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Partial differential equations

An L^p-theory for almost sure local well-posedness of the nonlinear Schrödinger equations

Une théorie L^p pour le problème de Cauchy de l'équation de Schrödinger non linéaire à données initiales aléatoires

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A R T I C L E I N F O

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A B S T R A C T

We consider the nonlinear Schrödinger equations (NLS) on $\mathbb{R}^d$ with random and rough initial data. By working in the framework of $L^p(\mathbb{R}^d)$ spaces, $p > 2$, we prove almost sure local well-posedness for rougher initial data than those considered in the existing literature. The main ingredient of the proof is the dispersive estimate.

R É S U M É

Dans cet article, nous considérons l'équation de Schrödinger non linéaire (NLS) sur $\mathbb{R}^d$ à données initiales aléatoires et surcritiques. En travaillant dans des espaces de $L^p(\mathbb{R}^d)$, $p > 2$, nous améliorons les résultats précédents de la littérature, en ce sens que nous prouvons que l'équation NLS est localement bien posée presque sûrement pour des données initiales à régularité plus basse. L'ingrédient principal de la preuve est l'estimation dispersive.

1. Introduction

We consider the nonlinear Schrödinger equation (NLS) with power-type nonlinearity on $\mathbb{R}^d$, $d \geq 1$:

$$\begin{cases} i\partial_t u + \Delta u = \pm |u|^{p-1} u \\ u|_{t=0} = u_0 \in H^s(\mathbb{R}^d), \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \tag{1.1}$$

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where $p > 1$. The equation (1.1) appears as a standard model in various physical contexts and has been studied extensively over the past decades. In this note, we consider the Cauchy problem (1.1) with random and rough initial data. In particular, by working in the framework of $L^p(\mathbb{R}^d)$, $p > 2$, and using the dispersive estimate, we establish almost sure local well-posedness of (1.1) with respect to random initial data of lower regularity than those considered in the existing results in the literature.

NLS (1.1) arises as a Hamiltonian evolution associated with energy

$$E(u) := \int_{\mathbb{R}^d} \frac{1}{2}|\nabla u|^2 \pm \frac{1}{p+1}|u|^{p+1}\mathrm{d}x. \tag{1.2}$$

In particular, $E(u)$ is conserved by the flow of NLS. The solution set to (1.1) possesses the following scaling symmetry:

$$u_\lambda(t,x) := \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x). \tag{1.3}$$

Note that $\|u_\lambda(0)\|_{\dot{H}^s(\mathbb{R}^d)} = \lambda^{s-(\frac{d}{2}-\frac{2}{p-1})}\|u_0\|_{\dot{H}^s(\mathbb{R}^d)}$. Associated with the scaling symmetry, one defines the so-called scaling-critical Sobolev index $s_{\text{crit}}(d,p) := \frac{d}{2} - \frac{2}{p-1}$ such that the homogeneous Sobolev norm $\|\cdot\|_{\dot{H}^{s_{\text{crit}}}(\mathbb{R}^d)}$ remains invariant under the scaling symmetry (1.3). We then say that the Cauchy problem (1.1) with an initial condition $u_0 \in H^s(\mathbb{R}^d)$ is subcritical, critical, or supercritical, depending on whether $s > s_{\text{crit}}(d,p)$, $s = s_{\text{crit}}(d,p)$, or $s < s_{\text{crit}}(d,p)$, respectively. When d and p are such that $s_{\text{crit}}(d,p) = 1$, the scaling symmetry (1.3) also leaves the energy $E(u)$ invariant, and in that case we say that (1.1) is energy-critical. We say that the Cauchy problem (1.1) is energy-subcritical or energy-supercritical, if $s_{\text{crit}}(d,p) < 1$ or $s_{\text{crit}}(d,p) > 1$, respectively.

In the deterministic setting, NLS (1.1) is known to be locally well-posed in the (sub)critical regime, see [17,7,9]. On the contrary, in the supercritical regime, it is known to be ill-posed, see for example [8,13]. In the last decade, a non-deterministic view point has been used, aiming to improve our understanding of NLS. It consists in studying the Cauchy problem (1.1) with random initial data. In this probabilistic setting, NLS is almost surely locally well-posed, even in certain supercritical regimes. See [1,2,4,3,15,11].

As in [19,12,1], in this note, we consider a randomization associated with the Wiener decomposition $\mathbb{R}^d_\xi = \bigcup_{n\in\mathbb{Z}^d}(n + (-\frac{1}{2},\frac{1}{2}]^d)$. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ satisfying

$$\operatorname{supp}\psi \subset [-1,1]^d \qquad \text{and} \qquad \sum_{n\in\mathbb{Z}^d}\psi(\xi - n) = 1 \quad \text{for any } \xi \in \mathbb{R}^d.$$

Given a function ϕ on $\mathbb{R}^d$, we have $\phi = \sum_{n\in\mathbb{Z}^d}\psi(D-n)\phi$. We then define the Wiener randomization of ϕ by

$$\phi^\omega := \sum_{n\in\mathbb{Z}^d} g_n(\omega)\psi(D-n)\phi, \tag{1.4}$$

where $\{g_n\}_{n\in\mathbb{Z}^d}$ is a sequence of independent mean zero complex-valued random variables on a probability space $(\Omega, \mathcal{F}, P)$. In the following, we assume that the real and imaginary parts of g_n are independent and endowed with probability distributions $\mu_n^{(1)}$ and $\mu_n^{(2)}$, satisfying the following exponential moment bound:

$$\left|\int_{\mathbb{R}} e^{\kappa x}\mathrm{d}\mu_n^{(j)}(x)\right| \le e^{c\kappa^2} \tag{1.5}$$

for all $\kappa \in \mathbb{R}$, $n \in \mathbb{Z}^d$, $j = 1, 2$. This condition is satisfied by the standard complex-valued Gaussian random variables and by the uniform distribution on the unit circle.

It is well known that the Wiener randomization (1.4) does not improve differentiability; see Lemma B.1 in [5]. However, its key advantage is improving integrability; see Lemma 2.3 in [1] and Lemma 2.2 below.

Given $d \ge 1$ and $p > 1$, we define $s_{d,p}$ by

(i) $s_{d,p} = 0$, if $p > 1 + \frac{4}{d}$, $d = 1, 2$ or $1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}$, $d \ge 3$,
(ii) $s_{d,p} = s_{\text{crit}}(d,p) - 1+$, if $p \ge 1 + \frac{4}{d-2}$, $d \ge 3$.

Note that we have $0 \le s_{d,p} < s_{\text{crit}}(d,p)$. Also, remark that (i) corresponds to the energy-subcritical case, while (ii) corresponds to the energy-(super)critical case.

In this note, given $\phi \in H^s(\mathbb{R}^d)$, $s_{d,p} \le s < s_{\text{crit}}(d,p)$, we study the Cauchy problem (1.1) with the random initial data ϕ^ω. We state our main result below.

Theorem 1.1 *(Almost sure local well-posedness). Given $d \ge 1$ and p an odd integer such that $p > 1 + \frac{4}{d}$, let $s_{d,p}$ be defined as above. Given $\phi \in H^s(\mathbb{R}^d)$ with $s_{d,p} \le s < s_{\text{crit}}(d,p)$, let ϕ^ω be its Wiener randomization defined in (1.4), satisfying (1.5). Then, (1.1) is* *almost surely locally well-posed with respect to the random initial data ϕ^ω. More precisely, there exist $C, c, \gamma > 0$ such that for each $0 < T \ll 1$, there exists a set $\Omega_T \subset \Omega$ with the following properties:*

(a1) $P(\Omega_T^c) < C \exp\big(-\frac{c}{T^\gamma \|\phi\|_{H^s}^2}\big)$,

(b1) *for each $\omega \in \Omega_T$, there exists a unique solution u^ω to (1.1) with $u^\omega|_{t=0} = \phi^\omega$ in the class*[1]

$$S(t)\phi^\omega + C([-T, T]; W^{s, r_{d,p}+1}(\mathbb{R}^d)),$$

where $r_{d,p}$ is defined by

$$r_{d,p} := \begin{cases} p, & \text{if} \quad p > 1 + \frac{4}{d}, \quad d = 1, 2 \quad \text{or} \quad 1 + \frac{4}{d} < p < 1 + \frac{4}{d-2}, \quad d \geq 3 \\ 1 + \frac{4}{d-2}-, & \text{if} \quad p \geq 1 + \frac{4}{d-2}, \quad d \geq 3. \end{cases}$$

Furthermore, for p non-integer in the energy-subcritical case (i)*, we also have almost sure local well-posedness of* (1.1) *with initial condition ϕ^ω, where $\phi \in H^s(\mathbb{R}^d)$, $s \in [0, s_{\text{crit}}(d, p))$, in the following sense. There exist $C, c, \gamma > 0$ such that for each $0 < T \ll 1$, there exists a set $\Omega'_T \subset \Omega$ with the following properties:*

(a2) $P((\Omega'_T)^c) < C \exp\big(-\frac{c}{T^\gamma \|\phi\|_{L^2}^2}\big)$,

(b2) *for each $\omega \in \Omega'_T$, there exists a unique solution u to* (1.1) *with $u|_{t=0} = \phi^\omega$ in the class*

$$S(t)\phi^\omega + C([-T, T]; L^{p+1}(\mathbb{R}^d)).$$

Here $S(t) = e^{it\Delta}$ denotes the linear propagator of the Schrödinger group. Let $z(t) = z^\omega(t) := S(t)\phi^\omega$ be the random linear solution with ϕ^ω as initial data. We reduce our analysis on (1.1) to the following Cauchy problem satisfied by the nonlinear part $v := u - z$ of a solution u:

$$\begin{cases} i\partial_t + \Delta v = \mathcal{N}(v + z^\omega) \\ v|_{t=0} = 0, \end{cases} \tag{1.6}$$

where $\mathcal{N}(u) := \pm|u|^{p-1}u$. In the Duhamel formulation, we have

$$v(t) = -i \int_0^t S(t - t')\mathcal{N}(v + z^\omega)(t')dt'. \tag{1.7}$$

The proof of Theorem 1.1 is based on a fixed-point argument for v. As a result, the uniqueness in Theorem 1.1 refers to uniqueness of the nonlinear part v of a solution u. The main idea of the proof is to exploit the improved integrability of the random linear solution z (see the probabilistic Strichartz estimates in Lemma 2.2) by working in the L^p-based Sobolev spaces, $p > 2$, (as opposed to L^2-based Sobolev spaces H^s) and by using the dispersive estimates (Lemma 2.1).

In recent years, there have been several results in the literature on almost sure local well-posedness of NLS with random initial data. In [1,2], the first author with Bényi and Oh considered the cubic NLS on $\mathbb{R}^d$, $d \geq 3$, with random initial data ϕ^ω defined as in (1.4). They proved almost sure local well-posedness of (1.1) with $p = 3$, for $\phi \in H^s(\mathbb{R}^d)$, $s_{\text{crit}} - 1 + \frac{3}{d+1} < s < s_{\text{crit}}$. In [4], Brereton considered the analogous problem for the quintic NLS on $\mathbb{R}^d$, $d \geq 3$, and proved almost sure local well-posedness for $\phi \in H^s(\mathbb{R}^d)$, $s_{\text{crit}} - \frac{1}{2} < s < s_{\text{crit}}$. In [15], the first author, with Oh and Okamoto, considered the energy-critical NLS on $\mathbb{R}^d$, $d = 5, 6$, and proved almost sure local well-posedness for $\phi \in H^s(\mathbb{R}^d)$, $1 - \frac{1}{d} < s < 1$. More recently, the first author with Bényi and Oh [3] proved almost sure local well-posedness of the cubic NLS on $\mathbb{R}^3$ based on a fixed point argument around a (modified) partial power series expansion, thus improving previous results in [2]. Theorem 1.1 is an improvement of all these results, in the sense that we are able to lower the regularity threshold for initial data that yield solutions to (1.1) almost surely.

Remark 1.2. All the above-mentioned results are based on the L^2-theory. This is not the case of Theorem 1.1 (in particular, of Corollary 2.5 where we use the more precise $\rho_{d,p}$ and $\sigma_{d,p}$, rather than $r_{d,p}$ and $s_{d,p}$), where v is constructed in $C([-T, T]; W^{\sigma, \rho_{d,p}+1}(\mathbb{R}^d))$, $\rho_{d,p} + 1 > 2$. For $\sigma \geq \sigma_{d,p}$, the space $W^{\sigma, \rho_{d,p}+1}(\mathbb{R}^d)$ is subcritical with respect to the scaling (1.3). In the above-cited results, v was also constructed in critical or subcritical L^2-based Sobolev spaces: $v \in H^\sigma(\mathbb{R}^d)$, $\sigma \geq s_{\text{crit}}(d, p)$. We remark that this required a gain of $\sigma - s$ derivatives for v, since the initial data is only in $H^s(\mathbb{R}^d)$,

[1] Arguing as in [14], one can easily choose Ω_T such that for each $\omega \in \Omega_T$ we have $u^\omega \in S(t)\phi^\omega + C([-T, T]; W^{s, r_{d,p}+1}(\mathbb{R}^d)) \cap C([-T, T]; H^s(\mathbb{R}^d))$. In particular, the solution u^ω constructed in Theorem 1.1 belongs to the L^2-based Sobolev space $C([-T, T]; H^s(\mathbb{R}^d))$.

$s < s_{\text{crit}}(d,p)$. Such a gain of derivatives was exhibited by using a case-by-case analysis and a bilinear refinement of the Strichartz estimates. In Theorem 1.1, v has at most the same differentiability as the initial data. In particular, a gain of derivatives is not needed and the analysis is much simpler. This is an advantage of working within the L^p-framework, $p > 2$, as opposed to the L^2-theory: it gives a more direct access to the gain of integrability given by the randomization.

Remark 1.3. In the deterministic setting, one cannot expect to obtain the local well-posedness of NLS in $C([-T,T]; W^{s,p}(\mathbb{R}^d))$, $p \neq 2$, because the linear propagator $S(t)$ is not bounded on $L^p(\mathbb{R}^d)$ for $p \neq 2$. We point out, however, the work of Zhou [20] in which he proposes an alternative notion of a solution (based on the interaction representation) and, in this new formulation, obtains the local well-posedness of the cubic NLS in $C([-T,T]; W^{s,p}(\mathbb{R}^d))$ for $p < 2$ (under additional restrictions on s and p).

Remark 1.4. For simplicity, we only stated the first part of Theorem 1.1 for p being an odd integer. In this case, the nonlinearity is algebraic, and we apply the fractional Leibnitz rule in estimating $\langle\nabla\rangle^s\mathcal{N}(v+z)$, $s > 0$. When p is not an odd integer, the analysis becomes more cumbersome and we prefer not to go into details. In particular, there are further restrictions on the pairs (d,p) for which one can obtain a result similar to that in Theorem 1.1. For instance, if we assume that d is arbitrarily large, while p is fixed, then $s_{\text{crit}}(d,p)$ is also arbitrarily large. As seen in Theorem 1.1, with the technique in this note, we can only hope to go down to $s > s_{\text{crit}}(d,p) - 1$, which is still very large. Then, in order to estimate $\langle\nabla\rangle^s\mathcal{N}(v+z)$, the nonlinearity $\mathcal{N}(u) = |u|^{p-1}u$ needs to be very smooth, which is not the case unless p is sufficiently large. We refer the readers to [18,15] for the study of NLS with non-algebraic nonlinearities.

In view of the time reversibility of NLS, we only consider positive times in the following.

2. Proof of Theorem 1.1

In this section, we prove the main result of the paper, Theorem 1.1. The main two tools are the dispersive estimate (that we recall for readers' convenience below, in Lemma 2.1) and the probabilistic Strichartz estimates (Lemma 2.2).

Lemma 2.1. *Let $p \in [2,\infty]$ and p' such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then, there exists $C > 0$ such that the following estimate holds*

$$\|S(t)\phi\|_{L^p_x(\mathbb{R}^d)} \le \frac{C}{|t|^{\frac{d}{2}-\frac{d}{p}}}\|\phi\|_{L^{p'}_x(\mathbb{R}^d)}. \tag{2.1}$$

Next, we recall some probabilistic Strichartz estimates. See [1,2] for the proofs.

Lemma 2.2 *([1,2]). Given $\phi \in L^2(\mathbb{R}^d)$, let ϕ^ω be its Wiener randomization defined in (1.4), satisfying (1.5). Then, given finite $q, r \ge 2$, there exist $C, c > 0$ such that*

$$P\Big(\|S(t)\phi^\omega\|_{L^q_t([0,T];L^r_x(\mathbb{R}^d))} > \lambda\Big) \le C\exp\left(-c\frac{\lambda^2}{T^{\frac{2}{q}}\|\phi\|^2_{L^2}}\right)$$

for all $T > 0$ and $\lambda > 0$.

In the following, we use the dispersive estimates and the probabilistic Strichartz inequalities in Lemmas 2.1 and 2.2 to prove the key nonlinear estimates needed to establish Theorem 1.1. Given $z(t) = S(t)\phi^\omega$, we define Γ by

$$\Gamma v(t) := -\mathrm{i}\int_0^t S(t-t')\mathcal{N}(v+z)(t')\mathrm{d}t'. \tag{2.2}$$

Proposition 2.3. *Given $d \ge 1$ and $p \ge 3$ an odd integer, let $\rho \in (1,p]$ for $d = 1,2$, while $\rho \in \big(1, 1+\frac{4}{d-2}\big)$ for $d \ge 3$. Let $\sigma \ge \sigma(d,p,\rho) := \frac{d(p-\rho)}{(\rho+1)(p-1)}$ and $0 < T \le 1$. Given $\phi \in H^\sigma(\mathbb{R}^d)$, let ϕ^ω be its Wiener randomization defined in (1.4), satisfying (1.5). Then $\theta := \frac{d}{2} - \frac{d}{\rho+1} \in (0,1)$, and for any $0 < \varepsilon < 1-\theta$, there exists $C_1, C_2 > 0$ such that the following estimates hold*

$$\|\Gamma v\|_{L^\infty([0,T];W^{\sigma,\rho+1}(\mathbb{R}^d))} \le C_1 T^{1-\theta-\varepsilon}\big(\|v\|^p_{L^\infty([0,T];W^{\sigma,\rho+1}(\mathbb{R}^d))} + R^p\big), \tag{2.3}$$

$$\begin{aligned}&\|\Gamma v_1 - \Gamma v_2\|_{L^\infty([0,T];W^{\sigma,\rho+1}(\mathbb{R}^d))}\\ &\quad\le C_2 T^{1-\theta-\varepsilon}\Big(\sum_{j=1}^2 \|v_j\|^{p-1}_{L^\infty([0,T];W^{\sigma,\rho+1}(\mathbb{R}^d))} + R^{p-1}\Big)\|v_1 - v_2\|_{L^\infty([0,T];W^{\sigma,\rho+1}(\mathbb{R}^d))},\end{aligned} \tag{2.4}$$

for all $v, v_1, v_2 \in L^\infty([0,T]; W^{\sigma,\rho+1}(\mathbb{R}^d))$ and all $R > 0$, outside a set of probability $\le C\exp\big(-c\frac{R^2}{T^{\frac{2\varepsilon}{p}}\|\phi\|^2_{H^\sigma}}\big)$.

Proof. Note that the hypothesis on ρ yields $\theta \in (0,1)$. For $\varepsilon \in (0, 1-\theta)$, we use the dispersive estimate (2.1) and Hölder's inequality to obtain

$$
\begin{aligned}
\|\Gamma v\|_{L^\infty([0,T];W^{\sigma,\rho+1}(\mathbb{R}^d))} &\le \sup_{t\in[0,T]} \int_0^t \frac{1}{|t-t'|^\theta} \|\langle\nabla\rangle^\sigma \mathcal{N}(v+z)(t')\|_{L_x^{1+\frac{1}{\rho}}(\mathbb{R}^d)} \mathrm{d}t' \\
&\le \sup_{t\in[0,T]} \left(\int_0^t \frac{1}{|t-t'|^{\frac{\theta}{1-\varepsilon}}} \mathrm{d}t' \right)^{1-\varepsilon} \|\langle\nabla\rangle^\sigma \mathcal{N}(v+z)\|_{L_T^{\frac{1}{\varepsilon}} L_x^{1+\frac{1}{\rho}}(\mathbb{R}^d)} \\
&\le C T^{1-\theta-\varepsilon} \|\langle\nabla\rangle^\sigma \mathcal{N}(v+z)\|_{L_T^{\frac{1}{\varepsilon}} L_x^{1+\frac{1}{\rho}}(\mathbb{R}^d)}.
\end{aligned}
\tag{2.5}
$$

Here and in the following, we use the short-hand notation $L_T^q L_x^r(\mathbb{R}^d) := L^q([0,T]; L^r(\mathbb{R}^d))$. Recalling that $p = 2k+1$ with $k \in \mathbb{N}$, we write $\mathcal{N}(v+z) = |v+z|^{p-1}(v+z)$ as the product $(v+z)^{k+1}(\overline{v}+\overline{z})^k$. Then, by the fractional Leibnitz rule (see, for example, [10]), the Sobolev embedding $W^{\sigma,\rho+1}(\mathbb{R}^d) \subset L^{\frac{(p-1)(\rho+1)}{\rho-1}}(\mathbb{R}^d)$ (which holds provided that $\sigma \ge \frac{d(p-\rho)}{(\rho+1)(p-1)}$), and Lemma 2.2, we have

$$
\begin{aligned}
\|\langle\nabla\rangle^\sigma \mathcal{N}(v+z)\|_{L_T^{\frac{1}{\varepsilon}} L_x^{1+\frac{1}{\rho}}(\mathbb{R}^d)} &= \left\| \langle\nabla\rangle^\sigma [(v+z)^{k+1}(\overline{v}+\overline{z})^k] \right\|_{L_T^{\frac{1}{\varepsilon}} L_x^{1+\frac{1}{\rho}}(\mathbb{R}^d)} \\
&\lesssim \|\langle\nabla\rangle^\sigma (v+z)\|_{L_T^{\frac{p}{\varepsilon}} L_x^{\rho+1}(\mathbb{R}^d)} \|v+z\|^{2k}_{L_T^{\frac{p}{\varepsilon}} L_x^{\frac{(p-1)(\rho+1)}{\rho-1}}(\mathbb{R}^d)} \\
&\lesssim \|\langle\nabla\rangle^\sigma (v+z)\|^p_{L_T^{\frac{p}{\varepsilon}} L_x^{\rho+1}(\mathbb{R}^d)} \\
&\lesssim \left(\|\langle\nabla\rangle^\sigma v\|^p_{L_T^{\frac{p}{\varepsilon}} L_x^{\rho+1}(\mathbb{R}^d)} + \|\langle\nabla\rangle^\sigma z\|^p_{L_T^{\frac{p}{\varepsilon}} L_x^{\rho+1}(\mathbb{R}^d)} \right) \\
&\lesssim \left(T^\varepsilon \|v\|^p_{L^\infty([0,T];W^{\sigma,\rho+1}(\mathbb{R}^d))} + R^p \right),
\end{aligned}
\tag{2.6}
$$

outside a set of probability

$$
\le C \exp\left(-c \frac{R^2}{T^{\frac{2\varepsilon}{p}} \|\phi\|^2_{H^\sigma}} \right).
$$

Estimate (2.3) then follows from (2.5) and (2.6). The proof of (2.4) is analogous. □

Remark 2.4. We remark that the proof of the nonlinear estimates in Proposition 2.3 is similar in spirit to the following works on almost sure local well-posedness for the nonlinear wave equation [5,6,16], in the sense that no case-by-case analysis is needed.

In Proposition 2.3, we have a degree of freedom in choosing ρ. It turns out that, to lower as much as possible the regularity σ of ϕ, one needs to take $\rho = p$ in the energy-subcritical case, while ρ needs to be arbitrarily close to $1 + \frac{4}{d-2}$ in the energy-(super)critical case. More precisely, the following corollary holds.

Corollary 2.5. *Given $d \ge 1$, $p \ge 3$ an odd integer, and $0 < \varepsilon \ll 1$, we define*

$$
\sigma_{d,p} := \begin{cases} 0, & \text{if } \ d = 1,2 \ \text{ or } \ 3 \le p < 1 + \frac{4}{d-2}, \quad d \ge 3 \\ \frac{d-2}{2-\varepsilon} \cdot \frac{p-1-\frac{4}{d-2}+\frac{\varepsilon d}{d-2}}{p-1}, & \text{if } \ p \ge 1 + \frac{4}{d-2}, \quad d \ge 3, \end{cases}
$$

and

$$
\rho_{d,p} := \begin{cases} p, & \text{if } \ d = 1,2 \ \text{ or } \ 3 \le p < 1 + \frac{4}{d-2}, \quad d \ge 3 \\ 1 + \frac{4-\varepsilon d}{d-2}, & \text{if } \ p \ge 1 + \frac{4}{d-2}, \quad d \ge 3. \end{cases}
$$

Given $\phi \in H^\sigma(\mathbb{R}^d)$ with $\sigma \ge \sigma_{d,p}$, let ϕ^ω be its Wiener randomization defined in (1.4), satisfying (1.5). Then there exist $C_1, C_2, \alpha > 0$ such that the following estimates hold

$$
\|\Gamma v\|_{L^\infty([0,T];W^{\sigma,\rho_{d,p}+1}(\mathbb{R}^d))} \le C_1 T^\alpha \left(\|v\|^p_{L^\infty([0,T];W^{\sigma,\rho_{d,p}+1}(\mathbb{R}^d))} + R^p \right),
\tag{2.7}
$$

$$\begin{aligned}&\|\Gamma v_1 - \Gamma v_2\|_{L^\infty([0,T];W^{\sigma,\rho_{d,p}+1}(\mathbb{R}^d))}\\&\quad\le C_2 T^\alpha\Big(\sum_{j=1}^{2}\|v_j\|^{p-1}_{L^\infty([0,T];W^{\sigma,\rho_{d,p}+1}(\mathbb{R}^d))} + R^{p-1}\Big)\|v_1 - v_2\|_{L^\infty([0,T];W^{\sigma,\rho_{d,p}+1}(\mathbb{R}^d))},\end{aligned}\tag{2.8}$$

for all $v, v_1, v_2 \in L^\infty([0,T]; W^{\sigma,\rho_{d,p}+1}(\mathbb{R}^d))$ and all $R > 0$, outside a set of probability $\le C\exp\big(-c\frac{R^2}{T^{\frac{2\varepsilon}{p}}\|\phi\|^2_{H^\sigma}}\big)$.

Proof. In the case when $d = 1, 2$ or $p < 1 + \frac{4}{d-2}$, $d \ge 3$, this follows by taking $\rho = p$ in Proposition 2.3 and by noticing that $\sigma(d, p, p) = 0$. For $p \ge 1 + \frac{4}{d-2}$, $d \ge 3$, one takes $\rho = \rho_{d,p}$ in Proposition 2.3 and a straightforward calculation shows that $\sigma(d, p, \rho_{d,p}) = \sigma_{d,p}$. □

In Proposition 2.3 and Corollary 2.5, we restricted our attention to the case when p is an odd integer. In the following, we remark that when $\phi \in L^2(\mathbb{R}^d)$, i.e. $\sigma = 0$, this assumption on p is redundant.

Remark 2.6. Let $d = 1, 2$ or $1 < p < 1 + \frac{4}{d-2}$ with $d \ge 3$. Given $\phi \in L^2(\mathbb{R}^d)$, let ϕ^ω be its Wiener randomization defined in (1.4), satisfying (1.5). Then there exist $C_1, C_2, \alpha > 0$ such that the following estimates hold

$$\|\Gamma v\|_{L^\infty([0,T];L^{p+1}(\mathbb{R}^d))} \le C_1 T^\alpha\big(\|v\|^p_{L^\infty([0,T];L^{p+1}(\mathbb{R}^d))} + R^p\big),\tag{2.9}$$

$$\begin{aligned}&\|\Gamma v_1 - \Gamma v_2\|_{L^\infty([0,T];L^{p+1}(\mathbb{R}^d))}\\&\quad\le C_2 T^\alpha\Big(\sum_{j=1}^{2}\|v_j\|^{p-1}_{L^\infty([0,T];L^{p+1}(\mathbb{R}^d))} + R^{p-1}\Big)\|v_1 - v_2\|_{L^\infty([0,T];L^{p+1}(\mathbb{R}^d))},\end{aligned}\tag{2.10}$$

for all $v, v_1, v_2 \in L^\infty([0,T]; L^{p+1}(\mathbb{R}^d))$ and all $R > 0$, outside a set of probability $\le C\exp\big(-c\frac{R^2}{T^{\frac{2\varepsilon}{p}}\|\phi\|^2_{L^2}}\big)$.

Proof. By Corollary 2.5, when $d = 1, 2$ or $1 < p < 1 + \frac{4}{d-2}$ with $d \ge 3$, estimates (2.9) and (2.10) hold when p is an odd integer. In this case, (2.9) and (2.10) are simply (2.7) and (2.8) with $\sigma = \sigma_{d,p} = 0$ and $\rho_{d,p} = p$. We then notice that the proof of Proposition 2.3 (and thus that of Corollary 2.5) when $\sigma = 0$ does not require the use of the fractional Leibnitz rule and, in particular, the assumption that p is an odd integer is redundant. □

We conclude this note with the proof of Theorem 1.1, which follows easily from Corollary 2.5 and Remark 2.6 via a fixed-point argument.

Proof of Theorem 1.1. Let $s \ge \sigma_{d,p}$. Fix $0 < T \le 1$ and define $M := M(T) = \min\Big\{\big(\frac{1}{2C_1}\big)^{\frac{1}{p-1}}, \big(\frac{1}{4C_2}\big)^{\frac{1}{p-1}}\Big\} T^{-\frac{\alpha}{p-1}}$, with C_1, C_2 as in (2.7) and (2.8). We also define $R \sim T^{-\frac{\alpha}{p-1}}$ such that

$$C_1 T^\alpha R^p \le \frac{M}{2} \quad \text{and} \quad C_2 T^\alpha R^{p-1} < \frac{1}{2}.$$

With these choices, it follows from Corollary 2.5 that Γ is a contraction on the ball of radius M centered at the origin in $L^\infty([0,T]; W^{s,\rho_{d,p}}(\mathbb{R}^d))$ outside a set of probability

$$\le C\exp\Big(-c\frac{R^2}{T^{\frac{2\varepsilon}{p}}\|\phi\|^2_{H^\sigma}}\Big) \sim \exp\Big(-c\frac{1}{T^\gamma\|\phi\|^2_{H^s}}\Big)$$

for some $\gamma > 0$. In view of (1.7) and (2.2), an application of the contraction mapping principle then concludes the proof of the first part of Theorem 1.1.

For the second part of Theorem 1.1, one argues similarly, using (2.9) and (2.10) to show that Γ is a contraction on a ball centered at the origin in $L^\infty([0,T], L^{p+1}(\mathbb{R}^d))$. □

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