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Super-multiplicativity and a lower bound for the decay of the signature of a path of finite length

Supermultiplicativité et une borne inférieure pour la décroissance de la signature d'un chemin de longueur finie

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ABSTRACT

For a path of length $L > 0$, if for all $n \geq 1$, we multiply the n-th term of the signature by $n!L^{-n}$, we say that the resulting signature is '*normalised*'. It has been established (T. J. Lyons, M. Caruana, T. Lévy, Differential equations driven by rough paths, Springer, 2007) that the norm of the n-th term of the normalised signature of a bounded-variation path is bounded above by 1. In this article, we discuss the super-multiplicativity of the norm of the signature of a path with finite length, and prove by Fekete's lemma the existence of a non-zero limit of the n-th root of the norm of the n-th term in the normalised signature as n approaches infinity.

RÉSUMÉ

Pour une trajectoire de longueur $L > 0$, si l'on multiplie le n-ième terme de la signature par $n!L^{-n}$ pour tout $n \geq 1$, la signature ainsi obtenue est dite « *normalisée* ». Il a été établi (T. J. Lyons, M. Caruana, T. Lévy, Differential equations driven by rough paths, Springer, 2007) que la norme du n-ième terme de la signature normalisée d'une trajectoire à variation bornée est majorée par 1. Dans cet article, nous étudions la super-multiplicativité de la norme de la signature d'une trajectoire de longueur finie, et nous démontrons, à l'aide du lemme de Fekete, l'existence d'une limite non nulle lorsque n tend l'infini pour la racine n-ième de la norme du n-ième terme de la signature normalisée.

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1. Super-multiplicativity of the signature in reasonable tensor algebra norms

Definition 1. Let $\{V_j\}_{j=1}^N$ be normed vector spaces over $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$. Their *algebraic tensor product space* is defined as the vector space

$$V_1 \otimes \ldots \otimes V_N = \left\{ \sum_{i \in I} v_i^1 \otimes \ldots \otimes v_i^N : v_i^j \in V_j, \quad \forall i \in I, |I| < \infty, \, J = 1, \ldots, N \right\},$$

where we identify $(u+v)\otimes w = u \otimes w + v \otimes w$, and $u \otimes (v+w) = u \otimes v + u \otimes w$.

Definition 2. If $\phi_j \in V_j'$ are bounded linear functionals on V_j, $j=1,\ldots,N$, then we define the dual action of $\phi_1 \otimes \ldots \otimes \phi_N$ on $V_1 \otimes \ldots \otimes V_N \to \mathbb{F}$ by

$$(\phi_1 \otimes \ldots \otimes \phi_N)(\sum_{i \in I} v_i^1 \otimes \ldots \otimes v_i^N) := \sum_{i \in I} \prod_{j=1}^N \phi(v_i^j)$$

for all $v_i^j \in V_j$, $j=1,\ldots,N, i \in I$, $|I| < \infty$. The map is well defined and independent of the representation on the right-hand side.

Now we state the properties of the norms on tensor products that are required for this article.

Definition 3 *(Reasonable tensor algebra norm).* Let $V, V \otimes V, \ldots, V^{\otimes n}$ be normed vector spaces. We assume that, for all $v \in V^{\otimes n}$, $w \in V^{\otimes m}$,

$$\|v \otimes w\| \le \|v\|\|w\| \tag{1}$$

and the norm induced on the dual spaces satisfies that for all $\phi \in (V^{\otimes m})'$, $\psi \in (V^{\otimes n})'$,

$$\|\phi \otimes \psi\| \le \|\phi\|\|\psi\|. \tag{2}$$

Moreover, if $S(n)$ denotes the symmetric group over $\{1,2,..,n\}$, we assume that, for all $n \ge 1$,

$$\|\sigma(v)\| = \|v\| \quad \forall \sigma \in S(n), \; v \in V^{\otimes n}.$$

Proposition 1 *(Ryan [4]). Let X and Y be normed vector spaces. If $\|\;\|$ is a tensor norm on $X \otimes Y$ that satisfies*

$$\|v \otimes w\| \le \|v\|\|w\| \quad \forall v \in X, w \in Y;$$

and the norm induced on the dual spaces satisfies

$$\|\phi \otimes \psi\| \le \|\phi\|\|\psi\| \quad \forall \phi \in X', \psi \in Y',$$

then $\|\;\|$ is called a reasonable cross norm, *and $\|x \otimes y\| = \|x\|\|y\|$ for every $x \in X$ and $y \in Y$; for every $\phi \in X'$ and $\psi \in Y'$, the norm of the linear functional $\phi \otimes \psi$ on $(X \otimes Y, \|\;\|)$ satisfies $\|\phi \otimes \psi\| = \|\psi\|\|\psi\|$.*

Using Proposition 1 implies that the inequalities in Equation (1) and (2) imply equality.

Remark 1. Note that under the assumptions of Definition 3 for all $a \in V^{\otimes m}$, $b \in V^{\otimes n}$, $c \in V^{\otimes l}$,

$$\|(a \otimes b) \otimes c\| = \|a \otimes (b \otimes c)\| = \|a\|\|b\|\|c\|.$$

We provide some examples of tensor norms that are reasonable tensor algebra norms.

Definition 4. Let $\{V_j\}_{j=1}^N$ be normed vector spaces over $\mathbb{F}$. The *projective tensor norm* on $V_1 \otimes \ldots \otimes V_N$ is defined such that, for $x \in V_1 \otimes \ldots \otimes V_N$,

$$\|x\|_\pi := \inf \left\{ \sum_{i \in I} \|v_i^1\| \ldots \|v_i^N\| : x = \sum_{i \in I} v_i^1 \otimes \ldots \otimes v_i^N, \, v_i^j \in V_j \, \forall i \in I, |I| < \infty \right\}.$$

The *injective tensor norm* on $V_1 \otimes \ldots \otimes V_N$ is defined such that for $x = \sum_{i \in I} v_i^1 \otimes \ldots \otimes v_i^N \in V_1 \otimes \ldots \otimes V_N$, $i \in I$, $|I| < \infty$,

$$\|x\|_\delta := \sup\{|\sum_{i\in I}\prod_{j=1}^{N}\phi_j(v_i^j)| : \phi_j \in V_j', \|\phi_j\| \le 1\, \forall j = 1, ..., N\}$$

for any representation of x.

Lemma 1. *The projective tensor norm and the injective tensor norm defined in Definition 4 both satisfy the properties stated in Definition 3. Moreover, if α is a reasonable cross norm on $X \otimes Y$, and $u \in X \otimes Y$, then*

$$\|x\|_\delta \le \alpha(x) \le \|x\|_\pi.$$

Furthermore, any reasonable tensor algebra norm is sandwiched between the injective and projective tensor norms.

The proof of Lemma 1 is omitted here.

Lemma 2. *The Hilbert–Schmidt norm is a reasonable tensor algebra norm.*

The proof of Lemma 2 is omitted here.

Definition 5. Let $V, V \otimes V, ..., V^{\otimes n}$ be Banach completed spaces equipped with a reasonable tensor algebra norm compatible with the norm on V, and $\gamma : J \to V$ be a continuous path with finite length. The *signature* of γ is denoted by

$$S = (1, S_1, S_2, ..., S_n, ...), \tag{3}$$

where, for each $n \ge 1$, $S_n = \int_{u_1<...<u_n, u_1,...,u_n \in J} \mathrm{d}\gamma_{u_1} \otimes ... \otimes \mathrm{d}\gamma_{u_n}$.

Remark 2. Note that the n-th term of S lives in the completed Banach space $V^{\otimes n}$ whenever the algebraic tensor product is completed with a reasonable tensor algebra norm.

From now on, we will fix a Banach space V, a reasonable tensor algebra norm, and we will take $V^{\otimes n}$ to be the completion of the algebraic tensor product with respect to that reasonable tensor algebra norm.

Definition 6 *(Shuffle product).* The shuffle product is defined inductively to be bilinear, and so that

$$u \otimes a \sqcup\!\sqcup v \otimes b := (u \sqcup\!\sqcup v \otimes b) \otimes a + (u \otimes a \sqcup\!\sqcup v) \otimes b$$

for any $a, b \in V$.

Definition 7 *(Group-like elements).* Define

$$\tilde{T}((V)) := \left\{(a_0, a_1, a_2, ...) : a_n \in V^{\otimes n}\, \forall n \ge 1, a_0 = 1\right\}.$$

An element $\mathbf{a} \in \tilde{T}((V))$ is called *group-like* if for all $\phi, \psi \in (\tilde{T}((V)))'$,

$$\phi \sqcup\!\sqcup \psi(\mathbf{a}) = \phi(\mathbf{a})\psi(\mathbf{a}).$$

Theorem 1. *Suppose $\gamma : J \to V$ is a path of finite length. Then, for $m, n \ge 0$, the signature of γ satisfies*

$$\|(m+n)!S_{m+n}\| \ge \|n!S_n\|\|m!S_m\| \quad \forall m, n \ge 0, \tag{4}$$

where $\|\ \|$ is any reasonable tensor algebra norm. $V^{\otimes 0}$ is defined to be $\mathbb{F}$, and $S_0 = 1$.

Proof. By Hahn–Banach's Theorem, there exists $\phi_n \in (V^{\otimes n})'$, $\phi_m \in (V^{\otimes m})'$ such that $\|\phi_n\| = 1$, $\|\phi_m\| = 1$, and

$$\phi_n(S_n) = \|S_n\|,\ \ \phi_m(S_m) = \|S_m\|.$$

Equivalently, we can write

$$\phi_n(S) = \|S_n\|,\ \ \phi_m(S) = \|S_m\|,$$

where we define $\phi_k(x) = 0$ for $x \notin V^{\otimes k}$ for all $k \ge 0$. From [3], we know that S is group-like; hence,

$$\phi_m \sqcup\!\sqcup \phi_n(S) = \phi_m(S)\phi_n(S) = \|S_m\|\|S_n\|.$$

Also,

$$\begin{aligned}\phi_m \sqcup\!\sqcup \phi_n(S_{m+n}) &= \sum_{\sigma\in\text{Shuffles}(m,n)} \sigma(\phi_m\otimes\phi_n)(S_{m+n}) \\ &= \sum_{\sigma\in\text{Shuffles}(m,n)} (\phi_m\otimes\phi_n)(\sigma^{-1}(S_{m+n})),\end{aligned}$$

so

$$|\phi_m \sqcup\!\sqcup \phi_n(S_{m+n})| \le \#\text{shuffles}(m,n)\|\phi_m\otimes\phi_n\|\|S_{m+n}\|.$$

Note that $\#\text{shuffles}(m,n) = \frac{(m+n)!}{n!m!}$, and by Definition 3 we know that

$$\|\phi_m\otimes\phi_n\| \le \|\phi_m\|\|\phi_n\| = 1.$$

Hence

$$\|(m+n)!S_{m+n}\| \ge \|n!S_n\|\|m!S_m\|$$

as expected. □

Corollary 1. *If $S_j = 0$, then $S_k = 0$ for $k = 1, ..., j$.*

Proof. The proof follows from Theorem 1. □

2. Limiting behaviour

We note the following lemma by Fekete [5].

Theorem 2 *(Fekete's Lemma). If a sequence of real numbers $\{a_n\}_{n\in\mathbb{N}}$ satisfies the sub-additivity condition*

$$a_{m+n} \le a_m + a_n \quad \forall m,n\in\mathbb{N}.$$

Then

$$\lim_{n\to\infty}\frac{a_n}{n} = \inf_{n\in\mathbb{N}}\frac{a_n}{n}.$$

Theorem 3 *(Asymptotic behaviour of the signature). If $\gamma : J \to V$ is a continuous tree-reduced path of finite length $L > 0$, then, under any reasonable tensor algebra norm $\|\ \|$, there exists a non-zero limit $\tilde{L}$ such that*

$$\begin{aligned}&\lim_{n\to\infty}\|n!S_n\|^{1/n} \\ &= \sup_{k\ge1}\|k!S_k\|^{1/k} \\ &= \tilde{L} > 0.\end{aligned}$$

Proof. By Theorem 1, we know that, for all $m, n \ge 0$,

$$\|(m+n)!S_{m+n}\| \ge \|n!S_n\|\|m!S_m\|.$$

Taking logarithm gives

$$-\log(\|(m+n)!S_{m+n}\| \le -\log(\|n!S_n\|) - \log(\|m!S_m\|.$$

So the function $f(n) := -\log(\|n!S_n\|/L^n)$ satisfies $f(m+n) \le f(m) + f(n)$ for all $m, n \in \mathbb{N}$. Then, by Fekete's lemma [5], $\frac{1}{n}\log(\|n!S_n\|)$ converges to $\sup_{k\ge1}\log(\|k!S_k\|)/k$; hence, $\|n!S_n\|^{1/n}$ converges to $\sup_{k\in\mathbb{N}}\|k!S_k\|^{1/k}$. Note that, by Hambly and Lyons [2], every path of finite length has a unique tree-reduced[1] version with the same signature if the tree-reduced path is non-trivial, then there will be at least one term in the signature of the path which is non-zero. Hence $\sup_{k\ge1}\|k!S_k\|^{1/k}$ is non-zero. Therefore, $\|n!S_n\|^{1/n}$ converges to a non-zero limit as n increases. □

[1] Roughly speaking, a tree-reduced path is a path where it does not go back on cancelling itself over any interval.

Corollary 2. *Let V be a Banach space. For any element*

$$\mathbf{a} = (a_0, a_1, a_2, ...) \in \left\{(b_0, b_1, b_2, ...) : b_0 = 1, b_n \in V^{\otimes n}\ \forall n \geq 1\right\}$$

which is group-like, we have

$$\|(m+n)!a_{m+n}\| \geq \|m!a_m\|\|n!a_n\| \quad \forall m, n \geq 0,$$

and $\|n!a_n\|^{1/n}$ converges to $\sup_{k\in\mathbb{N}}\|k!a_k\|^{1/k}$ as n increases under any reasonable tensor algebra norm $\|\ \|$.

Proof. Note that since **a** is group-like, the same arguments apply as in Theorem 1 and Theorem 3. □

Remark 3. It is an interesting question to ask whether there is a nice and simple form of the limit of $\|n!S_n\|^{1/n}$ mentioned in Theorem 3, and whether the limit is the same under any reasonable tensor algebra norm. Moreover, we know from [3] that for a path with finite length $L > 0$, an upper bound of $\|n!S_n\|$ is L^n. Furthermore, Lyons and Hambly [2] proved that, for a smooth enough path of finite length, the ratio $\|n!S_n\|/L^n$ converges to 1 under certain norms. Therefore, we have the following conjecture.

Conjecture 1. *Let V be a Banach space, and $\gamma : J \to V$ be a path with finite length $L > 0$. Then the signature of γ satisfies that*

$$\|n!S_n\|^{1/n} \to L \quad \text{as } n \to \infty,$$

under any reasonable tensor algebra norm.

Remark 4. An interesting tensor norm to consider is the *Haagerup tensor norm* [1]. Clearly, the Haagerup norm is not a reasonable tensor algebra norm; however, under the Haagerup norm, for a path of finite length $L > 0$, we still have $n!\|S_n\| \leq L^n$. Therefore, it is an interesting question to ask whether the signature will have the same behaviour as described in Theorem 3 under the Haagerup tensor norm, or the symmetrised forms of the Haagerup tensor norm.

Remark 5. Although it has been shown that $\|n!S_n\|$ eventually behaves like L^n under certain norms for well-behaved paths (see [2]), some simple examples show that, in general, for a path with finite length, $\|n!S_n\|/L^n$ does not necessarily converge to 1 as n increases. The result in Theorem 3 is the best description we have found about the decay of the signature for a path with finite length.

For a p-variation path where $p > 1$, by considering simple examples, we can see that we cannot have a non-zero limit for $\|(n/p)!S_n\|^{1/n}$ as n increases.

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