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Homological algebra

On the second homology group of extended Leibniz algebras

Sur le deuxième module d'homologie des algèbres de Leibniz étendues

Allahtan Victor Gnedbaye

Département de Mathématiques, FSEA, Université de N'Djaména, BP 1027, N'Djaména, Tchad

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ABSTRACT

Introduced by F. Chapoton, perm algebras allow us to define the notion of *extended Leibniz algebras*. We describe their second homology group in the particular case when the Leibniz algebra $\mathcal{G}$ is perfect and the perm algebra R satisfies $R = R^2$ (e.g., when the algebra R is unital). This gives rise to a comparison of the modules of differential 1-forms when the perm algebra R is an associative and commutative algebra with a unit-element.

RÉSUMÉ

Introduites par F. Chapoton, les algèbres perm donnent naissance à une notion d'*algèbres de Leibniz étendues*. Nous décrivons leur deuxième module d'homologie dans le cas particulier où l'algèbre de Leibniz $\mathcal{G}$ est parfaite et l'algèbre perm R vérifie $R = R^2$ (e.g., lorsque l'algèbre R est unitaire). Ceci nous permet de comparer les modules des 1-formes différentielles lorsque l'algèbre R est une algèbre associative et commutative avec unité.

Version française abrégée

Développée pour l'essentiel par Loday–Pirashvili (voir [9], [10]), l'homologie de Leibniz est une théorie d'homologie non commutative pour les algèbres de Lie qui s'étend à une classe plus large d'algèbres : les « algèbres de Leibniz », caractérisées par l'unique identité de Leibniz

$$[[x, y], z] = [[x, z], y] + [x, [y, z]].$$

Dans un premier temps (voir [3]), nous avions calculé l'homologie de Leibniz d'algèbres de Lie étendues $\mathcal{L} \otimes A$ (par une algèbre associative et commutative avec unité) en termes d'homologie de Hochschild de A et des coinvariants des puissances symétriques $(S^* \mathcal{L})_{\mathcal{L}}$. Dans cette Note, nous étudions le cas des algèbres de Leibniz étendues par une algèbre

E-mail address: gnedbaye_av@yahoo.fr.

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perm. Rappelons que ces dernières ont été introduites par F. Chapoton (voir [1], [2], [6]) comme algèbres binaires assujetties aux relations

$$(ab)c = a(bc) = a(cb).$$

Étant données une algèbre de Leibniz $\mathcal{G}$ et une algèbre perm R, le produit tensoriel linéaire $\mathcal{G}_R := R \otimes_{\mathbb{K}} \mathcal{G}$ (où $\mathbb{K}$ est le corps de base de caractéristique nulle sur lequel nous travaillons) admet une structure d'algèbre de Leibniz par le crochet suivant :

$$[r \otimes x, s \otimes y] := (rs) \otimes [x, y], \qquad \forall x, y \in \mathcal{G}, \quad \forall r, s \in R.$$

On obtient aisément en bas degrés $\mathrm{HL}_0(\mathcal{G}_R) \cong \mathbb{K}$ et $\mathrm{HL}_1(\mathcal{G}_R) \cong (\mathcal{G}_R)_{ab} := \mathcal{G}_R/[\mathcal{G}_R, \mathcal{G}_R]$. Par conséquent, lorsque l'algèbre de Leibniz $\mathcal{G}$ est parfaite ($\mathcal{G} = [\mathcal{G}, \mathcal{G}]$), il vient

$$\mathrm{HL}_1(\mathcal{G}_R) \cong (R/R^2) \otimes_{\mathbb{K}} \mathcal{G}.$$

On observera que si, de plus, l'algèbre perm R vérifie $R^2 = R$ (ce qui est le cas dès que l'algèbre R admet une unité), l'algèbre de Leibniz $\mathcal{G}_R$ est aussi parfaite ($\mathrm{HL}_1(\mathcal{G}_R) = 0$). Elle admet donc une « extension centrale universelle » dans la catégorie des algèbres de Leibniz (voir [4], [10]). D'après [5], cette dernière s'identifie au carré tensoriel non abélien $\mathcal{G}_R * \mathcal{G}_R$, et l'on sait qu'alors son noyau est canoniquement isomorphe au module d'homologie $\mathrm{HL}_2(\mathcal{G}_R)$. Nous décrivons ici ce module d'homologie en le reliant aux formes différentielles construites dans [7] dans le cadre des algèbres perm, et nous obtenons le résultat suivant.

Théorème. *Soit $\mathcal{G}$ une algèbre de Leibniz parfaite et soit R une algèbre perm telle que $R^2 = R$ (e.g., si R est unitaire). Alors, on a une suite exacte d'algèbres de Leibniz*

$$0 \to \mathrm{HL}_2(\mathcal{G}_R) \to \mathcal{G}_R * \mathcal{G}_R \xrightarrow{[-,-]} \mathcal{G}_R \to 0.$$

Si la semi-représentation $\mathcal{G}_R$ de $\mathcal{G}$ est complètement réductible, on a un isomorphisme de $\mathbb{K}$-modules

$$\mathrm{HL}_2(\mathcal{G}_R) \cong R \otimes_R \Omega^1_{\mathbb{K}}(R) \otimes_{\mathbb{K}} (\mathrm{S}^2\, \mathcal{G}_{\mathrm{Lie}})_{\mathcal{G}_{\mathrm{Lie}}}.$$

En particulier, lorsque l'algèbre perm $R = A$ est une algèbre associative et commutative avec unité, on a un isomorphisme de A-modules $\Omega^1_{A|\mathbb{K}} \cong \Omega^1_{\mathbb{K}}(A)$.

1. Introduction

It is well known (see [8]) that the homology of a Leibniz algebra $\mathcal{G}$ starts by $\mathrm{HL}_0(\mathcal{G}) = \mathbb{K}$ and $\mathrm{HL}_1(\mathcal{G}) \cong \mathcal{G}_{ab} := \mathcal{G}/[\mathcal{G}, \mathcal{G}]$. Therefore, when the Leibniz algebra $\mathcal{G}$ is perfect ($\mathcal{G} = [\mathcal{G}, \mathcal{G}]$), we have $\mathrm{HL}_1(\mathcal{G}) = 0$ and so $\mathcal{G}$ admits a universal central extension $\mathcal{U}$ whose kernel is precisely $\mathrm{HL}_2(\mathcal{G})$ (see [4], [10]).

In this Note, we give a description of the module $\mathrm{HL}_2(R \otimes \mathcal{G})$ where $\mathcal{G}$ is a perfect Leibniz algebra and R is a perm algebra such that $R = R^2$ (e.g., when R admits a unit-element). First we begin by defining general notions on Leibniz algebras, perm algebras and their semi-representations. We also recall a fundamental application of the non-abelian tensor square of a perfect Leibniz algebra, and that of differential forms of a perm algebra. At the end, we describe the module $\mathrm{HL}_2(R \otimes \mathcal{G})$ in terms of differential 1-forms of R and coinvariants of symmetric powers of $\mathcal{G}$.

2. Prerequisites on Leibniz algebras

Discovered by J.-L. Loday in 1989, *Leibniz algebras* are a non-commutative variation of classical Lie algebras. They consist in a bilinear bracket $[-,-] : \mathcal{G} \otimes \mathcal{G} \to \mathcal{G}$ satisfying the *Leibniz identity*:

$$[[x, y], z] = [[x, z], y] + [x, [y, z]], \ \forall\, x, y, z \in \mathcal{G}. \tag{2.1}$$

In other terms, the operator $[-, z]$ is a *derivation* for the bracket.

In the presence of the *skew-symmetry* condition $[x, x] = 0$ (which implies $[x, y] = -[y, x]$), the Leibniz identity (2.1) is equivalent to the *Jacobi relation* characterizing Lie algebras. These latter are then examples of Leibniz algebras. Conversely, with any Leibniz algebra $\mathcal{G}$, one can associate a Lie algebra $\mathcal{G}_{\mathrm{Lie}} := \mathcal{G}/([x, x])$ where $([x, x])$ denotes the two-sided ideal generated by all brackets $[x, x]$ when x spans $\mathcal{G}$.

Historically, Leibniz algebras were discovered from the following observation: the Chevalley–Eilenberg boundary $\mathrm{d} : \Lambda^n \mathcal{L} \to \Lambda^{n-1} \mathcal{L}$

$$\mathrm{d}(x_1 \wedge \cdots \wedge x_n) = \sum_{i<j} (-1)^{i+j} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \widehat{x_i} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_n$$
$$= \sum_{i<j} (-1)^{j} x_1 \wedge \cdots \wedge x_{i-1} \wedge [x_i, x_j] \wedge x_{i+1} \wedge \cdots \wedge \widehat{x_j} \wedge \cdots \wedge x_n$$

can be lifted to a well-defined differential $\mathrm{d} : \mathcal{L}^{\otimes n} \to \mathcal{L}^{\otimes n-1}$

$$\mathrm{d}(x_1 \otimes \cdots \otimes x_n) := \sum_{i<j} (-1)^{j} x_1 \otimes \cdots \otimes x_{i-1} \otimes [x_i, x_j] \otimes \cdots \otimes \widehat{x_j} \otimes \cdots \otimes x_n \tag{2.2}$$

and the fact that $\mathrm{d} \circ \mathrm{d} = 0$ does only use relation (2.1). Therefore, one has a new non-commutative (co)homology theory for Lie algebras, which can be extended to a larger class of algebras: Leibniz algebras. It is denoted by HL. There is a functor $\mathrm{HL}_*(\mathcal{L}) \to \mathrm{H}_*(\mathcal{L})$ that is an isomorphism in degrees 0 and 1 (and an epimorphism in degree 2):

$$\mathrm{HL}_0(\mathcal{L}) = \mathrm{H}_0(\mathcal{L}) = \mathbb{K} \text{ and } \mathrm{HL}_1(\mathcal{L}) \cong \mathrm{H}_1(\mathcal{L}) = \mathcal{L}_{ab} := \mathcal{L}/([x, y]).$$

For example, let $\mathcal{L}$ be a Lie algebra and let M be a representation of $\mathcal{L}$, seen as a right $\mathcal{L}$-module with an action denoted by m^x, where $m \in M$ and $x \in \mathcal{L}$. For any $\mathcal{L}$-equivariant morphism $f : M \to \mathcal{L}$, the bracket defined by $[m, m'] := m^{f(m')}$ induces a Leibniz algebra structure on the $\mathbb{K}$-module M. Any Leibniz algebra can be obtained in this way by considering the canonical projection $\mathcal{G} \to \mathcal{G}_{\mathrm{Lie}}$.

More generally, let V be a $\mathbb{K}$-module and let $\overline{\mathrm{T}}(V) := \bigoplus_{n>0} V^{\otimes n}$ be the non-unital tensor algebra free over V. Then the bracket inductively defined by

$$[x, v] := x \otimes v \quad \text{and} \quad [x, y \otimes v] := [x, y] \otimes v - [x \otimes v, y] \tag{2.3}$$

where $x, y \in \overline{\mathrm{T}}(V)$ and $v \in V$, satisfies Leibniz's identity (2.1). It is the *free Leibniz algebra* over V and we denote it by $\mathcal{F}(V)$. Observe that we have

$$v_1 \otimes \cdots \otimes v_n = [\cdots[[v_1, v_2], v_3], \cdots, v_n] \tag{2.4}$$

for all $v_1, \cdots, v_n \in V$. In particular, the free Lie algebra over V is nothing but $\mathcal{L} := \mathcal{F}(V)_{\mathrm{Lie}}$.

3. Perm algebras and differential forms

Introduced by F. Chapoton (see [1] and [6]), a "*perm algebra*" is a $\mathbb{K}$-module R equipped with a bilinear map $(-,-) : R \otimes R \longrightarrow R$ satisfying the (*right commutative and associative*) identities

$$(ab)c = a(bc) = a(cb), \qquad \forall a, b, c \in R. \tag{3.1}$$

For example, any associative and commutative algebra is a perm algebra. Conversely, a perm algebra with a unit-element is nothing but a unital associative and commutative algebra. But in sequel, we shall deal only with non-unital perm algebras.

More generally, a procedure for constructing perm algebras is the following. Let A be an associative and commutative algebra. Let R be a right A-module equipped with an A-module morphism $f : R \longrightarrow A$. Then the product given by $(r, r') \longmapsto rf(r')$ endows the $\mathbb{K}$-module R with a perm algebra structure that we denote by R_f.

Taking a $\mathbb{K}$-module V, denote by $A := S(V)$ and $R := V \otimes S(V)$ where $S(V)$ is the classical symmetric free algebra over V (with the obvious right A-module structure on R), the "*fusion map*" $f : R \longrightarrow A$

$$f(v) = v \qquad \text{and} \qquad f(v_0 \otimes v_1 \cdots \otimes v_n) := f(v_0) f(v_1) \cdots f(v_n),\ \forall v,\ v_i \in V,$$

endows $R := V \otimes S(V)$ with a perm algebra structure: it is the *free perm algebra* over V.

Recall from [7] that, for any perm algebra R, the module of derivations is represented by $\Omega^1_{\mathbb{K}}(R) := I/I^2$ where $I := \ker(\mu)$. Here $\mu : E(R) \to R$ is given by

$$\mu(\sum a \otimes b + a'_l + a''_r) := \sum ab + a' + a'',$$
$$E(R) := R \otimes R \quad \oplus \quad R_l \quad \oplus R_r$$

where R_l and R_r are two copies of R, seen as $R_l = R \otimes \mathbb{K}$ and $R_r = \mathbb{K} \otimes R$.

As a $\mathbb{K}$-module, $\Omega^1_{\mathbb{K}}(R)$ is generated by the symbols $a\,\mathrm{d}b + \mathrm{d}c,\ \ a, b, c \in R$ with $\mathrm{d}a := a_l - a_r$. It is a right-symmetric R-bimodule in the following sense:

$$\alpha \cdot a \cdot \mathrm{d}b = \alpha \cdot \mathrm{d}b \cdot a, \qquad \forall \alpha, a, b \in R.$$

Furthermore, we put for $n \geq 0$:

$$\Omega^0_{\mathbb{K}}(R) := R \quad \text{and} \quad \Omega^{n+1}_{\mathbb{K}}(R) := \Omega^1_{\mathbb{K}}(R) \otimes \Lambda^n_R \left(\Omega^1_{\mathbb{K}}(R)\right).$$

Then the module of differential n-forms $\Omega^n_{\mathbb{K}}(R)$ is generated by the symbols

$$\omega := a_0\, \mathrm{d}a_1 \otimes \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_n + \mathrm{d}b_1 \otimes \mathrm{d}b_2 \wedge \cdots \wedge \mathrm{d}b_n$$

together with the map

$$\mathrm{d} : \Omega^n_{\mathbb{K}}(R) \to \Omega^{n+1}_{\mathbb{K}}(R), \qquad \mathrm{d}(\omega) := \mathrm{d}a_0 \otimes \mathrm{d}a_1 \wedge \mathrm{d}a_2 \wedge \cdots \wedge \mathrm{d}a_n.$$

4. Semi-representations and Leibniz homology

Let $\mathcal{G}$ be a Leibniz algebra. A *semi-representation* of $\mathcal{G}$ is a $\mathbb{K}$-module M equipped with a right action $[-,-] : M \otimes \mathcal{G} \to M$ such that

$$[[m,x],y] = [[m,y],x] + [m,[x,y]] \qquad \forall m \in M, \quad \forall x, y \in \mathcal{G}. \tag{4.1}$$

For example, the $\mathbb{K}$-module $\mathcal{G}$ is a semi-representation over itself with its own bracket.

Now, let M be a semi-representation of a Leibniz algebra $\mathcal{G}$ and let $n \geq 0$ be any integer. Then the action $[-,-] : M \otimes \mathcal{G}^{\otimes n} \otimes \mathcal{G} \to M \otimes \mathcal{G}^{\otimes n}$ defined by

$$[(m, x_1, \cdots, x_n), x] := ([m,x], x_1, \cdots, x_n) \;+\; \sum_{i=1}^{n} [(m, x_1, \cdots, x_{i-1}, [x_i, x], x_{i+1}, \cdots, x_n) \tag{4.2}$$

induces a structure of semi-representation of $\mathcal{G}$ on the $\mathbb{K}$-module $M \otimes \mathcal{G}^{\otimes n}$. Therefore, one can define the *homology of* $\mathcal{G}$ *with coefficients in* M as the homology of the complex $(C_*(\mathcal{G}, M) := M \otimes \mathcal{G}^*, \mathrm{d}_*)$ where one inductively puts $\mathrm{d}_0(m) = 0$, $\mathrm{d}_1(m,x) := -[m,x]$ and

$$\mathrm{d}_{n+1}(m, x_1, \cdots, x_{n+1}) = \mathrm{d}_n(m, x_1, \cdots, x_n) \otimes x_{n+1} \;+ (-1)^{n+1}[(m, x_1, \cdots, x_n), x_{n+1}]. \tag{4.3}$$

It is denoted by $\mathrm{HL}_*(\mathcal{G}, M)$ and it is clear that we have

$$\mathrm{HL}_n(\mathcal{G}, \mathcal{G}) = \mathrm{HL}_{n+1}(\mathcal{G}, \mathbb{K}) =: \mathrm{HL}_{n+1}(\mathcal{G}) \tag{4.4}$$

This homology theory starts by $\mathrm{HL}_0(\mathcal{G}) = \mathbb{K}$, $\mathrm{HL}_1(\mathcal{G}) = \mathcal{G}_{ab} := \mathcal{G}/[\mathcal{G},\mathcal{G}]$. Therefore, if the Leibniz algebra $\mathcal{G}$ is *perfect* ($\mathcal{G} = [\mathcal{G},\mathcal{G}]$), then $\mathrm{HL}_1(\mathcal{G}) = 0$ and so we know that $\mathcal{G}$ admits a *universal central extension* $\mathcal{U}$ whose kernel is precisely $\mathrm{HL}_2(\mathcal{G})$ (see [4], [10]):

$$0 \to \mathrm{HL}_2(\mathcal{G}) \to \mathcal{U} \xrightarrow{[-,-]} \mathcal{G} \to 0.$$

Moreover, it is shown (see [5]) that the Leibniz algebra $\mathcal{U}$ is nothing but the non-abelian tensor square $\mathcal{U} \cong \mathcal{G} * \mathcal{G}$, which can be described as the quotient

$$\mathcal{U} \cong \mathcal{G} * \mathcal{G} \cong (\mathcal{G} \otimes \mathcal{G} \quad \oplus \quad \mathcal{G} \otimes \mathcal{G})/\mathcal{R}$$

where $\mathcal{R}$ is the relations $[x,y] \otimes z = [x,z] \otimes y + x \otimes [y,z]$ and $x \otimes [y,z] = -x \otimes [z,y]$. Since the Leibniz algebra $\mathcal{G}$ is perfect, one has readily

$$\mathcal{U} \cong \mathcal{G} * \mathcal{G} \cong (\mathcal{G} \otimes \mathcal{G}_{\mathrm{Lie}} \quad \oplus \quad \mathcal{G} \otimes \mathcal{G}_{\mathrm{Lie}})/\operatorname{Im} \mathrm{d}_3.$$

Therefore, we have an exact sequence of Leibniz algebras

$$0 \to \mathrm{HL}_2(\mathcal{G}) \to \mathcal{G} * \mathcal{G} \xrightarrow{[-,-]} \mathcal{G} \to 0$$

where the bracket acts by: $[x \otimes x', y \otimes y'] = [[x,x'],[y,y']]$, $\forall x, x', y, y' \in \mathcal{G}$.

5. Extended Leibniz algebras

Let $\mathcal{G}$ be a Leibniz algebra and let R be a perm algebra. Then we have the following.

Proposition 5.1. *The bracket defined by*

$$[r\otimes x, s\otimes y] := (rs)\otimes[x,y], \qquad \forall r,s\in R, \forall x,y\in\mathcal{G} \tag{5.1}$$

endows the $\mathbb{K}$*-module* $\mathcal{G}_R := R\otimes_{\mathbb{K}}\mathcal{G}$ *with a structure of Leibniz algebra that we call "extended Leibniz algebra".*

Proof. Indeed, for any $r,s,t\in R$ and $x,y,z\in\mathcal{G}$, we have

$$\begin{aligned}[][[r\otimes x, s\otimes y], t\otimes z] &= [(rs)\otimes[x,y], t\otimes z] = (rs)t\otimes[[x,y],z]\\ &= (rt)s\otimes[[x,z],y] + r(st)\otimes[x,[y,z]], \text{ by (5.1) and (2.1)}\\ &= [(rt)\otimes[x,z], s\otimes y] + [r\otimes x, (st)\otimes[y,z]], \text{ by (5.1)}\\ &= [[r\otimes x, t\otimes z], s\otimes y] + [r\otimes x, [s\otimes y, t\otimes z]], \text{ by (5.1)}\end{aligned}$$

which achieves proving the Leibniz identity. □

Definition 5.2. A "semi-representation" of a perm algebra R is a $\mathbb{K}$-module M equipped with a right action $(-,-): M\otimes R\to M$ satisfying the relations

$$(mr)s = m(rs) = m(sr), \qquad \forall m\in M, \quad \forall r,s\in R. \tag{5.2}$$

For example, any perm algebra is a semi-representation of itself with its own product as the right action.

Proposition 5.3. *Let* $\mathcal{G}$ *be a Leibniz algebra, let* R *be a perm algebra and let* M *be a semi-representation of* R*. Then the bracket defined by*

$$[m\otimes x, r\otimes y] := (mr)\otimes[x,y], \qquad \forall x,y\in\mathcal{G}, \forall r\in R, \ \forall m\in M \tag{5.3}$$

endows that $\mathbb{K}$*-module* $\mathcal{G}_M := M\otimes_{\mathbb{K}}\mathcal{G}$ *with a structure of semi-representation over the extended Leibniz algebra* $\mathcal{G}_R := R\otimes_{\mathbb{K}}\mathcal{G}$*.*

Proof. Indeed, for any $m\in M$, $r,s\in R$ and $x,y\in\mathcal{G}$, we have

$$\begin{aligned}[][[m\otimes x, r\otimes y], s\otimes z] &= [(mr)\otimes[x,y], s\otimes z] = (mr)s\otimes[[x,y],z]\\ &= (ms)r\otimes[[x,z],y] + m(rs)\otimes[x,[y,z]], \text{ by (2.1) and (5.2)}\\ &= [(ms)\otimes[x,z], r\otimes y] + [m\otimes x, (rs)\otimes[y,z]], \text{ by (5.3)}\\ &= [[r\otimes x, t\otimes z], s\otimes y] + [r\otimes x, [s\otimes y, t\otimes z]], \text{ by (5.3)}\end{aligned}$$

from whence we are done. □

Therefore, one can consider the homology groups $\mathrm{HL}_*(\mathcal{G}_R)$. For example, we have

$$\mathrm{HL}_0(\mathcal{G}_R) = \mathbb{K} \qquad \text{and} \qquad \mathrm{HL}_1(\mathcal{G}_R) \cong \mathcal{G}_R/[\mathcal{G}_R,\mathcal{G}_R] \tag{5.4}$$

where $[\mathcal{G}_R,\mathcal{G}_R]$ denotes the submodule of $\mathcal{G}_R$ generated by the elements

$$[r\otimes x, s\otimes y] := (rs)\otimes[x,y], \qquad \forall x,y\in\mathcal{G}, \forall r,s\in R.$$

It is clear that when the Leibniz algebra $\mathcal{G}$ is perfect ($\mathcal{G} = [\mathcal{G},\mathcal{G}]$), then we have

$$\mathrm{HL}_1(\mathcal{G}_R) \cong (R/R^2)\otimes_{\mathbb{K}}\mathcal{G}.$$

Therefore, if moreover the perm algebra R satisfies $R = R^2$ (e.g., when it admits a unit-element), then $\mathrm{HL}_1(\mathcal{G}_R) = 0$, that is, $\mathcal{G}_R$ is also perfect. Then in what follows, we focus on the second homology group $\mathrm{HL}_2(\mathcal{G}_R)$.

Theorem 5.4. *Let* $\mathcal{G}$ *be a perfect Leibniz algebra and let* R *be a perm algebra such that* $R = R^2$*. If the semi-representation* $\mathcal{G}_R$ *of* $\mathcal{G}$ *is completely reducible, then we have an isomorphism of* $\mathbb{K}$*-modules*

$$\mathrm{HL}_2(\mathcal{G}_R) \cong R\otimes_R \Omega^1_{\mathbb{K}}(R)\otimes_{\mathbb{K}} (\mathrm{S}^2\,\mathcal{G}_{\mathrm{Lie}})_{\mathcal{G}_{\mathrm{Lie}}}.$$

Proof. Consider the $\mathbb{K}$-linear map

$$\phi : \mathcal{G}_R \otimes \mathcal{G}_R \to R \otimes_R \Omega^1_{\mathbb{K}}(R) \otimes_{\mathbb{K}} (\mathrm{S}^2\, \mathcal{G}_{\mathrm{Lie}})_{\mathcal{G}_{\mathrm{Lie}}}, \quad rx \otimes sy \mapsto r \otimes \mathrm{d}s \otimes (xy).$$

Then we have:

$$\begin{aligned}\phi\, \mathrm{d}(rx, sy, tz) &= \phi((rs)[x, y], tz) - \phi((rt)[x, z], sy) - \phi(rx, (st)[y, z]) \\ &= (rs) \otimes \mathrm{d}t \otimes [x, y]z - (rt) \otimes \mathrm{d}s \otimes [x, z]y - r \otimes \mathrm{d}(st) \otimes x[y, z] \\ &= r \otimes s\, \mathrm{d}t \otimes [x, y]z - r \otimes t\, \mathrm{d}s \otimes [x, z]y - r \otimes (\mathrm{d}s{\cdot}t + s{\cdot}\mathrm{d}t) \otimes x[y, z] \\ &= r \otimes s\, \mathrm{d}t \otimes ([x, y]z - x[y, z]) - r \otimes t\, \mathrm{d}s \otimes [x, z]y - r \otimes \mathrm{d}s{\cdot}t \otimes x[y, z]. \quad \square\end{aligned}$$

But one has:

Lemma 5.5. *In the quotient* $(\mathrm{S}^2\, \mathcal{G}_{\mathrm{Lie}})_{\mathcal{G}_{\mathrm{Lie}}}$, *we have the equalities*

$$x[y, z] \equiv [x, y]z \equiv [z, x]y, \qquad \forall x, y, z \in \mathcal{G}_{\mathrm{Lie}}. \tag{5.5}$$

Therefore, we get

$$\phi\, \mathrm{d}(rx, sy, tz) = 0 - r \otimes (t\, \mathrm{d}s - \mathrm{d}s{\cdot}t) \otimes x[y, z] = r_1 \otimes [r_2(t\, \mathrm{d}s - \mathrm{d}s{\cdot}t)] \otimes x[y, z] = 0$$

because $R = R^2$ and because of the right symmetry of the bimodule $\Omega^1_{\mathbb{K}}(R)$, from whence we deduce a well-defined factorization:

$$\phi : \mathrm{HL}_2(\mathcal{G}_R) \to R \otimes_R \Omega^1_{\mathbb{K}}(R) \otimes_{\mathbb{K}} (\mathrm{S}^2\, \mathcal{G}_{\mathrm{Lie}})_{\mathcal{G}_{\mathrm{Lie}}}.$$

One easily checks that the map

$$\lambda : R \otimes_R \Omega^1_{\mathbb{K}}(R) \otimes_{\mathbb{K}} (\mathrm{S}^2\, \mathcal{G}_{\mathrm{Lie}})_{\mathcal{G}_{\mathrm{Lie}}} \to \mathrm{HL}_2((\mathcal{G}_R)_{\mathcal{G}}), \quad r \otimes \mathrm{d}s \otimes (xy) \mapsto (rx \otimes sy + ry \otimes sx)/2$$

is the inverse of ϕ. Here we identify the homology groups $\mathrm{HL}_2((\mathcal{G}_R)_{\mathcal{G}}) \cong \mathrm{HL}_2(\mathcal{G}_R)$ thanks to the reductivity of the semi-representation $\mathcal{G}_R$ with respect to the diagonal action of $\mathcal{G}$ (see [9, 10.6.6]):

$$[(r_1x_1, \cdots, r_nx_n), x] = \sum_{i=1}^{n} (r_1x_1, \cdots, r_{i-1}x_{i-1}, r_i[x_i, x], \cdots, r_nx_n)$$

Proof of the Lemma 5.5. Indeed, for any $x, y, z \in \mathcal{G}_{\mathrm{Lie}}$, we have

$$0 \equiv [xz, y] = [x, y]z + x[z, y]. \quad \square$$

Corollary 5.6. *For any associative and commutative* $\mathbb{K}$*-algebra* A *with unit, we have the isomorphism*

$$\Omega^1_{A|\mathbb{K}} \cong \Omega^1_{\mathbb{K}}(A).$$

Proof. Indeed, the above isomorphism yields

$$\mathrm{HL}_2(\mathcal{L}_A) \cong A \otimes_A \Omega^1_{\mathbb{K}}(A) \otimes_{\mathbb{K}} (\mathrm{S}^2\, \mathcal{L})_{\mathcal{L}} \cong \Omega^1_{\mathbb{K}}(A) \otimes_{\mathbb{K}} (\mathrm{S}^2\, \mathcal{L})_{\mathcal{L}}$$

for all perfect Lie algebras $\mathcal{L}$ such that the semi-representation $\mathcal{L}_A$ is completely reducible. But we have shown (see [3]) that under the same hypothesis, we have an isomorphism

$$\mathrm{HL}_2(\mathcal{L}_A) \cong \Omega^1_{A|\mathbb{K}} \otimes_{\mathbb{K}} (\mathrm{S}^2\, \mathcal{L})_{\mathcal{L}}.$$

So we are done. $\square$

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