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A characterizing property of commutative Banach algebras may not be sufficient only on the invertible elements

Une propriété caractérisant des algèbres de Banach commutatives qui ne peut être formulée sur les seuls éléments inversibles

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A B S T R A C T

In this article, we construct a commutative unital Banach algebra, in which the property $\|a^2\| = \|a\|^2$ is true for the invertible elements, but cannot be extended to the whole algebra.

R É S U M É

Dans cette Note, nous construisons une algèbre de Banach unitaire, commutative, dans laquelle l'identité $\|a^2\| = \|a\|^2$ est vraie pour les éléments inversibles, mais ne peut être étendue à toute l'algèbre.

1. Introduction

Let $(A, \|.\|)$ be a complex unital Banach algebra with unit 1. Let $G(A)$ and $Sing(A)$ denote the invertible and singular elements of A, respectively. The spectrum, the spectral radius, and the resolvent of an element a in A are denoted by $\sigma(a)$, $r(a)$, and $\rho(a)$, respectively. The fact that, in a Banach algebra, $r(a) = \|a\|$ if and only if $\|a^2\| = \|a\|^2$ for every $a \in A$ will be used throughout.

It is known that $G(A)$ is an open subset of A, as the open ball of radius $\frac{1}{\|a^{-1}\|}$ centered at any invertible element a, denoted by $B\left(a, \frac{1}{\|a^{-1}\|}\right)$, is contained in $G(A)$. Hence,

$$\frac{1}{\|a^{-1}\|} \leq \text{dist}(a, Sing(A)) \leq \frac{1}{r(a^{-1})}$$

for every $a \in G(A)$.

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Definition 1.1 *(Condition* (B)*).* An element $a \in G(A)$ is said to satisfy condition (B) if

$$\overline{B\left(a, \frac{1}{\|a^{-1}\|}\right)} \cap Sing(A) \neq \phi.$$

A Banach algebra A satisfies condition (B) if every $a \in G(A)$ satisfies condition (B).

Note that if $a \in G(A)$ satisfies condition (B), then the biggest open ball centered at a, contained in $G(A)$, is of radius $\frac{1}{\|a^{-1}\|}$. If $\|a^2\| = \|a\|^2$ for every invertible element $a \in A$, then A satisfies condition (B). C*-algebras also satisfy condition (B). For further examples and characterization of such algebras, see [4].

Let X be a compact Hausdorff space. Then the set of complex-valued, continuous functions on X, denoted by $C(X)$, with point wise multiplication and the uniform norm $\left(\|f\|_\infty = \sup_{x\in X} |f(x)|\right)$, is a commutative Banach algebra.

A *uniform algebra* U on a compact Hausdorff space X is a Banach subalgebra of $C(X)$ under the uniform norm, such that U separates the points of X and contains the constants. Note that $r(f) = \|f\|_\infty$ or equivalently $\|f^2\|_\infty = \|f\|_\infty^2$ for every element f in U. It is known that, if the spectral radius is a norm on a Banach algebra, then the Banach algebra must be commutative. This holds in particular if the spectral radius is equal to the original norm. Hence if $(A, \|.\|)$ is a unital Banach algebra such that $\|a^2\| = \|a\|^2$ for all $a \in A$, then there exists a compact Hausdorff space X such that A is *isometrically* isomorphic (via the Gelfand map) as a Banach algebra to a uniform algebra on X.

A Banach algebra may be isomorphic to a uniform algebra, but $r(a)$ may not be equal to $\|a\|$ or equivalently $\|a^2\| \neq \|a\|^2$, for some $a \in G(A)$, as seen in the following example.

Example 1.2. Let X be a locally compact Hausdorff space and $X^\infty = X \cup \{\infty\}$ denote the *one-point compactification* of X. Then X^∞ is a compact Hausdorff space and $(C(X^\infty), \|.\|_\infty)$ is a uniform algebra. Let $C_0(X)$ denote the vector space of all continuous functions on X that vanish at infinity. $C_0(X)$ is a Banach algebra with pointwise multiplication and the uniform norm. Since it is not unital, consider the standard *unitization* of $C_0(X)$, which is $C_0(X)^e := C_0(X) \times \mathbb{C}$ with multiplication

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu)$$

and norm defined as

$$\|(a, \lambda)\| = \|a\| + |\lambda|.$$

$(C_0(X)^e, \|.\|)$ is a Banach algebra with unit element $(0, 1)$. For $X = [1, \infty)$, consider the map $\psi : C_0([1, \infty))^e \to C([1, \infty)^\infty)$ defined by

$$\psi(f, \lambda) = f + \lambda e,$$

where $e(x) = 1$ for every $x \in [1, \infty)^\infty$, and each $f \in C_0([1, \infty))$ is extended by zero to $[1, \infty)^\infty$. We see that ψ is an isomorphism. But $\left\|\left(\frac{-1}{1+x^2}, 1\right)\right\|^2 \neq \left\|\left(\frac{-1}{1+x^2}, 1\right)^2\right\|$, where $\left(\frac{-1}{1+x^2}, 1\right)$ is invertible in $(C_0([1, \infty))^e, \|.\|)$, with the inverse $\left(\frac{1}{x^2}, 1\right)$.

In [4], Theorem 3, the authors proved that, if A is a commutative Banach algebra that satisfies condition (B), then A is isomorphic to a uniform algebra (via the Gelfand map) as $\|a\| \leq \exp(1)\, r(a)$ for every $a \in A$, with $r(a) = \|a\|$ for every $a \in G(A)$. In [4], Corollary 1, where the fact that invertible elements are dense in a finite dimensional Banach algebra was used, it is proved that if A is a finite-dimensional Banach algebra that satisfies condition (B), then A is commutative if and only if $\|a^2\| = \|a\|^2$ for every $a \in A$.

The authors raised a question that if Corollary 1 can be generalized to the following: let $(A, \|.\|)$ be a complex unital Banach algebra satisfying condition (B), is it true that A is commutative iff $\|a^2\| = \|a\|^2$ for every $a \in A$?

The answer to the above question is in the negative. We give an example of a commutative unital Banach algebra that satisfies condition (B) (and hence is isomorphic to a uniform algebra with $\|a^2\| = \|a\|^2$ for every $a \in G(A)$), but still there exists an $a \in A$ such that $\|a^2\| \neq \|a\|^2$. Note that the answer to the above question is positive if G(A) is dense in $(A, \|.\|)$. Regarding denseness, there is a complete characterization for commutative Banach algebra ([2] Proposition 3.1): let $(A, \|.\|)$ be a commutative Banach algebra, then A has dense invertible group if and only if the topological stable rank of A is one. Recently, Dawson and Feinstein [1] also investigated the condition that a complex commutative Banach algebra has dense invertible group.

2. Example

Example 2.1. Let $\bar{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ and $D = \{z \in \mathbb{C} : |z| < 1\}$. Let $A(\bar{D})$ be the set of those functions in $C(\bar{D})$ that are analytic on D. Consider $U = (A(\bar{D}), \|.\|_\infty)$, the disc algebra on $\bar{D}$, where $\|.\|_\infty$ denotes the uniform norm,

$$\|f\|_\infty = \sup\{|f(t)| : t \in \bar{D}\}$$

and multiplication is pointwise. U is a uniform algebra and $r(f) = \|f\|_\infty$ for every $f \in U$. Note that, in this case, $G(U)$ is not dense in U, as any element in $\overline{G(U)}$ is either identically zero or has all its zeros contained in the unit circle (see [3]).

For any $f \in U$, let

$$p(f) := \inf\left\{ \sum_{i=1}^{n} \|f_i\|_\infty : f = \sum_{k=1}^{n} f_i,\ f_i \in G(U) \right\}$$

where the infimum is taken over all representations of f as a finite combination of elements of $G(U)$. The set of such representations is non empty as $f = (f - \lambda 1) + \lambda 1$ for $f \in U$ and $\lambda \in \rho(f)$. Now $(A(\bar{D}), p(.))$ is a normed algebra with the following properties:

(1) $p(f) = \|f\|_\infty = r(f) \quad (f \in G(U))$
(2) $\|f\|_\infty \le p(f) \le 3\|f\|_\infty \quad (f \in A(\bar{D}))$. We have $p(f) \le 3\|f\|_\infty$, as every $0 \ne f \in A(\bar{D})$ can be expressed as $f + (1+\epsilon)\|f\|_\infty - (1+\epsilon)\|f\|_\infty$ for any $\epsilon > 0$.

Clearly, $(A(\bar{D}), p(.))$ is isomorphic to the disc algebra, but not isometrically isomorphic. That is, $\|f\|_\infty \ne p(f)$ for some $f \in A(\bar{D})$.

In fact, we have $p(f) \ge \frac{\mathrm{e}}{2}|f'(0)|$ for every f (where $\mathrm{e} = \exp(1)$).[1] To show this, consider a non-constant function $f \in G(U)$. Since $f(z) \ne 0$ for every $z \in \bar{D}$, there exists an analytic function $g : \bar{D} \longrightarrow \mathbb{C}$ such that $f = \mathrm{e}^g$. We can scale f such that $f(0) = 1$ (since we are estimating $\frac{\|f\|_\infty}{|f'(0)|}$). We may assume $g(0) = 0$. Therefore, $g(z) = \alpha z + \dots$, where $\alpha = f'(0)$. Now $\|f\|_\infty = \mathrm{e}^{\sup(\Re g)}$. Let $\beta = \sup(\Re g)$, and define the conformal map h from D onto the region $\Re z < \beta$ as $h(z) = \frac{2\beta z}{1+z}$. Note that $\beta > 0$ by the open mapping theorem. Applying Schwarz Lemma to $h^{-1} \circ g$, we get $|g'(0)| \le |h'(0)|$ i.e. $\frac{|\alpha|}{2} \le \beta$. Hence

$$\mathrm{e}^{\frac{|\alpha|}{2}} \le \mathrm{e}^{\beta} = \mathrm{e}^{\sup \Re g} = \|f\|_\infty .$$

Thus

$$\min_{|\alpha| \in \mathbb{R}} \frac{\mathrm{e}^{\frac{|\alpha|}{2}}}{|\alpha|} \le \min_{f \in G(U)} \frac{\|f\|_\infty}{|f'(0)|}.$$

Since $\min_{|\alpha| \in \mathbb{R}} \frac{\mathrm{e}^{\frac{|\alpha|}{2}}}{|\alpha|} = \frac{\mathrm{e}}{2}$, we have $\|f\|_\infty \ge \frac{\mathrm{e}}{2}|f'(0)|$ for every $f \in G(U)$.

Now let $f \in A(\bar{D})$ and let $f = \sum_{i=1}^{n} f_i$, where $f_i \in G(U)$. Then we have

$$\sum_{i=1}^{n} \|f_i\|_\infty \ge \frac{\mathrm{e}}{2} \sum_{i=1}^{n} |f_i{}'(0)| = \frac{\mathrm{e}}{2}|f'(0)|$$

and hence $p(f) \ge \frac{\mathrm{e}}{2}|f'(0)|$. Considering the function $f(z) = z$ for all $z \in \bar{D}$, we get $p(f) \ge \frac{\mathrm{e}}{2}|f'(0)| = \frac{\mathrm{e}}{2}\|f\|_\infty$, i.e. $p(f) \ne r(f)$.

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