Combinatorics

Symmetries on plabic graphs and associated polytopes

Symétries dans les graphes plan biclores et les polytopes associés

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\textbf{A B S T R A C T}

For Grassmann varieties, we explain how the duality between the Gelfand–Tsetlin polytopes and the Feigin–Fourier–Littelmann–Vinberg polytopes arises from different positive structures.
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\textbf{R É S U M É}

Nous expliquons, pour les variétés grassmanniennes, comment la dualité entre les polytopes de Gelfand–Tsetlin et les polytopes de Feigin–Fourier–Littelman–Vinberg émerge dans différentes structures positives.
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1. Introduction

Plabic graphs (planar bicoloured graphs) were introduced by Postnikov [8] to parametrize cells in the totally non-negative (TNN) Grassmannians \((\text{Gr}_{k,n}(\mathbb{R}))_{\geq 0}\). These graphs are drawn inside a disk with boundary vertices labelled by 1, 2, \ldots, \(n\) in a fixed orientation and internal vertices coloured black and white. For a reduced plabic graph \(\mathcal{G}\) corresponding to the top cell in the TNN-Grassmannian \((\text{Gr}_{k,n}(\mathbb{R}))_{\geq 0}\), Rietsch and Williams [10] constructed a family of polytopes for positive integers \(r\) as Newton–Okounkov bodies [5,7] associated with the line bundle \(r \in \mathbb{Z} \cong \text{Pic}(\text{Gr}_{k,n}(\mathbb{C}))\).

When the plabic graph \(\mathcal{G} := \text{Gr}_{k,n}^{\text{nc}}\) is chosen as in [10] (see Section 4.2), the corresponding Newton–Okounkov body \(\text{NO}_{\mathcal{G}}\) is unimodularly equivalent to the Gelfand–Tsetlin polytope \(\text{GT}_{k,n}^{1}\).

The Newton–Okounkov body is by definition a closed convex hull of points; even when it is a polytope, to read off its defining inequalities is a hard problem. In [10], the authors used mirror symmetry of Grassmannians to obtain these inequalities from the tropicalization of the super-potential on an open set of the mirror Grassmannian arising from the Landau–Ginzburg model. By applying this symmetry, they give explicit defining inequalities of \(\text{NO}_{\mathcal{G}}\).

Lattice points in Gelfand–Tsetlin polytopes parametrize the bases of finite-dimensional irreducible representations of the Lie algebra \(\mathfrak{sl}_n\). Motivated by a conjecture of Vinberg, another family of polytopes, called FFLV polytopes, is found by Feigin,
the second author, and Littelmann [3], whose lattice points also parametrize the bases of finite-dimensional irreducible representations of $s_{kn}$.

For a plabic graph $G$, its mirror $G^\vee$ is defined by swapping the black/white colouring of internal vertices in $G$. When the plabic graph $G$ corresponds to the top cell in $(Gr_{n-k,n}(\mathbb{R}))_{\geq 0}$, $G^\vee$ parametrizes the top cell in $(Gr_{k,n}(\mathbb{R}))_{\geq 0}$.

**Theorem 1.** The Newton–Okounkov body $NO_{G^\vee}$ is unimodularly equivalent to $F_{k,n}$ (see Section 4.1 for definition).

Another way to relate Gelfand–Tsetlin polytopes to FFLV polytopes is via a connection between the corresponding clusters in different cluster algebras. Each reduced plabic graph $G$ gives a cluster $\mathcal{C}$ consisting of Plücker coordinates $\Delta_I$, $I \in \binom{[n]}{k}$, where $I_1, \ldots, I_m$ are some $(n-k)$-element subsets of $[n] = \{1, 2, \ldots, n\}$.

For $I \subset [n]$, let $I^c$ denote its complement. Then the set $\mathcal{C}' = \{\Delta_{I_1}, \ldots, \Delta_{I_m}\}$ is a cluster for $Gr_{k,n}(\mathbb{C})$, corresponding to a plabic graph $G^\vee$.

**Corollary 1.** The Newton–Okounkov body $NO_{G^\vee}$ is unimodularly equivalent to $F_{k,n}$.

### 2. Plabic graphs

We recall the definition and basic properties of plabic graphs, following [8,10].

**Definition 1.** A plabic graph is an undirected planar graph $G$ satisfying:

1. $G$ is embedded in a closed disk and considered up to homotopy;
2. $G$ has $n$ vertices on the boundary of the disk, called boundary vertices, which are labelled clockwise by $1, 2, \ldots, n$;
3. all other vertices of $G$ are strictly inside the disk, they are called internal vertices and coloured in black and white;
4. each boundary vertex is incident to a single edge.

In [8] (see also [10]), there are three local moves defined on plabic graphs: gluing two vertices of the same colour, removing redundant vertices, and mutating a square. For a plabic graph $G$, let $\mathcal{F}(G)$ denote the set of its faces, which is invariant under the local moves.

**Definition 2.** A plabic graph $G$ is called reduced if there are no parallel edges $\begin{array}{|c|c|c|c|} \hline & \hspace{0.5em} & \hspace{0.5em} & \hspace{0.5em} \end{array}$ after applying any sequences of local moves.

**Definition 3.** Let $G$ be a reduced plabic graph. The trip $T_i$ starting from a boundary vertex $i$ is the path going through the edges of $G$, obeying the following rules:

1. at each internal black vertex, the path turns to the rightmost direction;
2. at each internal white vertex, the path turns to the leftmost direction.

The trip $T_i$ ends at a boundary vertex $\pi(i)$. We associate in this way a trip permutation $\pi_G := (\pi(1), \ldots, \pi(n))$ with $G$. Let $\pi_{k,n} = (n-k+1, n-k+2, \ldots, n, \pi(1), \ldots, \pi(n-k))$. The face labelling of $G$ is the injective map $\lambda_G : \mathcal{F}(G) \to \binom{[n]}{k}$ (the set of $k$-element subsets of $\{1, \ldots, n\}$) defined as follows: for a face $F \in \mathcal{F}(G)$, $\lambda_G(F)$ consists of those $i$ such that $F$ is to the left of the trip $T_i$. We set $\mathcal{V}_G := \lambda_G(\mathcal{F}(G))$.

See Fig. 1 for an example.

### 3. Polytopes arising from plabic graphs

We associate polytopes with plabic graphs following [10]. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ be the base field.

#### 3.1. Positive Grassmannians

For $0 < k < n$, let $\text{Mat}_{k,n}$ denote the set of $k \times n$-matrices with entries in $\mathbb{K}$. For $J \in \binom{[n]}{k}$ and $A \in \text{Mat}_{k,n}$, let $\Delta_J(A)$ denote the maximal minor of $A$ corresponding to columns in $J$.

Let $Gr_{k,n}$ be the Grassmann variety embedded into $\mathbb{P}^{N-1}$ via the Plücker embedding where $N = \binom{n}{k}$. The minors $\{\Delta_J | J \in \binom{[n]}{k}\}$ give the Plücker coordinates on $Gr_{k,n}$. When the base field is $\mathbb{R}$, the totally non-negative (resp. totally positive) Grassmannian $(Gr_{k,n}(\mathbb{R}))_{\geq 0}$ consists of those elements in $Gr_{k,n}$ having non-negative (resp. positive) Plücker coordinates.
3.2. Perfect orientations

To study flow models on plabic graphs, we fix a perfect orientation $O$ on $G$. Such an orientation requires that, at each black (resp. white) internal vertex, there is exactly one edge going out (resp. going in). It is shown in [9] that each reduced plabic graph admits an acyclic perfect orientation. Once such an orientation is fixed, we denote the source set by $I_O := \{i \in [n] | i \text{ is a boundary source of } O\}$; its complement $I_O^c$ is the set of boundary sinks.

For $I \in \binom{[n]}{k}$, let $x_I$ be a variable. For $i \in I_O$ and $j \in I_O^c$, let $P_{i,j}$ be the set of directed paths from $i$ to $j$. For such a directed path $\gamma$, let $F_\gamma(G)$ denote the set of faces to the left of $\gamma$. A flow $\mathfrak{f}$ from $I_O$ to $J \in \binom{[n]}{k}$ is a collection of pairwise vertex-disjoint directed paths in $G$ going from $I_O \setminus (I_O \cap J)$ to $J \setminus (I_O \cap J)$.

For a directed path $\gamma \in P_{i,j}$, we define the weight of $\gamma$ in $C(x_I | I \in \binom{[n]}{k})$ by:

$$\text{wt}(\gamma) := \prod_{F \in F_\gamma(G)} x_{k_{\gamma}(F)}.$$ 

The weight of a flow is the product of the weights of the paths it contains. For $J \in \binom{[n]}{k}$, we define $P_J$ to be the sum of the weights of all flows from $I_O$ to $J$.

For a reduced plabic graph $G$ of trip permutation $\pi_{n-k,k}$ with perfect orientation $O$, there exists only one face $F_{\emptyset}$ to the right of all directed paths with $\lambda_{\pi}(F_{\emptyset}) = [n-k+1, \ldots , n]$. We set $\mathcal{V}_{\emptyset}^G := \mathcal{V}_{\emptyset} \setminus \lambda_{\pi}(F_{\emptyset})$, $\Delta_{\emptyset} := \{ x_I | I \in \mathcal{V}_{\emptyset} \}$ and $\Delta_{\emptyset}^O := \{ x_I | I \in \mathcal{V}_{\emptyset} \}$.

**Theorem 2 ([8,12]):** Let $\mathbb{X} := \text{Gr}_{n,k}(\mathbb{C})$ and $C(\mathbb{X})$ be the field of rational functions on $\mathbb{X}$. There exists an isomorphism of fields:

$$C(\mathbb{X}) \cong C(x_I | x_I \in \Delta_{\emptyset}^O), \Delta_J \mapsto P_J.$$ 

The choice of the perfect orientation $O$ will only change the formula of $P_J$ by a scalar. We always assume that the choice $I_O = \{1, 2, \ldots , k\}$ is made.

Let $< \circ$ be a total order on $\Delta_{\emptyset}^O$. It induces a term order $< \circ$ on monomials in $\Delta_{\emptyset}^O$ by taking the lexicographic order. Let $f$ be a polynomial in Plücker coordinates of $\mathbb{X}$. By Theorem 2, $f$ can be written as a polynomial in $\Delta_{\emptyset}^O$:

$$f = \sum_{\mathbf{a} \in \mathbb{Z}^{\mathcal{V}_{\emptyset}^G}} c_{\mathbf{a}} x^{\mathbf{a}}, \text{ where } x^{\mathbf{a}} = \prod_{I \in \mathcal{V}_{\emptyset}^G} x_I^{a_I} \text{ if } \mathbf{a} = (a_I)_{I \in \mathcal{V}_{\emptyset}^G}.$$ 

Let $\nu_{\emptyset} : C(\mathbb{X})^* \rightarrow \mathbb{Z}^{\mathcal{V}_{\emptyset}^G}$ be the minimal term valuation on $C(\mathbb{X})$ with respect to the above total order.

Let $L_k$ denote the very ample line bundle on $\mathbb{X}$ generating $\text{Pic}(\mathbb{X})$. It gives the Plücker embedding. The space of global sections $H^0(\mathbb{X}, L_k^*)$, as a representation of $\text{GL}_n(\mathbb{C})$, is isomorphic to $V(r \sigma_k)^*$, where the latter is the dual of the finite-dimensional irreducible representation of highest weight $r \sigma_k$ ($\sigma_k$ is the $k$-th fundamental weight). The homogeneous coordinate ring $C[\mathbb{X}] := \bigoplus_{r \geq 0} H^0(\mathbb{X}, L_k^r)$ is embedded into $C(\mathbb{X})$ by sending $s \in H^0(\mathbb{X}, L_k^r)$ to $s/\Delta^O_{k}$. 

**Definition 4.** The Newton–Okounkov body associated with $L_k$, $\mathcal{V}_{\mathbb{X}}$, and the lexicographic order is defined by:

$$\text{NO}_{\emptyset} := \text{conv} \left( \bigcup_{r \geq 1} \left\{ \nu_{\emptyset}(s)/r \mid s \in H^0(\mathbb{X}, L_k^r) \setminus \{0\} \right\} \right).$$
We set $\text{NO}^1_k := \text{conv}(\{v^j_G((s) \mid s \in H^0(\mathcal{L}_k \setminus \emptyset)) \subseteq \text{NO}_k^1$. For the issue on whether this inclusion is proper (i.e., whether $\text{NO}_k^1$ is integral), see [10, Theorem 15.17].

### 4. Duality between Newton–Okounkov bodies

#### 4.1. Order polytopes and chain polytopes

Let $(P, \preceq_P)$ be a poset with covering relation $\prec$. Stanley [11] associated two Ehrhart equivalent polytopes, the order polytope and the chain polytope, with this poset. We recall here a dilated version of them.

For $r \in \mathbb{N}_{\geq 0}$, we denote the dilated order polytope $\mathcal{O}(P, r)$ to be the representation of the poset $P$ on the interval $[0, r]$ with the order on real numbers:

$$\mathcal{O}(P, r) := \text{Hom}_{\text{Poset}}(\{P, \preceq_P\}, ([0, r], \leq)) \subseteq \mathbb{R}^P.$$

The dilated chain polytope $\mathcal{C}(P, r) \subseteq \mathbb{R}^P$ has the following facets: for any $p \in P$, $x_p \geq 0$; for any maximal chain $p_1 < \cdots < p_s$, $x_{p_1} + \cdots + x_{p_s} \leq r$, where $x_p$ is the coordinate of $p \in P$ in $\mathbb{R}^P$.

Stanley [11] showed that the integral points of the chain polytope $\mathcal{C}(P, 1)$ are given by the characteristic functions of the anti-chains in $P$. In particular, the element $p \in P$ gives an integral point $x_p$ in $\mathcal{C}(P, 1)$.

In the following, we fix $1 \leq k \leq n - 1$, and let $(P_{k,n}, \leq)$ be the poset given by the elements $p_{i,j}$, where $1 \leq i \leq k$ and $k + 1 \leq j \leq n$, with covering relations

$$p_{i+1,j} \prec p_{i,j} \quad \text{and} \quad p_{i,j+1} \prec p_{i,j}.$$  

The polytope $\mathcal{O}(P_{k,n}, r)$ is the Gelfand–Tsetlin polytope $G_{k,n}^r$ for the representation $V(r\pi_k)$ of $sl_n ([4])$; while $\mathcal{C}(P_{k,n}, r)$ is the Feigin–Fourier–Littelmann–Vinberg polytope $\text{FLLV}^r_{k,n}$ ([1,3]) of the same representation.

For a polytope $Q \subseteq \mathbb{R}^m$, let $S(Q) := Q \cap \mathbb{Z}^m$ denote the set of integral points in it. The following integer decomposition properties hold: the $r$-fold Minkowski sum of $S(\mathcal{O}(P_{k,n}, 1))$ (resp. $S(\mathcal{C}(P_{k,n}, 1))$ coincides with $S(\mathcal{O}(P_{k,n}, r))$ (resp. $S(\mathcal{C}(P_{k,n}, r))$).

Moreover, if $a = \{p_{i_1,j_1}, \ldots, p_{i_s,j_s}\}$ is an anti-chain in $P_{k,n}$, then one has, for the corresponding lattice points, $\chi_a = \chi_{p_{i_1,j_1}} + \cdots + \chi_{p_{i_s,j_s}} \in \mathcal{C}(P_{k,n}, 1)$.

**Proposition 1.** Suppose that $Q$ is an integral polytope in $\mathbb{R}^{P_{k,n}}$ such that

- $\#S(Q) = \#S(\text{FLLV}^1_{k,n})$;
- there is a parametrization of the lattice points in $Q$ by anti-chains in $P_{k,n}$ sending an anti-chain $a$ to $y_a \in \mathbb{R}^{P_{k,n}}$ such that, for any anti-chain $a = \{p_{i_1,j_1}, \ldots, p_{i_s,j_s}\}$, the relation $y_a = y_{p_{i_1,j_1}} + \cdots + y_{p_{i_s,j_s}}$ holds;
- there is a linear map of determinant 1 expressing $y_a$ in terms of $\chi_a$.

Then the assignment $\chi_a \rightarrow y_a$ induces a unimodularly equivalence from $\text{FLLV}^1_{k,n}$ to $Q$.

#### 4.2. Duality of polytopes from positive structures

We refer the reader to [10, Section 7.1] for the definition of the rec-plabic graph $G_{k,n}^{\text{rec}}$. For example, the plabic graph in Fig. 1 is $G_{k,n}^{\text{rec}}$.

The following has been shown in [10, Lemma 15.2]:

**Proposition 2.** The Newton–Okounkov body $\text{NO}^{\text{rec}}_{G_{k,n}^{\text{rec}}}$ is unimodularly equivalent to the Gelfand–Tsetlin polytope $G_{k,n}^{\text{rec}}$.

We define the dual rec-plabic graph $(G_{k,n}^{\text{rec}})^\vee$ by swapping the black/white colour of the internal vertices, reversing the perfect orientation and changing the boundary labelling $r \rightarrow r + n - k \mod n$. The dual rec-plabic graph is a plabic graph of trip permutation $\pi_{k,n}$ with a perfect orientation. The face labelling in $(G_{k,n}^{\text{rec}})^\vee$ of a face $F$ in $G_{k,n}^{\text{rec}}$ is given by the complement:

$$\lambda_{(G_{k,n}^{\text{rec}})^\vee}(F) = (\lambda_{G_{k,n}^{\text{rec}}}(F))^\vee.$$

Notice that in $(G_{k,n}^{\text{rec}})^\vee$, for a boundary source $i$ and a boundary sink $j$, the flow from $i$ to $j$ of strongly minimal weight (we borrow the notion of strongly minimal from [10, Definition 5.13]) is given by a “vertical” path starting from $i$ followed by a “horizontal” path ending in $j$. We denote this path by $\gamma_{i,j}^{\text{min}}$ (see Fig. 2 for an example for $\gamma_{i,j}^{\text{min}}$).

**Proposition 3.** In the dual rec-plabic graph $(G_{k,n}^{\text{rec}})^\vee$, let $\{i_1 < \cdots < i_t\}$ be a subset of the sources and $\{j_1 > \cdots > j_t\}$ be a subset of the sinks. Let $J = \{i_1, \ldots, i_r, j_1, \ldots, j_t\}$. Then the unique flow $F(J)$ of strongly minimal weight is given by $\gamma_{i_1,j_1}^{\text{min}}, \ldots, \gamma_{i_r,j_r}^{\text{min}}$.  

Proof. Since the paths of strongly minimal weight do not intersect, the flow of minimal weight is given by the union of these paths. □

Theorem 3. The Newton–Okounkov body $NO_{(G_{k,n}^{	ext{rec}})}$ is unimodularly equivalent to the FFLV polytope FFLV$_{k,n}^1$.

Proof. We first set $Q = NO_{(G_{k,n}^{	ext{rec}})}^1$ and verify the conditions in Proposition 1 to show that $Q$ is unimodularly equivalent to FFLV$_{k,n}^1$ by a linear map.

The polytope $Q$ is a lattice polytope satisfying $\# S(Q) = \# S(\text{FFLV}_{k,n}^1)$ (the valuation images of the Plücker coordinates are different). Let $f_{i \times j} := v_{(G_{k,n}^{	ext{rec}})}(y_{i,j}^{\min})$. We define a linear map

$$\psi : \text{FFLV}_{k,n}^1 \rightarrow Q, \quad \xi_{p_{i,j}} \mapsto f_{i \times j}.$$

We label a basis on the right-hand side indexed by the faces of the plabic graph and a basis on the left-hand side indexed by the elements $p_{i,j}$. Using row operations, one can show straightforwardly, that the matrix of $\psi$ corresponding to these bases has determinant 1.

Since $\psi$ is linear, $NO_{(G_{k,n}^{	ext{rec}})}$ is unimodularly equivalent to FFLV$_{k,n}^1$. □

Remark 1. We set $(G_{k,n}^{	ext{rec}})_{n_{ij}}$ to be the plabic graph obtained from $G_{k,n}^{	ext{rec}}$ by replacing each $I = \{i_1, \ldots, i_{n-k}\}$ by $I_{w_{ij}} = \{n+1 - i_{n-k}, \ldots, n+1 - i_1\}$. This is nothing but applying a maximal Green sequence of mutations [6] to the cluster variables in $G_{k,n}^{	ext{rec}}$. Then one can show similarly to the theorem above, that the Newton–Okounkov body $NO_{(G_{k,n}^{	ext{rec}})}^1$ is unimodularly equivalent to FFLV$_{n-k,n}^1$.

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