^{\hat{r}} b_r \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{M=-\infty}^{\infty} \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right]^M \frac{e^{-q^{MA+j}(\pm i\gamma_r)t}}{q^{(MA+j)(MA+j-\mu)/\lambda}}\right]. \tag{188}$$

Evaluating (188) at zero gives

$$\sum_{r=0}^{\hat{r}} b_r h(c_r, 0) = \sum_{r=0}^{\hat{r}} b_r \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{M=-\infty}^{\infty} \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right]^M \frac{1}{q^{(MA+j)(MA+j-\mu)/\lambda}}\right] \tag{189}$$

$$= \sum_{r=0}^{\hat{r}} b_r \sum_{j=0}^{A-1} a_{j,r} q^{-j(j-\mu)/\lambda} \theta\left(q^{2A^2/\lambda}; \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R}\right] q^{A[\mu-2j-A]/\lambda}\right) \tag{190}$$

where equality in (190) follows from (61). From (189)–(190), one sees that the non-vanishing of the real part of (190) in the hypothesis (184) has the equivalent form

$$\mathcal{R}\left(\sum_{r=0}^{\hat{r}} b_r h(c_r, 0)\right) \neq 0\,. \tag{191}$$

Next, examine the related sum

$$\sum_{r=0}^{\hat{r}} |b_r| \left[\sum_{j=0}^{A-1} |a_{j,r}| \sum_{M=-\infty}^{\infty} \left[\frac{1}{\gamma_r^R}\right]^M \frac{1}{q^{(MA+j)(MA+j-\mu)/\lambda}}\right] \tag{192}$$

$$= \sum_{r=0}^{\hat{r}} |b_r| \sum_{j=0}^{A-1} |a_{j,r}| q^{-j(j-\mu)/\lambda} \theta\left(q^{2A^2/\lambda}; \left[\frac{1}{\gamma_r^R}\right] q^{A[\mu-2j-A]/\lambda}\right), \tag{193}$$

where (193) again follows from (61). Thus there is an $N > 0$, dependent on ϵ, such that the sum (of the tails in (192)) is bounded by

$$\sum_{r=0}^{\hat{r}} |b_r| \left[\sum_{j=0}^{A-1} |a_{j,r}| \sum_{|M|>N} \left[\frac{1}{\gamma_r^R} \right]^M \frac{1}{q^{(MA+j)(MA+j-\mu)/\lambda}} \right] < \epsilon/3 \,. \tag{194}$$

Next, for each $k \in \mathbb{N}$, define

$$t_k \equiv (-k) \cdot 2\pi \cdot a^{AN} \cdot b^{AN+A-1} \cdot \left[\prod_{r=0}^{\hat{r}} \beta_r \right] \cdot |T| \;<\; T \,, \tag{195}$$

where N is as in (194), $q = a/b$, $\gamma_r = \alpha_r/\beta_r$, and T is as in (187). Observe that for each M in the range $-N \le M \le N$, and for each $0 \le j \le A-1$ where $0 \le r \le \hat{r}$, one has

$$-AN \le MA + j \le AN + A - 1 \,. \tag{196}$$

One then has that for $-N \le M \le N$ the expression

$$\begin{aligned} -q^{MA+j}(\pm i\gamma_r)t_k \;&=\; -(a/b)^{MA+j}(\pm i\alpha_r/\beta_r) \left((-k) \cdot 2\pi \cdot a^{AN} \cdot b^{AN+A-1} \cdot \left[\prod_{r=0}^{\hat{r}} \beta_r \right] \cdot |T| \right) \\ &=\; \pm k(2\pi i) a^{AN+MA+j} b^{AN+A-1-MA-j} \alpha_r \left[\prod_{m=0, m\neq r}^{\hat{r}} \beta_r \right] \cdot |T| \end{aligned} \tag{197}$$

is an integral multiple of $2\pi i$, which follows from (196) as the exponents of a and b in (197) are non-negative integers. It follows that for M in the range $-N \le M \le N$ one has

$$e^{-q^{MA+j}(\pm i\gamma_r)t_k} = 1 \,. \tag{198}$$

Thus for each $k \in \mathbb{N}$

$$\begin{aligned} h(t_k) &= \sum_{r=0}^{\hat{r}} b_r h(c_r, t_k) + \sum_{r=\hat{r}+1}^{R-1} b_r h(c_r, t_k) && (199) \\ &= \sum_{r=0}^{\hat{r}} b_r \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{M=-N}^{N} \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R} \right]^M \frac{e^{-q^{MA+j}(\pm i\gamma_r)t_k}}{q^{(MA+j)(MA+j-\mu)/\lambda}} \right] && (200) \\ &\quad + \sum_{r=0}^{\hat{r}} b_r \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{|M|>N} \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R} \right]^M \frac{e^{-q^{MA+j}(\pm i\gamma_r)t_k}}{q^{(MA+j)(MA+j-\mu)/\lambda}} \right] && (201) \\ &\quad + \sum_{r=\hat{r}+1}^{R-1} b_r h(c_r, t_k) \,, \end{aligned}$$

where (199) follows from (186); and (200)–(201) follow from (188). Thus

$$\begin{aligned} h(t_k) &= \sum_{r=0}^{\hat{r}} b_r \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{M=-N}^{N} \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R} \right]^M \frac{1}{q^{(MA+j)(MA+j-\mu)/\lambda}} \right] && (202) \\ &\quad + \sum_{r=0}^{\hat{r}} b_r \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{|M|>N} \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R} \right]^M \frac{1}{q^{(MA+j)(MA+j-\mu)/\lambda}} \right] && (203) \\ &\quad - \sum_{r=0}^{\hat{r}} b_r \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{|M|>N} \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R} \right]^M \frac{1}{q^{(MA+j)(MA+j-\mu)/\lambda}} \right] && (204) \\ &\quad + \sum_{r=0}^{\hat{r}} b_r \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{|M|>N} \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R} \right]^M \frac{e^{-q^{MA+j}(\pm i\gamma_r)t_k}}{q^{(MA+j)(MA+j-\mu)/\lambda}} \right] \\ &\quad + \sum_{r=\hat{r}+1}^{R-1} b_r h(c_r, t_k) \,, \end{aligned}$$

where: (202) follows from an application of (198) to (200); and (203) combined with (204) is an addition of 0. From (189), combining (202) with (203) yields $\sum_{r=0}^{\hat{r}} b_r h(c_r, 0)$, and the above becomes

$$h(t_k) = \sum_{r=0}^{\hat{r}} b_r h(c_r, 0)$$

$$- \sum_{r=0}^{\hat{r}} b_r \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{|M|>N} \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R} \right]^M \frac{1}{q^{(MA+j)(MA+j-\mu)/\lambda}} \right] \tag{205}$$

$$+ \sum_{r=0}^{\hat{r}} b_r \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{|M|>N} \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R} \right]^M \frac{e^{-q^{MA+j}(\pm i\gamma_r)t_k}}{q^{(MA+j)(MA+j-\mu)/\lambda}} \right] \tag{206}$$

$$+ \sum_{r=\hat{r}+1}^{R-1} b_r h(c_r, t_k), \tag{207}$$

where, first by an application of (194) to each of the sums in (205)–(206), and second by an application of (187) to the sum in (207), each of the sums in (205)–(207) is bounded in absolute value by $\epsilon/3$. For instance, from (194), one has the following bound on absolute value of the subtracted sum in (205):

$$\left| \sum_{r=0}^{\hat{r}} b_r \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{|M|>N} \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R} \right]^M \frac{1}{q^{(MA+j)(MA+j-\mu)/\lambda}} \right] \right|$$

$$\leq \sum_{r=0}^{\hat{r}} |b_r| \left[\sum_{j=0}^{A-1} |a_{j,r}| \sum_{|M|>N} \left[\frac{1}{\gamma_r^R} \right]^M \frac{1}{q^{(MA+j)(MA+j-\mu)/\lambda}} \right] < \epsilon/3\,.$$

Taking real parts of each term in the equation containing (205)–(207), taking absolute values, and applying the triangle inequality gives that

$$|\mathcal{R}(h(t_k))| \geq \left| \mathcal{R}\left(\sum_{r=0}^{\hat{r}} b_r h(c_r, 0) \right) \right|$$

$$- \left| \mathcal{R}\left(\sum_{r=0}^{\hat{r}} b_r \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{|M|>N} \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R} \right]^M \frac{1}{q^{(MA+j)(MA+j-\mu)/\lambda}} \right] \right) \right| \tag{208}$$

$$- \left| \mathcal{R}\left(\sum_{r=0}^{\hat{r}} b_r \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{|M|>N} \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R} \right]^M \frac{e^{-q^{MA+j}(\pm i\gamma_r)t_k}}{q^{(MA+j)(MA+j-\mu)/\lambda}} \right] \right) \right| \tag{209}$$

$$- \left| \mathcal{R}\left(\sum_{r=\hat{r}+1}^{R-1} b_r h(c_r, t_k) \right) \right|. \tag{210}$$

Applying the fact that $|\mathcal{R}(z)| \leq |z|$ to (208)–(210) gives

$$|\mathcal{R}(h(t_k))| \geq \left| \mathcal{R}\left(\sum_{r=0}^{\hat{r}} b_r h(c_r, 0) \right) \right|$$

$$- \left| \sum_{r=0}^{\hat{r}} b_r \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{|M|>N} \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R} \right]^M \frac{1}{q^{(MA+j)(MA+j-\mu)/\lambda}} \right] \right|$$

$$- \left| \sum_{r=0}^{\hat{r}} b_r \left[\sum_{j=0}^{A-1} a_{j,r} \sum_{|M|>N} \left[\frac{(-1)^{A-\ell_r}}{\gamma_r^R} \right]^M \frac{e^{-q^{MA+j}(\pm i\gamma_r)t_k}}{q^{(MA+j)(MA+j-\mu)/\lambda}} \right] \right|$$

$$- \left| \sum_{r=\hat{r}+1}^{R-1} b_r h(c_r, t_k) \right|$$

$$\geq \left| \mathcal{R}\left(\sum_{r=0}^{\hat{r}} b_r h(c_r, 0) \right) \right| - \epsilon/3 - \epsilon/3 - \epsilon/3 \tag{211}$$
$$= \left| \mathcal{R}\left(\sum_{r=0}^{\hat{r}} b_r h(c_r, 0) \right) \right| - \epsilon \ ,$$

where the $\epsilon/3$ bounds in (211) follow from (194) and (187), respectively. Thus, from (191), if one chooses ϵ to be less than $\left|\mathcal{R}\left(\sum_{r=0}^{\hat{r}} b_r h(c_r, 0)\right)\right| > 0$ the expressions $|\mathcal{R}(h(t_k))|$ are bounded away from 0 for all $k \in \mathbb{N}$. Thus $\mathcal{R}(h(t))$ does not decay as $t \to -\infty$, and the proposition is proven. □

Remark 17. If $\hat{r} = 0$ in Proposition 8.2, (that is if $c_0 = \pm i\gamma_0$ is the only term with $\mathcal{R}(c_0) = 0$), then one can drop the assumption in Proposition 8.2 that γ_0 be rational. In this setting one defines t_k for $k \in \mathbb{N}$ by

$$t_k \equiv (-k) \cdot 2\pi \cdot a^{AN} \cdot b^{AN+A-1} \cdot [|T|/\gamma_0] \cdot (\lfloor \gamma_0 \rfloor + 1) \ ,$$

where $\lfloor x \rfloor$ denotes the greatest integer function. Then $t_k < T$ and

$$-q^{MA+j_0}(\pm i\gamma_0)t_k = -(a/b)^{MA+j_0}(\pm i\gamma_0)\left[(-k) \cdot 2\pi \cdot a^{AN} \cdot b^{AN+A-1} \cdot |T|/\gamma_0\right] \cdot (\lfloor \gamma_0 \rfloor + 1)$$
$$= \pm k(2\pi i) a^{AN+MA+j_0} b^{AN+A-1-MA-j_0} \cdot |T| \cdot (\lfloor \gamma_0 \rfloor + 1)$$

is an integral multiple of $2\pi i$ and the proof proceeds as above.

Remark 18. Though $\mathcal{R}(h(t))$ need not decay if any $\mathcal{R}(c_r) = 0$, it does remain bounded, via (55)–(57).

9. Proof of the relation of Fourier transforms to Jacobi theta functions

This section is devoted to proving Theorems 6.3 and 6.5. We first present the proof of Theorem 6.3 relating $\mathcal{F}[F_{2N+1,2n}(t)](x)$ to the Jacobi theta function via a series of lemmas. For $\mu, \lambda \in \mathbb{R}$ with $\lambda > 0$, and $x \in \mathbb{R}$, Proposition 6.1 gives

$$\int_0^\infty e^{-ixt} f_{\mu,\lambda}(t)\, dt = \sum_{k=-\infty}^{\infty} \left[\frac{(-1)^k}{q^{k(k-\mu)/\lambda}} \frac{1}{(ix + q^k)} \right] = \sum_{k=-\infty}^{\infty} \left[\frac{(-1)^k}{q^{k(k+1)/\lambda}} \frac{q^{k(\mu+1)/\lambda}}{(ix + q^k)} \right]$$
$$= \sum_{k=-\infty}^{\infty} \left[\frac{(-1)^k}{\left(q^{2/\lambda}\right)^{k(k+1)/2}} \frac{\left(q^{2/\lambda}\right)^{k(\mu+1)/2}}{(ix + \left(q^{2/\lambda}\right)^{k\lambda/2})} \right]$$
$$= \sum_{k=-\infty}^{\infty} \left[\frac{(-1)^k}{Q^{k(k+1)/2}} \frac{Q^{k(\mu+1)/2}}{(ix + Q^{k\lambda/2})} \right] , \tag{212}$$

where $Q \equiv q^{2/\lambda}$. Letting $\mu = 2N + 1$ and $\lambda = 2n$ in (212) and multiplying by $1/\sqrt{2\pi}$ yields

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ixt} f_{2N+1,2n}(t)\, dt = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \left[\frac{(-1)^k}{Q^{k(k+1)/2}} \frac{\left[Q^k\right]^{(N+1)}}{\left(ix + \left[Q^k\right]^n\right)} \right] , \tag{213}$$

where $Q = q^{1/n}$. We shall see shortly that the "alternating Q-combinatoric" $(-1)^k/Q^{k(k+1)/2}$ in (213) will be given by the residue of $1/[u\theta(Q;u)]$ at $u = -Q^k$, and that the term $\left[Q^k\right]^{(N+1)}/\left(ix + \left[Q^k\right]^n\right)$ in (213) will then be obtained from $[-u]^{(N+1)}/\left(ix + [-u]^n\right)$ by evaluation at $u = -Q^k$. Therefore, we will be interested in integrating the expression

$$\frac{1}{u\theta(Q;u)} \frac{[-u]^{N+1}}{(z + [-u]^n)} \tag{214}$$

around an appropriate contour in $\mathbb{C}$, where we set $z = ix$ later.

In anticipation of the residue computation of the expression (214), we begin by examining $\theta(Q;u)$ and removing one appropriate factor from the product. That is, note that from (22), one has that for $k \geq 0$

$$\theta(Q;u) = \mu_Q \prod_{n=0}^{\infty}\left(1+\frac{u}{Q^n}\right)\left(1+\frac{1}{uQ^{n+1}}\right)$$
$$= \left(1+\frac{u}{Q^k}\right)\left[\mu_Q \prod_{n=0, n\neq k}^{\infty}\left(1+\frac{u}{Q^n}\right)\prod_{n=0}^{\infty}\left(1+\frac{1}{uQ^{n+1}}\right)\right] \tag{215}$$
$$\equiv \left(1+\frac{u}{Q^k}\right)\theta(k\,|\,Q;u) = \left(\frac{Q^k+u}{Q^k}\right)\theta(k\,|\,Q;u)\,, \tag{216}$$

where for $k \geq 0$, the expression $\theta(k\,|\,Q;u)$ in (216) is defined to be the expression in square brackets in (215). Similarly, for $k < 0$ one has that

$$\theta(Q;u) = \mu_Q \prod_{n=0}^{\infty}\left(1+\frac{u}{Q^n}\right)\left(1+\frac{1}{uQ^{n+1}}\right)$$
$$= \left(1+\frac{1}{uQ^{|k|}}\right)\left[\mu_Q \prod_{n=0}^{\infty}\left(1+\frac{u}{Q^n}\right)\prod_{n=0, n\neq |k|-1}^{\infty}\left(1+\frac{1}{uQ^{n+1}}\right)\right] \tag{217}$$
$$\equiv \left(1+\frac{1}{uQ^{|k|}}\right)\theta(k\,|\,Q;u) = \left(\frac{u+Q^k}{u}\right)\theta(k\,|\,Q;u)\,, \tag{218}$$

where for $k < 0$, the expression $\theta(k\,|\,Q;u)$ in (218) is defined to be the expression in square brackets in (217). Thus, via (216) and (218), $\theta(k\,|\,Q;u)$ is defined for each $k \in \mathbb{Z}$.

From (216), one sees that for $k \geq 0$ the residue at $-Q^k$ of $1/[u\theta(Q;u)]$ is $Q^k/[-Q^k\theta(k\,|\,Q;-Q^k)]$. Also, from (218), for $k < 0$ the residue at $-Q^k$ of $1/[u\theta(Q;u)]$ is $-Q^k/[-Q^k\theta(k\,|\,Q;-Q^k)]$. The next lemma will re-express $\theta(k\,|\,Q;-Q^k)$, and thus these residues, in terms of an expression that we call an alternating Q-combinatoric.

Lemma 9.1. *For $k \geq 0$ and $\theta(k\,|\,Q;u)$ as in (216), one has*

$$\theta(k\,|\,Q;-Q^k) = \left[\mu_Q \prod_{n=0, n\neq k}^{\infty}\left(1-\frac{Q^k}{Q^n}\right)\prod_{n=0}^{\infty}\left(1-\frac{1}{Q^{k+n+1}}\right)\right] = (-1)^k \mu_Q^3\, Q^{k(k+1)/2}\,. \tag{219}$$

And for $k < 0$ and $\theta(k\,|\,Q;u)$ as in (218), one has

$$\theta(k\,|\,Q;-Q^k) = \left[\mu_Q \prod_{n=0}^{\infty}\left(1-\frac{Q^k}{Q^n}\right)\prod_{n=0, n\neq |k|-1}^{\infty}\left(1-\frac{1}{Q^{k+n+1}}\right)\right] = -(-1)^k \mu_Q^3\, Q^{k(k+1)/2}\,. \tag{220}$$

Proof. Let $k \geq 0$. The first equality in (219) follows from (215)–(216). By re-indexing the product formula in (219), we obtain

$$\theta(k\,|\,Q;-Q^k) = \mu_Q \prod_{j=1}^{k}(1-Q^j)\prod_{m=1}^{\infty}(1-1/Q^m)\prod_{p=k+1}^{\infty}(1-1/Q^p)\frac{\prod_{p=1}^{k}(1-1/Q^p)}{\prod_{p=1}^{k}(1-1/Q^p)}$$
$$= \mu_Q \frac{\prod_{j=1}^{k}(1-Q^j)}{\prod_{p=1}^{k}(1-1/Q^p)}\prod_{m=1}^{\infty}(1-1/Q^m)\prod_{p=1}^{\infty}(1-1/Q^p)$$
$$= \mu_Q \frac{\prod_{j=1}^{k}(1-Q^j)}{\prod_{p=1}^{k}(Q^p-1)}\prod_{p=1}^{k}(Q^p)\prod_{m=1}^{\infty}(1-1/Q^m)\prod_{p=1}^{\infty}(1-1/Q^p) \tag{221}$$
$$= \mu_Q \left(\prod_{j=1}^{k}(-1)\right)\left(\prod_{p=1}^{k}Q^p\right)\mu_Q^2 = (-1)^k\mu_Q^3\, Q^{k(k+1)/2} \tag{222}$$

which is the last expression in (219). Recall from (23) that one has $\mu_Q = \prod_{p=1}^{\infty}(1-1/Q^p)$, which justifies moving from (221) to (222).

Let $k < 0$. The first equality in (220) follows from (217)–(218). From the product formula in (220), we obtain (223), and then re-index to obtain (224) below. Thus

$$\theta(k\,|\,Q\,;-Q^k) = \mu_Q \prod_{n=0}^{\infty}\left(1-\frac{1}{Q^{n+|k|}}\right)\prod_{n=0,n\neq|k|-1}^{\infty}\left(1-\frac{1}{Q^{-|k|+n+1}}\right) \tag{223}$$

$$= \mu_Q \prod_{j=|k|}^{\infty}(1-1/Q^j)\prod_{m=1}^{|k|-1}(1-Q^m)\prod_{p=1}^{\infty}(1-1/Q^p)\frac{\prod_{j=1}^{|k|-1}(1-1/Q^j)}{\prod_{j=1}^{|k|-1}(1-1/Q^j)} \tag{224}$$

$$= \mu_Q \prod_{j=1}^{\infty}(1-1/Q^j)\frac{\prod_{m=1}^{|k|-1}(1-Q^m)}{\prod_{m=1}^{|k|-1}(1-1/Q^m)}\prod_{p=1}^{\infty}(1-1/Q^p)$$

$$= \mu_Q \prod_{j=1}^{\infty}(1-1/Q^j)\frac{\prod_{m=1}^{|k|-1}(1-Q^m)}{\prod_{m=1}^{|k|-1}(Q^m-1)}\prod_{m=1}^{|k|-1}Q^m\prod_{p=1}^{\infty}(1-1/Q^p)$$

$$= \mu_Q\mu_Q\left(\prod_{m=1}^{|k|-1}(-1)\right)Q^{(|k|-1)|k|/2}\mu_Q = (-1)^{-k-1}\mu_Q^3\,Q^{k(k+1)/2}\,,$$

which is the last expression in (220). The lemma is proven. □

From the preceding lemma, one deduces the following corollary to evaluate relevant residues.

Corollary 9.2. *For $k\in\mathbb{Z}$ and $G(u)$ analytic in a neighborhood of $-Q^k$ the residue of*

$$\frac{1}{u\theta(Q\,;u)}G(u) \tag{225}$$

at $u=-Q^k$ is given by

$$Res(-Q^k) = \left[\frac{(-1)}{\mu_Q^3}\frac{(-1)^k}{Q^{k(k+1)/2}}\right]G(-Q^k)\,. \tag{226}$$

In particular, for $k\in\mathbb{Z}$ the residue of (214), namely of

$$\frac{1}{u\theta(Q\,;u)}\frac{[-u]^{N+1}}{(z+[-u]^n)}\,,$$

at $u=-Q^k$ is given by

$$Res(-Q^k) = \left[\frac{(-1)}{\mu_Q^3}\frac{(-1)^k}{Q^{k(k+1)/2}}\right]\frac{[Q^k]^{N+1}}{\left(z+[Q^k]^n\right)}\,. \tag{227}$$

Proof. From the remarks immediately preceding Lemma 9.1, for $k\geq 0$ one has

$$Res(-Q^k) = \frac{Q^k}{-Q^k\theta(k\,|\,Q\,;-Q^k)}G(-Q^k) \tag{228}$$

$$= \frac{Q^k}{-Q^k(-1)^k\mu_Q^3\,Q^{k(k+1)/2}}G(-Q^k) = \left[\frac{(-1)}{\mu_Q^3}\frac{(-1)^k}{Q^{k(k+1)/2}}\right]G(-Q^k)\,, \tag{229}$$

where the $k\geq 0$ case (219) of Lemma 9.1 has been used to move from (228) to (229).

From the remarks immediately preceding Lemma 9.1, for $k<0$ one has

$$Res(-Q^k) = \frac{-Q^k}{-Q^k\theta(k\,|\,Q\,;-Q^k)}G(-Q^k) \tag{230}$$

$$= \frac{-Q^k}{-Q^k(-1)(-1)^k\mu_Q^3\,Q^{k(k+1)/2}}G(-Q^k) = \left[\frac{(-1)}{\mu_Q^3}\frac{(-1)^k}{Q^{k(k+1)/2}}\right]G(-Q^k)\,, \tag{231}$$

where the $k<0$ case (220) of Lemma 9.1 has been used to move from (230) to (231). Thus (226) is shown for all $k\in\mathbb{Z}$. Setting $G(u)=[-u]^{N+1}\big/(z+[-u]^n)$ in (225) then gives (227) via (226). The corollary is now proven. □

Note that the expression in (227) matches the summand in (213) up to the scaling $-1/\mu_Q^3$ when $z = \mathrm{i}x$, via choice of expression (214). Expression (214) will have further zeros (other than $u = -Q^k$) in the denominator when the factor $\left(z + [-u]^n\right)$ vanishes, that is when $u = -(-z)^{1/n}$. Let z_j for $0 \le j \le n-1$ denote the n values of $-(-z)^{1/n}$, and note then that

$$(-z_j)^n = -z\,. \tag{232}$$

Relying on (232), one has

$$\begin{aligned} z + [-u]^n = [-u]^n + z = [-u]^n - (-z_j)^n &= \big([-u] - [-z_j]\big)\left[\sum_{j=0}^{n-1}[-u]^{n-1-j}[-z_j]^j\right] \\ &= (-1)\left(u - z_j\right)\left[\sum_{j=0}^{n-1}[-u]^{n-1-j}[-z_j]^j\right]. \end{aligned} \tag{233}$$

From (233), the residue of $1/\left(z + [-u]^n\right)$ at z_j for $0 \le j \le n-1$ is given by

$$\begin{aligned} (-1)\left[\sum_{j=0}^{n-1}[-z_j]^{n-1-j}[-z_j]^j\right]^{-1} = (-1)\left[\sum_{j=0}^{n-1}[-z_j]^{n-1}\right]^{-1} &= (-1)[-z_j]\left[\sum_{j=0}^{n-1}[-z_j]^n\right]^{-1} \\ = (-1)[-z_j]\left[n[-z_j]^n\right]^{-1} = z_j\left[n[-z]\right]^{-1} &= \left[\frac{-z_j}{nz}\right], \end{aligned} \tag{234}$$

where (232) was used to obtain the second equality in (234). This fact is applied in the following lemma.

Lemma 9.3. *Given $z \neq 0$, let z_j for $0 \le j \le n-1$ denote the n values of $-(-z)^{1/n}$, where (232) holds for z_j. Then the residue of (214), namely of*

$$\frac{1}{u\theta(Q;u)}\frac{[-u]^{N+1}}{(z + [-u]^n)}\,,$$

at $u = z_j$, is given by

$$Res(z_j) = \frac{-[-z_j]^{N+1}}{nz\theta(Q;z_j)} = \frac{(-1)^N Q^{(N+1)(N)/2}}{nz\theta(Q;z_j/Q^{N+1})}\,. \tag{235}$$

Proof. From (234), one has

$$Res(z_j) = \frac{1}{z_j\theta(Q;z_j)}[-z_j]^{N+1}\left[\frac{-z_j}{nz}\right] = \frac{-[-z_j]^{N+1}}{nz\theta(Q;z_j)}\,, \tag{236}$$

giving the first equality in (235). One next obtains

$$\frac{-[-z_j]^{N+1}}{nz\theta(Q;z_j)} = \frac{(-1)^N}{nz[z_j]^{-(N+1)}\theta(Q;z_j)} = \frac{(-1)^N Q^{(-N-1)(-N)/2}}{nzQ^{(-N-1)(-N)/2}[z_j]^{-(N+1)}\theta(Q;z_j)} \tag{237}$$

$$= \frac{(-1)^N Q^{(N+1)(N)/2}}{nz\theta(Q;Q^{-N-1}z_j)}\,, \tag{238}$$

where (24) is used to move from (237) to (238). This gives the second equality in (235). □

Lemma 9.4. *Let $\Gamma_M = C_M - c_M$ be the oriented boundary of the annular region A_M in $\mathbb{C}$ enclosed by the circular paths $C_M = \frac{Q^{M+1}+Q^M}{2}e^{\mathrm{i}\alpha}$ and $c_M = \frac{Q^{-M-1}+Q^{-M}}{2}e^{\mathrm{i}\alpha}$ where α increases from 0 to 2π. Given $z \in \mathbb{C}\backslash\{0\}$, let z_j for $0 \le j \le n-1$ denote the n values of $-(-z)^{1/n}$. Choose M sufficiently large so that $z_j \in A_M$ for $0 \le j \le n-1$. Then one has*

$$\begin{aligned} &\int_{\Gamma_M} \frac{1}{u\theta(Q;u)}\frac{[-u]^{N+1}}{(z + [-u]^n)}\mathrm{d}u \\ &= \frac{-2\pi\mathrm{i}}{\mu_Q^3}\sum_{k=-M}^{M}\left[\frac{(-1)^k}{Q^{k(k+1)/2}}\frac{\left[Q^k\right]^{(N+1)}}{\left(z + \left[Q^k\right]^n\right)}\right] + (-1)^N Q^{(N+1)(N)/2}\frac{2\pi\mathrm{i}}{nz}\sum_{j=0}^{n-1}\frac{1}{\theta(Q;z_j/Q^{N+1})}\,. \end{aligned} \tag{239}$$

Proof. The integral over Γ_M yields $2\pi\mathrm{i}$ times the enclosed residues, which occur at $u = z_j$ for $0 \le j \le n-1$ and at the zeroes of $\theta(Q;u)$ in A_M, which by construction of Γ_M are $u = -Q^k$ for $k \in \{-M, -M+1, \ldots, M-1, M\}$. So the residue theorem gives

$$\int_{\Gamma_M} \frac{1}{u\theta(Q;u)} \frac{[-u]^{N+1}}{(z+[-u]^n)} \mathrm{d}u = 2\pi\mathrm{i} \sum_{k=-M}^{M} Res(-Q^k) + 2\pi\mathrm{i} \sum_{j=0}^{n-1} Res(z_j)$$

$$= 2\pi\mathrm{i} \sum_{k=-M}^{M} \left[\frac{(-1)}{\mu_Q^3} \frac{(-1)^k}{Q^{k(k+1)/2}} \frac{[Q^k]^{(N+1)}}{\left(z+[Q^k]^n\right)} \right] + 2\pi\mathrm{i} \sum_{j=0}^{n-1} \frac{(-1)^N Q^{(N+1)(N)/2}}{nz\theta(Q;z_j/Q^{N+1})}, \tag{240}$$

where $Res(-Q^k)$ has been replaced by (227) to obtain the first summation in (240), and where $Res(z_j)$ has been replaced by (235) to obtain the second summation (240). These expressions combine and simplify to (239). The lemma is proven. □

Lemma 9.5. *Let $\Gamma_M = C_M - c_M$ be the oriented boundary of the annular region A_M in $\mathbb{C}$ enclosed by the circular paths $C_M = \frac{Q^{M+1}+Q^M}{2} e^{\mathrm{i}\alpha}$ and $c_M = \frac{Q^{-M-1}+Q^{-M}}{2} e^{\mathrm{i}\alpha}$ where α increases from 0 to 2π. Then,*

$$\lim_{M\to\infty} \int_{\Gamma_M} \frac{1}{u\theta(Q;u)} \frac{[-u]^{N+1}}{(z+[-u]^n)} \mathrm{d}u = 0 .$$

Proof. The result follows by showing:

$$\lim_{M\to\infty} \int_{C_M} \frac{1}{u\theta(Q;u)} \frac{[-u]^{N+1}}{(z+[-u]^n)} \mathrm{d}u = 0 = \lim_{M\to\infty} \int_{c_M} \frac{1}{u\theta(Q;u)} \frac{[-u]^{N+1}}{(z+[-u]^n)} \mathrm{d}u ,$$

which holds if $\theta(Q;u)$ grows sufficiently rapidly as M approaches infinity for $u \in C_M$ or $u \in c_M$. This growth follows directly from the identity (24). Let $C = \{v \mid v = \frac{(Q+1)}{2} e^{\mathrm{i}\alpha}, \alpha \in [0, 2\pi]\}$ be a reference circle of radius $\rho := (Q+1)/2 > 1$ which by construction contains no zeros of $\theta(Q;u)$. Observe that $\rho/Q = (1+Q^{-1})/2 < 1$. By continuity of $\theta(Q;v)$, $\exists\, b, B$ such that for $v \in C$

$$0 < b \le |\theta(Q;v)| \le B < \infty .$$

Note that $u \in C_M$ implies $\exists\, v \in C$ with $u = Q^M v$, and by (24) one has $\theta(Q;u) = \theta(Q;Q^M v) = Q^{M(M+1)/2} v^M \theta(Q;v)$ with $|\theta(Q;u)| = Q^{M(M+1)/2} \rho^M |\theta(Q;v)|$. Then one has

$$\left| \int_{C_M} \frac{1}{u\theta(Q;u)} \frac{[-u]^{N+1}}{(z+[-u]^n)} \mathrm{d}u \right| \le \int_{C_M} \frac{1}{|\theta(Q;u)|} \frac{[|u|]^{N+1}}{|z+[-u]^n|} \frac{|\mathrm{d}u|}{|u|}$$

$$\le \frac{1}{(Q^{(M)(M+1)/2}\rho^M b)} \frac{[Q^M\rho]^{N+1}}{([Q^M\rho]^n - |z|)} 2\pi$$

$$= \frac{1}{(Q^{(M)(M-2N-1)/2}\rho^M b)} \frac{2\pi\rho^{N+1}}{([Q^M\rho]^n - |z|)}$$

which approaches 0 as M approaches infinity. Similarly, $u \in c_M$ implies $\exists\, v \in C$ with $u = Q^{-M-1}v$, and by (24) one has

$$\theta(Q;u) = \theta(Q;Q^{-M-1}v) = Q^{(-M-1)(-M)/2} v^{-M-1} \theta(Q;v)$$

with

$$|\theta(Q;u)| = Q^{(-M-1)(-M)/2} \rho^{-M-1} |\theta(Q;v)| .$$

Thus

$$\left| \int_{c_M} \frac{1}{u\theta(Q;u)} \frac{[-u]^{N+1}}{(z+[-u]^n)} \mathrm{d}u \right| \le \int_{c_M} \frac{1}{|\theta(Q;u)|} \frac{[|u|]^{N+1}}{|z+[-u]^n|} \frac{|\mathrm{d}u|}{|u|}$$

$$\le \frac{1}{(Q^{(-M-1)(-M)/2}\rho^{-M-1} b)} \frac{[Q^{-M-1}\rho]^{N+1}}{(|z| - [Q^{-M-1}\rho]^n)} 2\pi$$

$$= \frac{1}{(Q^{(M+1)(M+2N)/2}b)} \frac{(\rho/Q)^{M+1}\rho^{N+1}}{(|z| - [\rho/Q^{M+1}]^n)} 2\pi$$

which also vanishes as M approaches infinity. The lemma is shown. □

Proof of Theorem 6.3. By Lemma 9.4 with z_j being the solutions of $(-z_j)^n = -z$ for $0 \le j \le n-1$, we have

$$\int_{\Gamma_M} \frac{1}{u\theta(Q;u)} \frac{[-u]^{N+1}}{(z+[-u]^n)} du$$

$$= \frac{-2\pi i}{\mu_Q^3} \sum_{k=-M}^{M} \left[\frac{(-1)^k}{Q^{k(k+1)/2}} \frac{[Q^k]^{(N+1)}}{\left(z+[Q^k]^n\right)} \right] + (-1)^N Q^{(N+1)(N)/2} \frac{2\pi i}{nz} \sum_{j=0}^{n-1} \frac{1}{\theta(Q;z_j/Q^{N+1})}.$$

Taking the limit as M approaches infinity, and applying Lemma 9.5, we obtain

$$0 = \frac{-2\pi i}{\mu_Q^3} \sum_{k=-\infty}^{\infty} \left[\frac{(-1)^k}{Q^{k(k+1)/2}} \frac{[Q^k]^{(N+1)}}{\left(z+[Q^k]^n\right)} \right] + (-1)^N Q^{(N+1)(N)/2} \frac{2\pi i}{nz} \sum_{j=0}^{n-1} \frac{1}{\theta(Q;z_j/Q^{N+1})}.$$

Multiplying through by $\mu_q^3/[2\pi i\sqrt{2\pi}]$ yields the identity

$$\frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \left[\frac{(-1)^k}{Q^{k(k+1)/2}} \frac{[Q^k]^{(N+1)}}{\left(z+[Q^k]^n\right)} \right] = (-1)^N Q^{(N+1)(N)/2} \frac{\mu_Q^3}{\sqrt{2\pi} nz} \sum_{j=0}^{n-1} \frac{1}{\theta(Q;z_j/Q^{N+1})}, \tag{241}$$

which reduces the infinite sum on the left in (241) to the finite sum on the right. From (213), and from (241) with z set equal to ix, one has that

$$\mathcal{F}[F_{2N+1,2n}(t)](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ixt} F_{2N+1,2n}(t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ixt} f_{2N+1,2n}(t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \left[\frac{(-1)^k}{Q^{k(k+1)/2}} \frac{[Q^k]^{(N+1)}}{\left(ix+[Q^k]^n\right)} \right] = (-1)^N Q^{(N+1)(N)/2} \frac{\mu_Q^3}{\sqrt{2\pi} n(ix)} \sum_{j=0}^{n-1} \frac{1}{\theta(Q;z_j/Q^{N+1})},$$

where the z_j for $0 \le j \le n-1$ are the n distinct solutions of $[-z_j]^n = -ix$. Substituting $Q = q^{1/n}$, as per (213), and expressing dependence of the roots z_j on x as $z_j = z_j(x)$ gives (148), which finishes the proof of Theorem 6.3. □

We remark on Theorem 6.3 above, that, in the special case $N = -1$ and $n = 1$ one recovers Theorem 2 of [22].

We finish the paper with proof that the Fourier transform of the even/odd extension of $f_{\mu,1}(t)$ can be expressed in terms of theta functions.

Proof of Theorem 6.5. We proceed first with the proof of the even case. If $f_{\mu,1}(t)$ is to be extended to be an even differentiable function, then, at $t = 0$ its odd order derivatives $f_{\mu,1}^{(2\ell+1)}(0) = (-1)^{2\ell+1} f_{\mu+2\ell+1,1}(0)$ must vanish. From (29), this only occurs when $\mu + 2\ell + 1 = 2k + 1$ is an odd integer. Thus $\mu = 2(k-\ell) = 2n$ is an even integer. The extension methods of Theorem 3.2, as deployed in Example 2 Case A, now produce an even extension $F_{2n,1}(t)$ of $f_{2n,1}(t)$ to the negative reals by taking $\ell_0 = 0 = \ell_1$ and $p_0 = 1 = p_1$ with $\gamma_0 = 1$ and generic choice of $\gamma_1 \ne 1$. Thus we have the even extensions

$$F_{2n,1}(t) = \begin{cases} \displaystyle\sum_{k=-\infty}^{\infty} (-1)^k \frac{e^{-q^k t}}{q^{k(k-2n)}} & \text{if } t \ge 0 \\ \displaystyle\sum_{k=-\infty}^{\infty} (-1)^k \frac{e^{q^k t}}{q^{k(k-2n)}} = \sum_{k=-\infty}^{\infty} (-1)^k \frac{e^{-q^k |t|}}{q^{k(k-2n)}} & \text{if } t \le 0. \end{cases} \tag{242}$$

Relying on (242), the Fourier transform is computed as

$$\mathcal{F}[F_{2n,1}(t)](x)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{-ixt} \sum_{k=-\infty}^{\infty} (-1)^k \frac{e^{q^k t}}{q^{k(k-2n)}} dt + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-ixt} \sum_{k=-\infty}^{\infty} (-1)^k \frac{e^{-q^k t}}{q^{k(k-2n)}} dt$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{q^{k(k-2n)}} \frac{1}{(-\mathrm{i}x+q^k)} + \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{q^{k(k-2n)}} \frac{-1}{(-\mathrm{i}x-q^k)} \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{q^{k(k-2n)}} \frac{2q^k}{x^2+q^{2k}} = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{q^{k(k-2n)}} \frac{2q^k}{x^2+q^{2k}} \left[\frac{q^{k(2n+1)}}{q^{k(2n+1)}}\right] \\
&= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(q^2)^{k(k+1)/2}} \frac{2q^{2k(n+1)}}{x^2+q^{2k}} = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \frac{(-1)^k}{Q^{k(k+1)/2}} \frac{2Q^{k(n+1)}}{x^2+Q^k} \,, \qquad (243)
\end{aligned}
$$

where $Q = q^2$. From (243) and Corollary 9.2, one sees that an expression of the form

$$
\frac{1}{u\theta(Q;u)} \frac{2[-u]^{n+1}}{(x^2-u)} \qquad (244)
$$

should be deployed in a contour integral. Let Γ_M be the oriented boundary of the annulus given in Lemma 9.5 with M sufficiently large that x^2 is contained in the interior of the annulus. One has via (225)–(226) of Corollary 9.2, with $G(u) = 2[-u]^{n+1}/(x^2-u)$, that

$$
\begin{aligned}
&\int_{\Gamma_M} \frac{1}{u\theta(Q;u)} \frac{2[-u]^{n+1}}{(x^2-u)} \mathrm{d}u = 2\pi\mathrm{i} \sum_{k=-M}^{M} Res(-Q^k) + 2\pi\mathrm{i} Res(x^2) \\
&= \frac{-2\pi\mathrm{i}}{\mu_Q^3} \sum_{k=-M}^{M} \left[\frac{(-1)^k}{Q^{k(k+1)/2}} \frac{2[Q^k]^{n+1}}{\left(x^2+Q^k\right)}\right] + 2\pi\mathrm{i} \frac{(-2)[-x^2]^{n+1}}{x^2\theta(Q;x^2)} \,. \qquad (245)
\end{aligned}
$$

By an argument parallel to that in Lemma 9.5, one has that

$$
\lim_{M\to\infty} \int_{\Gamma_M} \frac{1}{u\theta(Q;u)} \frac{2[-u]^{n+1}}{(x^2-u)} \mathrm{d}u = 0 \,,
$$

due to the rapid growth of $\theta(Q;u)$ for $u \in \Gamma_M$ as $M \to \infty$. Taking the limit of (245) as M approaches infinity, and dividing by $2\pi\mathrm{i}$ yields

$$
0 = \frac{-1}{\mu_Q^3} \sum_{k=-\infty}^{\infty} \left[\frac{(-1)^k}{Q^{k(k+1)/2}} \frac{2[Q^k]^{n+1}}{\left(x^2+Q^k\right)}\right] + \frac{2[-x^2]^n}{\theta(Q;x^2)} \,. \qquad (246)
$$

Isolating the expression containing $\theta(Q;x^2)$, multiplying by $\mu_Q^3/\sqrt{2\pi}$, and writing $-x^2 = (-\mathrm{i}x)^2$ yields

$$
\frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \left[\frac{(-1)^k}{Q^{k(k+1)/2}} \frac{2[Q^k]^{n+1}}{\left(x^2+Q^k\right)}\right] = \frac{\mu_Q^3}{\sqrt{2\pi}} \frac{2[(-\mathrm{i}x)^2]^n}{\theta(Q;x^2)} \,. \qquad (247)
$$

Substituting $Q = q^2$ in (247) and comparing with (243) yields

$$
\mathcal{F}[F_{2n,1}(t)](x) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \left[\frac{(-1)^k}{q^{k(k+1)}} \frac{2[q^{2k}]^{n+1}}{\left(x^2+q^{2k}\right)}\right] = \frac{2\mu_{q^2}^3}{\sqrt{2\pi}} \frac{(-\mathrm{i}x)^{2n}}{\theta(q^2;x^2)} \,, \qquad (248)
$$

giving the theorem in the $N = 2n$ case.

On the other hand, if $f_{\mu,1}(t)$ is to be extended to be an odd function, then at $t = 0$ its even order derivatives $f_{\mu,1}^{(2\ell)}(0) = (-1)^{2\ell} f_{\mu+2\ell,1}(0)$ must vanish. From (29), this only occurs when $\mu + 2\ell = 2k+1$ is an odd integer. Thus $\mu = 2(k-\ell)+1 = 2n+1$ is an odd integer. The extension methods of Theorem 3.2, as deployed in Example 2 Case A, now produce an odd extension $F_{2n+1,1}(t)$ of $f_{2n+1,1}(t)$ to the negative reals by taking $\ell_0 = 0 = \ell_1$ and $p_0 = 1 = p_1$ with $\gamma_0 = 1$ and generic choice of $\gamma_1 \neq 1$. Thus we have the odd extensions

$$
F_{2n+1,1}(t) = \begin{cases} \displaystyle\sum_{k=-\infty}^{\infty} (-1)^k \frac{\mathrm{e}^{-q^k t}}{q^{k(k-2n-1)}} & \text{if } t \geq 0 \\ \displaystyle -\sum_{k=-\infty}^{\infty} (-1)^k \frac{\mathrm{e}^{q^k t}}{q^{k(k-2n-1)}} = -\sum_{k=-\infty}^{\infty} (-1)^k \frac{\mathrm{e}^{-q^k |t|}}{q^{k(k-2n-1)}} & \text{if } t \leq 0 \,. \end{cases} \qquad (249)
$$

From (242) and (249), observe that $F'_{2n,1}(t) = -F_{2n+1,1}(t)$. Thus

$$\mathcal{F}[F_{2n+1,1}(t)](x) = \mathcal{F}[-F'_{2n,1}(t)](x) = -(\mathrm{i}\,x)\mathcal{F}[F_{2n,1}(t)](x) = \frac{2\mu_{q^2}^3}{\sqrt{2\pi}} \frac{(-\mathrm{i}\,x)^{2n+1}}{\theta(q^2;x^2)}, \tag{250}$$

where the last equality in (250) follows from (248). This gives the theorem in the $N = 2n+1$ case.

Alternatively, one could integrate an expression of the form

$$\frac{1}{u\theta(Q;u)} \frac{(-2\,\mathrm{i}\,x)[-u]^{n+1}}{(x^2-u)} \tag{251}$$

over the contour Γ_M, mirroring the approach of the even case, to also obtain (250). □

Remark 19. In [24], the $F_{N,1}(t)$ were denoted by $f_N(t)$, and the computation of the Fourier transforms $\mathcal{F}[F_{N,1}(t)](x)$ in Theorem 6.5 here recovers the computation of the $\mathcal{F}[f_N(t)](x)$ (in Theorem 8 of [24]) via an entirely different and more general method of contour integration.

References

[1] L. Amerio, Almost-periodic functions and functional equations, Boll. Unione Mat. Ital. (3) 20 (1965) 287–334.
[2] S. Bochner, Beitrage zur Theorie der fastperiodischen Funktionen, Math. Ann. 96 (1926) 119–147.
[3] H. Bohr, Zur Theorie der fastperiodischen Funktionen I, Acta Math. 45 (1925) 29–127.
[4] M. Brownik, Anisotropic Hardy spaces and wavelets, Mem. Amer. Math. Soc. 164 (781) (2003).
[5] O. Costin, M. Huang, Behavior of lacunary series at the natural boundary, Adv. Math. 222 (2009) 1370–1404.
[6] I. Daubechies, Ten Lectures on Wavelets, CBMS–NSF Regional Conference Series in Applied Mathematics, vol. 61, SIAM, Philadelphia, PA, USA, 1992.
[7] N.T. Dung, Asymptotic behavior of linear advanced differential equations, Acta Math. Sci. 35 (3) (2015) 610–618.
[8] S.W. Goode, Differential Equations and Linear Algebra, second ed., Prentice Hall, Upper Saddle River, NJ, USA, 2000.
[9] G.H. Hardy, N. Riesz, The General Theory of Dirichlet's Series, Cambridge University Press, Cambridge, UK, 1915.
[10] S. Krantz, Handbook of Complex Variables, Birkhäuser, 1951.
[11] A. Lastra, S. Malek, On q-Gevrey asymptotics for singularly perturbed q-difference-differential problems with an irregular singularity, Abstr. Appl. Anal. 2012 (2012) 860716.
[12] A. Lastra, S. Malek, On parametric Gevrey asymptotics for singularly perturbed partial differential equations with delays, Abstr. Appl. Anal. 2013 (2013) 723040.
[13] A. Lastra, S. Malek, Parametric Gevrey asymptotics for some Cauchy problems in quasiperiodic function spaces, Abstr. Appl. Anal. 2014 (2014) 153169.
[14] A. Lastra, S. Malek, On parametric multilevel q-Gevrey asymptotics for some linear q-difference-differential equations, Adv. Differ. Equ. 2015 (2015) 344.
[15] A. Lastra, S. Malek, On multiscale Gevrey and q-Gevrey asymptotics for some linear q-difference differential initial value Cauchy problems, J. Differ. Equ. Appl. 23 (8) (2017) 1397–1457.
[16] A. Lastra, S. Malek, J. Sanz, On q-asymptotics for linear q-difference-differential equations with Fuchsian and irregular singularities, J. Differ. Equ. 252 (10) (2012) 5185–5216.
[17] A. Lastra, S. Malek, J. Sanz, Gevrey solutions of threefold singular nonlinear partial differential equations, J. Differ. Equ. 255 (10) (2013) 3205–3232.
[18] S. Malek, On complex singularity analysis for linear partial q-difference-differential equations using nonlinear differential equations, J. Dyn. Control Syst. 19 (1) (2013) 69–93.
[19] Y. Meyer, Wavelets and Operators, Cambridge University Press, Cambridge, UK, 1992.
[20] D. Pravica, M. Spurr, Analytic continuation int the FUTURE, Discrete Contin. Dyn. Syst. Suppl. Vol. 2002 (2002) 709–716.
[21] D. Pravica, M. Spurr, Unique summing of formal power series solutions to advanced and delayed differential equations, Discrete Contin. Dyn. Syst. Suppl. Vol. 2005 (2005) 730–737.
[22] D. Pravica, N. Randriampiry, M. Spurr, Applications of an advanced differential equation in the study of wavelets, Appl. Comput. Harmon. Anal. 27 (2009) 2–11.
[23] D. Pravica, N. Randriampiry, M. Spurr, Theta function identities in the study of wavelets satisfying advanced differential equations, Appl. Comput. Harmon. Anal. 29 (2010) 134–155.
[24] D. Pravica, N. Randriampiry, M. Spurr, Reproducing kernel bounds for an advanced wavelet frame via the theta function, Appl. Comput. Harmon. Anal. 33 (1) (2012) 79–108.
[25] D. Pravica, N. Randriampiry, M. Spurr, q-Advanced models for tsunamis and rogue waves, Abstr. Appl. Anal. 2012 (2012) 414060.
[26] D. Pravica, N. Randriampiry, M. Spurr, Smooth wavelet approximations of truncated Legendre polynomials via the Jacobi theta function, Abstr. Appl. Anal. 2014 (2014) 890456.
[27] D. Pravica, N. Randriampiry, M. Spurr, On q-advanced spherical Bessel functions of the first kind and perturbations of the Haar wavelet, Appl. Comput. Harmon. Anal. 44 (2) (2018) 350–413.
[28] W. Rudin, Real and Complex Analysis, second ed., McGraw–Hill Series in Higher Mathematics, McGraw–Hill, New York, 1974.
[29] C.-C. Tseng, S.-C. Pei, C.-C. Hsia, Computation of fractional derivatives using Fourier transform and digital FIR differentiator, Signal Process. 80 (2000) 151–159.
[30] C. Zhang, Analytic continuation of solutions of the pantograph equation by means of θ-modular forms, http://arxiv.org/abs/1202.0423.