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Differential geometry

Conformally flat real hypersurfaces in nonflat complex planes

Hypersurfaces réelles conformément plates dans les plans complexes non plats

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ABSTRACT

In this paper, we prove that there are no conformally flat real hypersurfaces in nonflat complex space forms of complex dimension two provided that the structure vector field is an eigenvector field of the Ricci operator. This extends some recent results by Cho (Conformally flat normal almost contact 3-manifolds, Honam Math. J. 38 (2016) 59–69) and Kon (3-dimensional real hypersurfaces with η-harmonic curvature, in: Hermitian–Grassmannian Submanifolds, Springer, Singapore, 2017, pp. 155–164).

RÉSUMÉ

Dans cette note, nous démontrons qu'il n'existe pas d'hypersurface réelle conformément plate dans les espaces de formes complexes de dimension deux, non plats, pourvu que le champ de vecteurs structurel soit champ de vecteur propre de l'opérateur de Ricci. Ceci étend des résultats récents de Cho (Conformally flat normal almost contact 3-manifolds, Honam Math. J. 38 (2016) 59–69) et Kron (3-dimensional real hypersurfaces with η-harmonic curvature, in : Hermitian–Grassmannian Submanifolds, Springer, Singapore, 2017, pp. 155–164).

1. Introduction

A complex n-dimensional Kählerian manifold with constant holomorphic sectional curvature c is said to be a *complex space form* and is denoted by $M^n(c)$. A complete and simply connected complex space form is complex analytically isometric to a complex projective space $\mathbb{C}P^n(c)$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $\mathbb{C}H^n(c)$ according to $c > 0$, $c = 0$ or $c < 0$, respectively. Let M be a real hypersurface in a complex space form $M^n(c)$, $c \neq 0$, whose Kähler metric and complex structure are denoted by $\overline{g}$ and J, respectively. On M there exists an *almost contact metric structure* (ϕ, ξ, η, g) induced from $\overline{g}$ and J (see Section 2), where ξ is called a *structure vector field*. Let D be the distribution determined by

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tangent vectors orthogonal to ξ at each point of M. Let A be the shape operator of M in $M^n(c)$. If the structure vector field ξ is *principal*, that is, $A\xi = \alpha\xi$, where $\alpha = \eta(A\xi)$, then M is called a *Hopf hypersurface* and α is called *Hopf principal curvature*.

Let us recall some known results regarding the Weyl conformal tensor on real hypersurfaces. The Riemannian curvature tensor R is harmonic (i.e. $\mathrm{div} R = 0$) if and only if the associated Ricci operator Q is of Codazzi type, i.e.

$$(\nabla_X Q)Y = (\nabla_Y Q)X$$

for any vector fields X, Y. The parallelism of the Ricci tensor implies naturally the harmonicity, but the converse is not necessarily true (see [5]).

Theorem 1.1 *([10,15]). There are no real hypersurfaces with harmonic curvature tensor in a nonflat complex space form $M^n(c)$, $n \geq 2$, on which ξ is principal.*

Theorem 1.1 extends Kimura [11, Theorem 2], who says that there are no real hypersurfaces in $\mathbb{C}P^n(c)$ with parallel Ricci tensor on which ξ is principal. Such conclusion is still true even when the condition "ξ is principal" is removed and the ambient space is generalized to any nonflat space from (see [6, Theorem A]). The curvature tensor is said to be η-harmonic if it satisfies $g((\nabla_X Q)Y - (\nabla_Y Q)X, Z) = 0$ for any vector fields X, Y and Z in D (see [7]). In fact, the η-harmonicity of the curvature tensor on a real hypersurface in complex planes implies η-parallelism of the Ricci tensor under some other restrictions (see Kon [14, Theorem 1]). We remark that the conclusion of Theorem 1.1 is still true if the condition "ξ is principal" is weakened to "ξ is an eigenvector field of the Ricci operator" for dimension three.

Theorem 1.2 *([14]). There are no real hypersurfaces with harmonic curvature tensor in a nonflat complex space form $M^2(c)$ of complex dimension two on which the Ricci operator Q satisfies $Q\xi = \beta\xi$, where β is a function.*

The Weyl conformal tensor W on a Riemannian manifold of dimension greater than three is harmonic (i.e. $\mathrm{div} W = 0$) if the associated Ricci operator satisfies $(\nabla_X Q)Y - (\nabla_Y Q)X = \frac{1}{2(n-1)}(X(r)Y - Y(r)X)$ for any vector fields X, Y, where r denotes the scalar curvature. Therefore, the harmonicity of the Riemannian curvature tensor can be viewed as a special case of that of the Weyl tensor. Such two notions are the same, under condition that the scalar curvature is a constant. Notice that there are Riemannian manifolds on which the Weyl tensor is harmonic but the curvature tensor is not harmonic (see [1]). We observe that Theorem 1.1 was generalized to the following one for dimensions greater than three.

Theorem 1.3 *([9]). There are no real hypersurfaces with harmonic Weyl tensor in a nonflat complex space form $M^n(c)$, $n \geq 3$.*

Generalizing Theorem 1.1, Ki, Kim and Nakagawa in [7] considered a weaker condition named η-harmonicity of the Weyl conformal tensor (i.e. $g((\nabla_X Q)Y - (\nabla_Y Q)X, Z) = \frac{1}{2(n-1)} g(X(r)Y - Y(r)X, Z)$ for any vector fields X, Y, Z orthogonal to the structure vector field ξ, where n is the dimension of the manifold). The authors in [7] also classified real hypersurfaces in a nonflat complex space form $M^n(c)$, $n \geq 3$, provided that ξ is principal and the Weyl tensor is η-harmonic.

Note that the Weyl tensor vanishes on a 3-dimensional Riemannian manifold M^3. Therefore, one always consider another conformal invariant, which is named the Cotton tensor and defined by

$$C(X, Y) = (\nabla_X Q)Y - (\nabla_Y Q)X - \frac{1}{4}\{X(r)Y - Y(r)X\} \tag{1.1}$$

for any vector fields X, Y on M^3. A Riemannian 3-manifold is conformally flat if and only if the Cotton tensor C vanishes identically. From Theorem 1.3, we know there are no conformally flat real hypersurfaces in a nonflat complex space form $M^n(c)$, $n \geq 3$. Except for the above result, conformally flat hypersurfaces of dimension greater than three in a conformally flat Riemannian manifold were investigated in [19]. However, as far as we know, the studies on conformal flatness on a three-dimensional real hypersurface in complex planes are few. In this paper, we study this problem and prove the following.

Theorem 1.4. *There are no conformally flat real hypersurfaces in nonflat complex space forms of complex dimension two provided that the structure vector field is an eigenvector field of the Ricci operator.*

The condition "ξ is an eigenvector field of the Ricci operator" is rather weak. Such condition was also studied by many authors in recent papers (for example, see [8], [12–14] and [16] and references therein).

Our main result extends naturally Theorems 1.1 and 1.2 in [10,14,15] and is a nice complement of Theorem 1.3 in [9] for dimension three.

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M^n(c)$ and N be a unit normal vector field of M. We denote by $\overline{\nabla}$ the Levi-Civita connection of the metric $\overline{g}$ of $M^n(c)$ and J the complex structure. Let g and ∇ be the induced metric from the ambient space and the Levi-Civita connection of g, respectively. Then the Gauss and Weingarten formulas are given respectively as follows:

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N,\ \overline{\nabla}_X N = -AX \tag{2.1}$$

for any vector fields X and Y tangent to M, where A denotes the shape operator of M in $M^n(c)$. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N,\ JN = -\xi. \tag{2.2}$$

We can define on M an almost contact metric structure (ϕ, ξ, η, g) satisfying

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi,\ \eta(\xi) = 1,\ \phi\xi = 0, \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\ \eta(X) = g(X, \xi) \tag{2.4}$$

for any vector fields X and Y on M. Moreover, applying the parallelism of the complex structure (i.e. $\overline{\nabla} J = 0$) of $M^n(c)$ and using (2.1), (2.2), we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.5}$$

$$\nabla_X \xi = \phi AX \tag{2.6}$$

for any vector fields X and Y. We denote by R the Riemannian curvature tensor of M. Since $M^n(c)$ is assumed to be of constant holomorphic sectional curvature c, then the Gauss and Codazzi equations of M in $M^n(c)$ are given respectively as follows:

$$\begin{aligned} R(X, Y)Z = \frac{c}{4}\{&g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &- 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.7}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\} \tag{2.8}$$

for any vector fields X, Y on M.

From (2.7) we see that the Ricci operator Q is given by

$$QX = \frac{c}{4}((2n+1)X - 3\eta(X)\xi) + mAX - A^2 X \tag{2.9}$$

for any vector field X tangent to the hypersurface, where $m := \mathrm{trace} A$ is the mean curvature.

In this paper, all manifolds are assumed to be connected and of class C^∞.

3. Conformally flat real hypersurfaces in $\mathbb{C}P^2$ and $\mathbb{C}H^2$

Let M be a real hypersurface in a complex space form $M^n(c)$. We put

$$A\xi = \alpha\xi + \beta U, \tag{3.1}$$

where $\alpha = \eta(A\xi)$, U a unit vector field orthogonal to ξ and β a smooth function. Applying (2.1) and (2.2), we see that $\beta U = -\phi\nabla_\xi \xi$. We put

$$\Omega = \{p \in M \mid \beta(p) \neq 0\}.$$

Then Ω is an open subset of M.

Lemma 3.1 *([21, Lemma 1]). Let M be a three-dimensional real hypersurface in a nonflat complex plane $M^2(c)$. Then the following relations hold:*

$$\begin{aligned} &AU = \gamma U + \delta\phi U + \beta\xi,\ A\phi U = \delta U + \mu\phi U, \\ &\nabla_U \xi = -\delta U + \gamma\phi U,\ \nabla_{\phi U}\xi = -\mu U + \delta\phi U,\ \nabla_\xi \xi = \beta\phi U, \\ &\nabla_U U = \kappa_1 \phi U + \delta\xi,\ \nabla_{\phi U} U = \kappa_2 \phi U + \mu\xi,\ \nabla_\xi U = \kappa_3 \phi U, \\ &\nabla_U \phi U = -\kappa_1 U - \gamma\xi,\ \nabla_{\phi U}\phi U = -\kappa_2 U - \delta\xi,\ \nabla_\xi \phi U = -\kappa_3 U - \beta\xi, \end{aligned} \tag{3.2}$$

where $\gamma, \delta, \mu, \kappa_i$, $i=\{1,2,3\}$, are smooth functions on M and $\{\xi, U, \phi U\}$ is an orthonormal basis of the tangent space of M at a point of M.

Applying this lemma, from the Codazzi equation (2.8) for $X=U$ or $X=\phi U$ and $Y=\xi$, we have

$$U(\beta)-\xi(\gamma)=\alpha\delta-2\delta\kappa_3. \tag{3.3}$$

$$\xi(\delta)=\alpha\gamma+\beta\kappa_1+\delta^2+\mu\kappa_3+\frac{c}{4}-\gamma\mu-\gamma\kappa_3-\beta^2. \tag{3.4}$$

$$U(\alpha)-\xi(\beta)=-3\beta\delta. \tag{3.5}$$

$$\xi(\mu)=\alpha\delta+\beta\kappa_2-2\delta\kappa_3. \tag{3.6}$$

$$\phi U(\alpha)=\alpha\beta+\beta\kappa_3-3\beta\mu. \tag{3.7}$$

$$\phi U(\beta)=\alpha\mu-2\gamma\mu+2\delta^2+\frac{c}{2}+\alpha\gamma+\beta\kappa_1. \tag{3.8}$$

Similarly, from the Codazzi equation for $X=U$ and $Y=\phi U$, we have

$$U(\delta)-\phi U(\gamma)=\mu\kappa_1-\gamma\kappa_1-\beta\gamma-2\delta\kappa_2-2\beta\mu. \tag{3.9}$$

$$U(\mu)-\phi U(\delta)=\gamma\kappa_2+\beta\delta-\kappa_2\mu-2\delta\kappa_1. \tag{3.10}$$

Moreover, applying again Lemma 3.1, from the Gauss equation (2.7) and the definition of the Riemannian curvature tensor $R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z$, we have

$$U(\kappa_2)-\phi U(\kappa_1)=2\delta^2-2\gamma\mu-\kappa_1^2-\gamma\kappa_3-\kappa_2^2-\mu\kappa_3-c. \tag{3.11}$$

$$\phi U(\kappa_3)-\xi(\kappa_2)=2\beta\mu-\mu\kappa_1+\delta\kappa_2+\kappa_3\kappa_1+\beta\kappa_3. \tag{3.12}$$

The above relations can also be seen in [20,21].

Lemma 3.2. *Let M be a real hypersurface in a nonflat complex plane such that ξ is an eigenvector field of the Ricci operator. If the Cotton tensor vanishes, then ξ is principal.*

Proof. All we need to do is prove that Ω is empty under the above hypotheses. The applications of (3.1) and (3.2) in (2.9) give

$$Q\xi=\left(\frac{1}{2}c+\alpha(\gamma+\mu)-\beta^2\right)\xi+\beta\mu U-\beta\delta\phi U.$$

Next we suppose that Ω is nonempty. It follows that $\mu=\delta=0$ because of $\beta\neq 0$ and the assumption that ξ is an eigenvector field of the Ricci operator. Now, (2.9) can be expressed by

$$Q\xi=\left(\frac{1}{2}r-c\right)\xi,\ QU=\left(\frac{1}{2}r-\frac{1}{4}c\right)U,\ Q\phi U=\frac{5}{4}c\phi U \tag{3.13}$$

with respect to an orthonormal basis $\{\xi, U, \phi U\}$, where r is the scalar curvature. Actually, by (3.1) and (3.2), we have $r=3c+2\alpha\gamma-2\beta^2$. The application of (3.2) and (3.13) in (1.1) implies

$$C(\xi,U)=-\frac{1}{4}U(r)\xi+\frac{1}{4}\xi(r)U+\left(\frac{1}{2}r\kappa_3-\frac{3}{2}c\kappa_3-\frac{1}{2}r\gamma+\frac{9}{4}c\gamma\right)\phi U. \tag{3.14}$$

$$C(\xi,\phi U)=\left(\frac{1}{2}r\beta-\frac{9}{4}\beta c-\frac{1}{4}\phi U(r)\right)\xi+\left(\frac{1}{2}r-\frac{3}{2}c\right)\kappa_3 U-\frac{1}{4}\xi(r)\phi U. \tag{3.15}$$

$$\begin{aligned}C(U,\phi U)=&\left(\frac{1}{2}r-\frac{9}{4}c\right)\gamma\xi+\left(\frac{1}{2}r\kappa_1-\frac{3}{2}c\kappa_1-\frac{1}{4}\phi U(r)\right)U\\&+\left(\frac{3}{2}c\kappa_2-\frac{1}{2}r\kappa_2-\frac{1}{4}U(r)\right)\phi U.\end{aligned} \tag{3.16}$$

In view of $C=0$, from (3.15) we acquire $(r-3c)\kappa_3=0$. If $\kappa_3\neq 0$ holds on some open subset of Ω. On this subset, we have $r=3c$, a constant, and hence the vanishing of the Cotton tensor C implies that the Ricci operator is of Codazzi type. However, this is impossible, because Kon [14, Theorem 2] proved that there are no real hypersurfaces in complex planes with harmonic curvature tensor and ξ an eigenvector field of the Ricci operator. Therefore, it follows that $\kappa_3=0$, which is combined with (3.14) and $C=0$ yielding $\gamma=0$ or $2r=9c$. Obviously, the latter case can not occur as discussed before.

Applying again $C = 0$ on (3.14), we obtain $\xi(r) = U(r) = 0$. Similarly, from (3.16) and $C = 0$, we have $6c\kappa_2 - 2r\kappa_2 - U(r) = 0$. Because r cannot be a constant, it follows directly that $\kappa_2 = 0$.

Based on the above analyses, (3.4), (3.8) and (3.11) become

$$\beta\kappa_1 + \frac{1}{4}c = \beta^2,\ \phi U(\beta) = \frac{1}{2}c + \beta\kappa_1 \text{ and } \phi U(\kappa_1) = \kappa_1^2 + c \tag{3.17}$$

respectively. By virtue of $C = 0$, from (3.15), we also have $2r\beta - 9\beta c - \phi U(r) = 0$. Recall that now the scalar curvature is given by $r = 3c - 2\beta^2$. Making use of this in the previous relation, together with the second term of (3.17), we obtain

$$\beta(4\beta\kappa_1 - c - 4\beta^2) = 0.$$

Because of $\beta \neq 0$, it follows directly that $4\beta\kappa_1 - c - 4\beta^2 = 0$ and, by comparing this with the first term of (3.17), we obtain $c = 0$, a contradiction. This means that M is a Hopf hypersurface. □

Now we are ready to present the proof of our main result.

Proof of Theorem 1.4. According to Lemma 3.2, $\beta = 0$, the applications of (3.1) and (3.2) in (2.9) give

$$\begin{aligned} Q\xi &= \left(\frac{1}{2}c + \alpha(\gamma+\mu)\right)\xi, \\ QU &= \left(\frac{1}{2}r - \frac{1}{4}c - \alpha\mu\right)U + \alpha\delta\phi U, \\ Q\phi U &= \alpha\delta U + \left(\frac{1}{2}r - \frac{1}{4}c - \alpha\gamma\right)\phi U, \end{aligned} \tag{3.18}$$

where the scalar curvature is given by $r = 3c + 2(\alpha\gamma + \alpha\mu + \gamma\mu - \delta^2)$. Making use of $\beta = 0$ in (3.3) and (3.6), we have

$$\xi(\gamma) = \delta(2\kappa_3 - \alpha) \text{ and } \xi(\mu) = \delta(\alpha - 2\kappa_3), \tag{3.19}$$

respectively. Moreover, it is known that α on a Hopf real hypersurface is always a constant (see [17, Lemma 2.4] or [18, Theorem 2.1]). Thus, the applications of (3.18) and (3.19) in (1.1), together with Lemma 3.1 and $\beta = 0$, imply the following three equations.

$$\begin{aligned} C(\xi, U) = &\left(\frac{1}{4}U(r) - \alpha U(\gamma+\mu)\right)\xi \\ &+ \left(\frac{1}{4}\xi(r - 4\alpha\mu) + \delta\left(2\alpha\gamma + 2\alpha\mu - 2\alpha\kappa_3 - \frac{1}{2}r + \frac{3}{4}c\right)\right)U \\ &+ \left(\alpha\kappa_3(\gamma-\mu) + \alpha\xi(\delta) - \alpha\delta^2 + \gamma\left(\frac{1}{2}r - 2\alpha\gamma - \alpha\mu - \frac{3}{4}c\right)\right)\phi U. \\ C(\xi, \phi U) = &\left(\frac{1}{4}\phi U(r) - \alpha\phi U(\gamma+\mu)\right)\xi \\ &+ \left(\alpha\kappa_3(\gamma-\mu) + \alpha\xi(\delta) + \alpha\delta^2 + \mu\left(\frac{3}{4}c + \alpha\gamma + 2\alpha\mu - \frac{1}{2}r\right)\right)U \\ &+ \left(\frac{1}{4}\xi(r - 4\alpha\gamma) + \delta\left(2\alpha\kappa_3 - 2\alpha\gamma - 2\alpha\mu + \frac{1}{2}r - \frac{3}{4}c\right)\right)\phi U. \\ C(U, \phi U) = &\left((\gamma+\mu)\left(\frac{3}{4}c - \frac{1}{2}r\right) + 2\alpha(\delta^2 + \mu^2 + \gamma^2 + \mu\gamma)\right)\xi \\ &+ \left(\alpha\kappa_1(\gamma-\mu) + 2\alpha\delta\kappa_2 - \frac{1}{4}\phi U(r - 4\alpha\mu) + \alpha U(\delta)\right)U \\ &+ \left(\alpha\kappa_2(\mu-\gamma) + 2\alpha\delta\kappa_1 + \frac{1}{4}U(r - 4\alpha\gamma) - \alpha\phi U(\delta)\right)\phi U. \end{aligned}$$

If M is conformally flat, from the above three relations, we have

$$\begin{aligned} &U(r) - 4\alpha U(\gamma+\mu) = 0, \\ &\xi(r - 4\alpha\mu) + \delta(3c + 8\alpha\gamma + 8\alpha\mu - 8\alpha\kappa_3 - 2r) = 0, \\ &4\alpha\kappa_3(\gamma-\mu) + 4\alpha\xi(\delta) - 4\alpha\delta^2 + \gamma(2r - 3c - 8\alpha\gamma - 4\alpha\mu) = 0, \end{aligned} \tag{3.20}$$

$$\begin{aligned} &\phi U(r) - 4\alpha\phi U(\gamma+\mu) = 0,\\ &4\alpha\kappa_3(\gamma-\mu) + 4\alpha\xi(\delta) + 4\alpha\delta^2 + \mu(3c - 2r + 8\alpha\mu + 4\alpha\gamma) = 0,\\ &\xi(r - 4\alpha\gamma) + \delta(8\alpha\kappa_3 - 3c - 8\alpha\gamma - 8\alpha\mu + 2r) = 0, \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} &(\gamma+\mu)(3c-2r) + 8\alpha(\delta^2+\mu^2+\gamma^2+\gamma\mu) = 0,\\ &4\alpha\kappa_1(\gamma-\mu) + 8\alpha\delta\kappa_2 - \phi U(r - 4\alpha\mu) + 4\alpha U(\delta) = 0,\\ &4\alpha\kappa_2(\mu-\gamma) + 8\alpha\delta\kappa_1 + U(r - 4\alpha\gamma) - 4\phi U(\delta) = 0. \end{aligned} \tag{3.22}$$

The addition of the second term of (3.20) to the third term of (3.21) implies $\xi(r) = 0$, where we have applied (3.19) and the fact that α is a constant. In this context, using the first term of (3.19) in the third term of (3.21), we acquire

$$\delta(2r - 3c + 4\alpha^2 - 8\alpha(\gamma+\mu)) = 0. \tag{3.23}$$

In view of (3.23), next we first consider the case $2r = 3c - 4\alpha^2 + 8\alpha(\gamma+\mu)$. Recall that the scalar curvature is given by $r = 3c + 2(\alpha\gamma + \alpha\mu + \gamma\mu - \delta^2)$ with the aid of $\beta = 0$. It follows directly from the previous two equations that

$$4\alpha\gamma + 4\alpha\mu - 4\gamma\mu - 4\alpha^2 + 4\delta^2 - 3c = 0.$$

On the other hand, making use of $\beta = 0$ in (3.8), we get

$$\alpha(\gamma+\mu) + \frac{1}{2}c - 2\gamma\mu + 2\delta^2 = 0.$$

Subtracting the last equation multiple of two from the previous one, we obtain $\alpha(\gamma+\mu) = 2c + 2\alpha^2$. Now, the scalar curvature becomes $r = \frac{19}{2}c + 6\alpha^2$, which is a constant. Consequently, the conformal flatness of M means that the curvature tensor is harmonic. However, this is impossible because of Theorem 1.2, and we conclude that (3.23) implies only one case, i.e. $\delta = 0$.

The application of $\delta = 0$ in the third term of (3.20) and the second term of (3.21) give

$$4\alpha\kappa_3(\gamma-\mu) + \gamma(2r - 3c - 8\alpha\gamma - 4\alpha\mu) = 0,$$

and

$$4\alpha\kappa_3(\gamma-\mu) + \mu(3c + 4\alpha\gamma + 8\alpha\mu - 2r) = 0,$$

respectively. The addition of one of the above two equations to the other one gives

$$(\gamma-\mu)(2r - 3c + 8\alpha(\kappa_3 - \gamma - \mu)) = 0. \tag{3.24}$$

If $\gamma = \mu$, using $\delta = \beta = 0$ in (3.8), we have $4\alpha\mu - 4\mu^2 + c = 0$. It follows that either μ does not exist or it is a constant. For the latter case, we observe that the scalar curvature $r = 3c + 2(\alpha\gamma + \alpha\mu + \gamma\mu)$ is still a constant, which is a contradiction.

Finally, by $\gamma \neq \mu$, it follows from (3.24) that

$$r = \frac{3}{2}c - 4\alpha\kappa_3 + 4\alpha(\gamma+\mu). \tag{3.25}$$

Taking differentiation of (3.25) along U, together with the first term of (3.20), we obtain $U(\kappa_3) = 0$, where we have applied the fact that α is a nonzero constant. Actually, if $\alpha = 0$, by (3.25), the scalar curvature is a constant, which is a contradiction. Similarly, taking differentiation of (3.25) along ϕU, together with the first term of (3.21), we obtain $\phi U(\kappa_3) = 0$. Moreover, with the aid of (3.19) and $\xi(r) = 0$, taking differentiation of (3.25) along ξ, we obtain $\xi(\kappa_3) = 0$. This means that κ_3 is a constant. By means of $\beta = \delta = 0$, now (3.8) becomes

$$\alpha(\gamma+\mu) + \frac{1}{2}c - 2\gamma\mu = 0. \tag{3.26}$$

Recall that the scalar curvature is given by $r = 3c + 2(\alpha\gamma + \alpha\mu + \gamma\mu)$. Comparing this with (3.25) gives $4\alpha(\gamma+\mu) = 8\alpha\kappa_3 + 3c + 4\gamma\mu$. Obviously, substituting this relation in (3.26), we acquire $4\gamma\mu = 8\alpha\kappa_3 + 5c$, a constant. Thus, it follows from (3.26) that $\gamma + \mu$ is a constant and hence by (3.25) the scalar curvature r is also a constant, which is a contradiction. This completes the proof. □

Remark 3.1. Let M be an almost contact metric manifold of dimension three such that the Reeb vector field ξ is an eigenvector field of the Ricci operator. M can be conformally flat if it is a contact metric manifold (see [2]), an almost Kenmotsu manifold (see [22]) or an almost coKähler manifold (see [4,23]). What is interesting is that, however, by Theorem 1.4, as an almost contact metric manifold, a real hypersurface in $\mathbb{C}P^2$ or $\mathbb{C}H^2$ with ξ an eigenvector field of the Ricci operator cannot be conformally flat.

Remark 3.2. A real hypersurface in a nonflat complex space form is said to be *totally η-umbilical* if the shape operator is given by $A = a\mathrm{id} + b\eta \otimes \xi$ for some constants a, b and id denotes the identity transformation. Cho in [3, Proposition 5.6] proved that a totally η-umbilical real hypersurface in a nonflat complex space form of complex dimension two does not admit conformally flat structure. Our Theorem 1.4 is an extension of the above result because the total η-umbilication of the shape operator A implies that ξ is an eigenvector field of the Ricci operator.

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