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Differential geometry

Hamilton–Souplet–Zhang's gradient estimates and Liouville theorems for a nonlinear parabolic equation ☆

Estimations du gradient de Hamilton–Souplet–Zhang et théorèmes de Liouville pour une équation non linéaire parabolique

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ABSTRACT

In this paper, we study Hamilton–Souplet–Zhang's gradient estimates for positive solutions to the nonlinear parabolic equation

$$u_t = \Delta u + \lambda u^\alpha$$

on noncompact Riemannian manifolds, where λ, α are two real constants. As an application, we obtain a Liouville-type theorem.

RÉSUMÉ

Dans la présente Note, nous étudions les estimations du gradient de Hamilton–Souplet–Zhang pour les solutions positives de l'équation non linéaire parabolique

$$u_t = \Delta u + \lambda u^\alpha$$

sur une variété riemannienne non compacte, où λ et α sont deux constantes réelles. Nous en déduisons, comme application, un théorème de type Liouville.

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1. Introduction

After Cheng–Yau's work in [1] and Li–Yau's work in [7] on gradient estimates of the heat equation

$$u_t = \Delta u \tag{1.1}$$

on a complete Riemannian manifold, there have been plenty of results obtained concerning gradient estimates, for example, [2–6,8–10,12,13,16] and the references therein. Generalizing Hamilton's estimates in [3], Souplet and Zhang in [11] proved the following theorem.

Theorem A. *[11] Let (M^n, g) be an n-dimensional Riemannian manifold with $\mathrm{Ric}(M^n) \geq -K$, where K is a non-negative constant. Suppose that u is a positive solution to the equation (1.1) in $Q_{R,T} := B_{x_0}(R) \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty)$. Let $u_{\max} = \max_{x \in Q_{R,T}} u(x)$. Then, in $Q_{R,T}$,*

$$\frac{|\nabla u|}{u} \leq C\left(\sqrt{K} + \frac{1}{R} + \frac{1}{\sqrt{T}}\right)\left(1 + \log\frac{u_{\max}}{u}\right), \tag{1.2}$$

where the constant C depends only on the dimension n.

In this paper, we consider the following nonlinear parabolic equation

$$u_t = \Delta u + \lambda u^{\alpha}, \tag{1.3}$$

where λ, α are two real constants. The gradient estimate of the elliptic equation $\Delta u + \lambda u^{\alpha} = 0$ has been studied by Yang [14] and Zhang and Ma in [15]. When $\lambda < 0$ and M^n is a bounded smooth domain in $\mathbb{R}^n$, the equation $\Delta u + \lambda u^{\alpha} = 0$ is known as the thin film equation, which describes a steady state of the thin film.

In this paper, we study Hamilton–Souplet–Zhang's gradient estimates for positive solutions to the nonlinear parabolic equation (1.3) and obtain the following theorem.

Theorem 1.1. *Let (M^n, g) be an n-dimensional Riemannian manifold with $\mathrm{Ric}(M^n) \geq -K$, where K is a non-negative constant. Suppose that u is a positive solution to the equation (1.3) in $Q_{R,T} := B_{x_0}(R) \times [0, T] \subset M^n \times (-\infty, \infty)$. Let $u_{\max} = \max_{x \in Q_{R,T}} u(x)$ and $u_{\min} = \min_{x \in Q_{R,T}} u(x)$. Then in $Q_{R,T}$, we have:*

1) if $\lambda < 0$ and $\alpha \in (-\infty, 0) \cup (0, 1)$,

$$\frac{|\nabla u|^2}{u^2} \leq C\left(K + \frac{1}{R^2} + \frac{1}{T} + \lambda(\alpha - 1)u_{\min}^{\alpha-1}\right)\left(1 + \log\frac{u_{\max}}{u}\right)^2; \tag{1.4}$$

2) if $\lambda < 0$ and $\alpha \in (1, +\infty)$,

$$\frac{|\nabla u|^2}{u^2} \leq C\left(K + \frac{1}{R^2} + \frac{1}{T} + \lambda(\alpha - 1)u_{\min}^{\alpha-1}\right)\left(1 + \log\frac{u_{\max}}{u}\right)^2; \tag{1.5}$$

3) if $\lambda > 0$ and $\alpha \in (-\infty, 0) \cup (1, +\infty)$,

$$\frac{|\nabla u|^2}{u^2} \leq C\left(K + \frac{1}{R^2} + \frac{1}{T} + \lambda\alpha u_{\max}^{\alpha-1}\right)\left(1 + \log\frac{u_{\max}}{u}\right)^2; \tag{1.6}$$

4) if $\lambda > 0$ and $\alpha \in (0, 1)$,

$$\frac{|\nabla u|^2}{u^2} \leq C\left(K + \frac{1}{R^2} + \frac{1}{T} + \lambda\alpha u_{\min}^{\alpha-1}\right)\left(1 + \log\frac{u_{\max}}{u}\right)^2. \tag{1.7}$$

Here, the constant C depends only on the dimension n.

As an application, we get the following Liouville-type theorem.

Corollary 1.2. *Let (M^n, g) be an n-dimensional complete noncompact Riemannian manifold with nonnegative Ricci curvature. Let u be a positive ancient solution to the equation (1.3) such that*

$$u(x,t) = \mathrm{e}^{o(d(x) + \sqrt{|t|})}$$

near infinity. If $\lambda < 0$ and $\alpha \in (1, +\infty)$ (or $\lambda > 0$ and $\alpha \in (-\infty, 0)$), then u must be a constant.

Remark 1.1. When $\lambda \to 0$, the equation (1.3) becomes the heat equation (1.1). In particular, from the estimates (1.5) or (1.6), we have

$$\frac{|\nabla u|^2}{u^2} \le C\left(K + \frac{1}{R^2} + \frac{1}{T}\right)\left(1 + \log\frac{u_{\max}}{u}\right)^2, \tag{1.8}$$

which gives

$$\frac{|\nabla u|}{u} \le C\left(\sqrt{K} + \frac{1}{R} + \frac{1}{\sqrt{T}}\right)\left(1 + \log\frac{u_{\max}}{u}\right). \tag{1.9}$$

Noticing that the estimate (1.9) is the same as (1.2) of Souplet and Zhang. Therefore, our Theorem 1.1 generalizes the Theorem A of Souplet and Zhang in [11].

Remark 1.2. In [16], Zhu considered Souplet and Zhang's gradient estimates for positive solutions to the nonlinear parabolic

$$u_t = \Delta u + \lambda(x,t)u^{\alpha}, \tag{1.10}$$

where $\lambda(x,t)$ is a nonnegative function defined on $M^n \times (-\infty, 0]$, and α a real constant satisfying $0 < \alpha < 1$. The results in this paper can be seen as complementary to those of Zhu in [16].

2. Proof of the theorem

Let $\tilde{u} = \frac{u}{u_{\max}}$. Then we have $0 < \tilde{u} \le 1$. From (1.3), we obtain that $\tilde{u}$ satisfies the following equation

$$\tilde{u}_t = \Delta\tilde{u} + \tilde{\lambda}\,\tilde{u}^{\alpha}, \tag{2.1}$$

where $\tilde{\lambda} = \lambda\, u_{\max}^{\alpha-1}$. Let $f = \log\tilde{u} \le 0$ and

$$w = |\nabla \log(1-f)|^2.$$

Then,

$$f_t = \Delta f + |\nabla f|^2 + \tilde{\lambda}\,\mathrm{e}^{(\alpha-1)f}. \tag{2.2}$$

A direct calculation yields

$$\begin{aligned} w_t &= \frac{2}{(1-f)^2} f_i (f_t)_i + \frac{2}{(1-f)^3} f_i^2 f_t \\ &= \frac{2}{(1-f)^2} f_i (f_{jj} + f_j^2 + \tilde{\lambda}\,\mathrm{e}^{(\alpha-1)f})_i \\ &\quad + \frac{2}{(1-f)^3} f_i^2 (f_{jj} + f_j^2 + \tilde{\lambda}\,\mathrm{e}^{(\alpha-1)f}) \\ &= \frac{2}{(1-f)^2}[f_{jji} f_i + 2 f_{ji} f_i f_j + \tilde{\lambda}\,(\alpha-1)\,\mathrm{e}^{(\alpha-1)f} f_i^2] \\ &\quad + \frac{2}{(1-f)^3} f_i^2 (f_{jj} + f_j^2 + \tilde{\lambda}\,\mathrm{e}^{(\alpha-1)f}). \end{aligned} \tag{2.3}$$

From the definition of w, we have

$$w = \frac{1}{(1-f)^2} f_j^2,$$

which shows

$$w_i = \frac{2}{(1-f)^3} f_i f_j^2 + \frac{2}{(1-f)^2} f_j f_{ji} \tag{2.4}$$

and

$$\begin{aligned} w_{ii} &= \frac{6}{(1-f)^4} f_i^2 f_j^2 + \frac{2}{(1-f)^3} f_{ii} f_j^2 + \frac{8}{(1-f)^3} f_{ji} f_i f_j \\ &\quad + \frac{2}{(1-f)^2} f_{ji}^2 + \frac{2}{(1-f)^2} f_j f_{jii} \\ &= \frac{6}{(1-f)^4} f_i^2 f_j^2 + \frac{2}{(1-f)^3} f_{ii} f_j^2 + \frac{8}{(1-f)^3} f_{ji} f_i f_j \\ &\quad + \frac{2}{(1-f)^2} f_{ji}^2 + \frac{2}{(1-f)^2} f_j f_{iij} + \frac{2}{(1-f)^2} R_{ij} f_i f_j, \end{aligned} \tag{2.5}$$

where, in the second equality, we used the Ricci formula:

$$f_{jii} = f_{iji} = f_{iij} + R_{ij} f_i.$$

By (2.3) and (2.5), we obtain

$$\begin{aligned}\Delta w - w_t = &\frac{2}{(1-f)^2} f_{ji}^2 + \Big[\frac{6}{(1-f)^4} - \frac{2}{(1-f)^3}\Big] f_i^2 f_j^2 \\ &+ \Big[\frac{8}{(1-f)^3} - \frac{4}{(1-f)^2}\Big] f_{ji} f_i f_j + \frac{2}{(1-f)^2} R_{ij} f_i f_j \\ &- 2\tilde{\lambda}\Big[\frac{\alpha-1}{(1-f)^2} + \frac{1}{(1-f)^3}\Big] \mathrm{e}^{(\alpha-1)f} f_i^2.\end{aligned} \tag{2.6}$$

Note that from (2.4), we have

$$\langle \nabla f, \nabla w\rangle = f_i w_i = \frac{2}{(1-f)^3} f_i^2 f_j^2 + \frac{2}{(1-f)^2} f_{ji} f_i f_j. \tag{2.7}$$

Therefore, (2.6) can be written as

$$\begin{aligned}&\Delta w - w_t - \varepsilon\langle \nabla f, \nabla w\rangle \\ &= \frac{2}{(1-f)^2} f_{ji}^2 + 2\Big[\frac{3}{(1-f)^4} - \frac{1+\varepsilon}{(1-f)^3}\Big] f_i^2 f_j^2 \\ &\quad + 2\Big[\frac{4}{(1-f)^3} - \frac{2+\varepsilon}{(1-f)^2}\Big] f_{ji} f_i f_j + \frac{2}{(1-f)^2} R_{ij} f_i f_j \\ &\quad - 2\tilde{\lambda}\Big[\frac{\alpha-1}{(1-f)^2} + \frac{1}{(1-f)^3}\Big] \mathrm{e}^{(\alpha-1)f} f_i^2,\end{aligned} \tag{2.8}$$

where $\varepsilon = \varepsilon(f)$ is a function depending on f. Applying

$$\begin{aligned}&\frac{2}{(1-f)^2} f_{ji}^2 + 2\Big[\frac{4}{(1-f)^3} - \frac{2+\varepsilon}{(1-f)^2}\Big] f_{ji} f_i f_j \\ &\quad = \frac{2}{(1-f)^2}\Big\{ f_{ji}^2 + \Big[\frac{4}{1-f} - (2+\varepsilon)\Big] f_{ji} f_i f_j \Big\} \\ &\quad \geq -\frac{1}{2(1-f)^2}\Big[\frac{4}{1-f} - (2+\varepsilon)\Big]^2 f_i^2 f_j^2\end{aligned}$$

into (2.8) gives

$$\begin{aligned}&\Delta w - w_t - \varepsilon\langle \nabla f, \nabla w\rangle \\ &\geq \Big[-\frac{2}{(1-f)^4} + \frac{6+2\varepsilon}{(1-f)^3} - \frac{(2+\varepsilon)^2}{2(1-f)^2}\Big] f_i^2 f_j^2 \\ &\quad + \frac{2}{(1-f)^2} R_{ij} f_i f_j - 2\tilde{\lambda}\Big[\frac{\alpha-1}{(1-f)^2} + \frac{1}{(1-f)^3}\Big] \mathrm{e}^{(\alpha-1)f} f_i^2 \\ &= \frac{1}{(1-f)^4}\Big\{ -\frac{1}{2}(1-f)^2\varepsilon^2 - 2[(1-f)^2 - (1-f)]\varepsilon \\ &\quad - [2(1-f)^2 - 6(1-f) + 2]\Big\} f_i^2 f_j^2 + \frac{2}{(1-f)^2} R_{ij} f_i f_j \\ &\quad - 2\tilde{\lambda}\Big[\frac{\alpha-1}{(1-f)^2} + \frac{1}{(1-f)^3}\Big] \mathrm{e}^{(\alpha-1)f} f_i^2.\end{aligned} \tag{2.9}$$

Taking

$$\varepsilon = -2 + \frac{2}{1-f}$$

in (2.9), we derive

$$\begin{aligned}
&\Delta w - w_t + 2\,\frac{-f}{1-f}\langle \nabla f, \nabla w\rangle \\
&\quad \geq \frac{2}{(1-f)^3}|\nabla f|^4 + \frac{2}{(1-f)^2}R_{ij} f_i f_j \\
&\quad\quad - 2\,\tilde{\lambda}\Big[\frac{\alpha-1}{(1-f)^2} + \frac{1}{(1-f)^3}\Big]\mathrm{e}^{(\alpha-1)f} f_i^2 \\
&\quad \geq 2\,(1-f)\,w^2 - 2\,K\,w - 2\,\tilde{\lambda}\left(\alpha - \frac{-f}{1-f}\right)\mathrm{e}^{(\alpha-1)f}\,w.
\end{aligned} \tag{2.10}$$

Denote by $B_p(R)$ the geodesic ball centered at p with radius R. Take a cut-off function ϕ of Li–Yau [7] satisfying $\mathrm{supp}(\phi) \subset B_p(2R)$, $\phi|_{B_p(R)} = 1$ and

$$\begin{aligned}
&\frac{|\nabla \phi|^2}{\phi} \leq \frac{C}{R^2}, \\
-\Delta\phi \leq \frac{C}{R^2}&\left(1 + \sqrt{K}\coth(\sqrt{K}R)\right) \leq \frac{C}{R^2}\left(1 + \sqrt{K} + \frac{1}{R}\right),
\end{aligned} \tag{2.11}$$

where C is a constant depending only on n.

Let $G = t\,\phi\,w$. Next, we are going to apply the maximum principle to G on $B_p(2R) \times [0, T]$. Assume G achieves its maximum at the point $(x_0, s) \in B_p(2R) \times [0, T]$ and assume $G(x_0, s) > 0$ (otherwise the proof is trivial), which implies $s > 0$. Then, at the point (x_0, s), it holds that

$$(\Delta - \partial_t)G \leq 0, \quad \nabla w = -\frac{w}{\phi}\nabla\phi$$

and

$$\begin{aligned}
0 &\geq (\Delta - \partial_t)G \\
&= s\,\phi(\Delta - \partial_t)\,w + \frac{\Delta\phi}{\phi}G + 2\,s\nabla\phi\,\nabla w - \frac{G}{s} \\
&\geq s\,\phi\Big\{2\,(1-f)w^2 - 2\,K\,w - 2\,\tilde{\lambda}\left(\alpha - \frac{-f}{1-f}\right)\mathrm{e}^{(\alpha-1)f}\,w \\
&\quad - 2\frac{-f}{1-f}\langle\nabla f, \nabla w\rangle\Big\} + \frac{\Delta\phi}{\phi}G + 2\,s\nabla\phi\nabla w - \frac{G}{s} \\
&= 2\,(1-f)\frac{G^2}{s\phi} - 2KG - 2\,\tilde{\lambda}\Big(\alpha - \frac{-f}{1-f}\Big)\mathrm{e}^{(\alpha-1)f}G \\
&\quad + 2\,\frac{-f}{1-f}\langle\nabla f, \nabla\phi\rangle\frac{G}{\phi} + \frac{\Delta\phi}{\phi}G - 2\frac{|\nabla\phi|^2}{\phi^2}G - \frac{G}{s},
\end{aligned} \tag{2.12}$$

where in the second inequality we used (2.10). Thus, multiplying both sides of (2.12) by $\frac{\phi}{G}$ yields

$$\begin{aligned}
0 &\geq 2\,(1-f)\frac{G}{s} - 2K\phi - 2\,\tilde{\lambda}\Big(\alpha - \frac{-f}{1-f}\Big)\mathrm{e}^{(\alpha-1)f}\phi \\
&\quad + 2\,\frac{-f}{1-f}\langle\nabla f, \nabla\phi\rangle + \Delta\phi - 2\,\frac{|\nabla\phi|^2}{\phi} - \frac{\phi}{s}.
\end{aligned} \tag{2.13}$$

Applying the Cauchy inequality

$$2\,\frac{-f}{1-f}\langle\nabla f, \nabla\phi\rangle \geq -(1-f)\frac{G}{s} - \frac{f^2}{1-f}\frac{|\nabla\phi|^2}{\phi} \tag{2.14}$$

into (2.13) gives

$$\begin{aligned}
0 &\geq (1-f)\frac{G}{s} - 2\,K\,\phi - 2\,\tilde{\lambda}\Big(\alpha - \frac{-f}{1-f}\Big)\mathrm{e}^{(\alpha-1)f}\phi \\
&\quad + \Delta\phi - \Big(2 + \frac{f^2}{1-f}\Big)\frac{|\nabla\phi|^2}{\phi} - \frac{\phi}{s}.
\end{aligned} \tag{2.15}$$

Thus, we have

$$\begin{aligned}(1-f)\,G(x,T) &\leq (1-f)\,G(x_0,s)\\ &\leq 2\,Ks\,\phi + 2\tilde{\lambda}\Big(\alpha - \frac{-f}{1-f}\Big)\mathrm{e}^{(\alpha-1)f}s\,\phi\\ &\quad - s\Delta\phi + \Big(2+\frac{f^2}{1-f}\Big)s\frac{|\nabla\phi|^2}{\phi} + \phi.\end{aligned} \tag{2.16}$$

Case one: $\alpha < 0$.

If $\lambda < 0$, then we have

$$0 < 1-(1-f)^{-1} = \frac{-f}{1-f} < 1.$$

Therefore, we obtain from (2.16) that

$$\begin{aligned}G(x,T) &\leq 2\,K\,T + 2\,\tilde{\lambda}\,(\alpha-1)\,\tilde{u}_{\min}^{\alpha-1}\,T\\ &\quad + \frac{C}{R^2}\left(1+\sqrt{K}+\frac{1}{R}\right)T+1\\ &\leq C\left(KT + \frac{T}{R^2} + 1 + \tilde{\lambda}\,(\alpha-1)\tilde{u}_{\min}^{\alpha-1}T\right).\end{aligned} \tag{2.17}$$

Notice that $\phi = 1$ in $B_p(R)$, $w = \frac{|\nabla f|^2}{(1-f)^2}$. Therefore, we obtain from (2.17)

$$\left.\frac{|\nabla f|^2}{(1-f)^2}\right|_{(x,t)} \leq C\left(K + \frac{1}{R^2} + \frac{1}{T} + \tilde{\lambda}\,(\alpha-1)\,\tilde{u}_{\min}^{\alpha-1}\right). \tag{2.18}$$

Since $f = \log(\frac{u}{u_{\max}})$ and $\tilde{\lambda} = \lambda\,u_{\max}^{\alpha-1}$, we have from (2.18)

$$\left.\frac{|\nabla u|^2}{u^2}\right|_{(x,t)} \leq C\left(K + \frac{1}{R^2} + \frac{1}{T} + \lambda\,(\alpha-1)\,u_{\min}^{\alpha-1}\right)\left(1+\log\frac{u_{\max}}{u}\right)^2. \tag{2.19}$$

If $\lambda > 0$, then we have from (2.16) that

$$\begin{aligned}G(x,T) &\leq 2\,K\,T + 2\,\tilde{\lambda}\,\alpha\,\tilde{u}_{\max}^{\alpha-1}\,T\\ &\quad + \frac{C}{R^2}\left(1+\sqrt{K}+\frac{1}{R}\right)T+1\\ &\leq C\left(KT + \frac{T}{R^2} + 1 + \tilde{\lambda}\,\alpha\,\tilde{u}_{\max}^{\alpha-1}\,T\right).\end{aligned} \tag{2.20}$$

Therefore, we obtain from (2.20)

$$\left.\frac{|\nabla u|^2}{u^2}\right|_{(x,t)} \leq C\left(K + \frac{1}{R^2} + \frac{1}{T} + \lambda\,\alpha\,u_{\max}^{\alpha-1}\right)\left(1+\log\frac{u_{\max}}{u}\right)^2. \tag{2.21}$$

Case two: $0 < \alpha < 1$.

If $\lambda < 0$, similarly, we obtain from (2.16) that

$$\begin{aligned}G(x,T) &\leq 2\,K\,T + 2\,\tilde{\lambda}\,(\alpha-1)\,\tilde{u}_{\min}^{\alpha-1}\,T\\ &\quad + \frac{C}{R^2}\left(1+\sqrt{K}+\frac{1}{R}\right)T+1\\ &\leq C\left(K\,T + \frac{T}{R^2} + 1 + \tilde{\lambda}\,(\alpha-1)\,\tilde{u}_{\min}^{\alpha-1}\,T\right).\end{aligned} \tag{2.22}$$

Therefore, we obtain from (2.22)

$$\left.\frac{|\nabla u|^2}{u^2}\right|_{(x,t)} \leq C\left(K + \frac{1}{R^2} + \frac{1}{T} + \lambda\,(\alpha-1)\,u_{\min}^{\alpha-1}\right)\left(1+\log\frac{u_{\max}}{u}\right)^2. \tag{2.23}$$

If $\lambda > 0$, then we have from (2.16) that

$$G(x,T) \le 2\,K\,T + 2\,\tilde{\lambda}\,\alpha\,\tilde{u}_{\min}^{\alpha-1}\,T + \frac{C}{R^2}\left(1+\sqrt{K}+\frac{1}{R}\right)T+1 \le C\left(K\,T+\frac{T}{R^2}+1+\tilde{\lambda}\,\alpha\,\tilde{u}_{\min}^{\alpha-1}\,T\right). \tag{2.24}$$

Therefore, we obtain from (2.24)

$$\left.\frac{|\nabla u|^2}{u^2}\right|_{(x,t)} \le C\left(K+\frac{1}{R^2}+\frac{1}{T}+\lambda\,\alpha\,u_{\min}^{\alpha-1}\right)\left(1+\log\frac{u_{\max}}{u}\right)^2. \tag{2.25}$$

Case three: $\alpha > 1$.

If $\lambda < 0$, similarly, we obtain from (2.16) that

$$G(x,T) \le 2\,K\,T + 2\,\tilde{\lambda}\,(\alpha-1)\,\tilde{u}_{\min}^{\alpha-1}\,T + \frac{C}{R^2}\left(1+\sqrt{K}+\frac{1}{R}\right)T+1 \le C\left(K\,T+\frac{T}{R^2}+1+\tilde{\lambda}\,(\alpha-1)\,\tilde{u}_{\min}^{\alpha-1}\,T\right). \tag{2.26}$$

Therefore, we obtain from (2.26)

$$\left.\frac{|\nabla u|^2}{u^2}\right|_{(x,t)} \le C\left(K+\frac{1}{R^2}+\frac{1}{T}+\lambda\,(\alpha-1)\,u_{\min}^{\alpha-1}\right)\left(1+\log\frac{u_{\max}}{u}\right)^2. \tag{2.27}$$

If $\lambda > 0$, then we have from (2.16) that

$$G(x,T) \le 2\,K\,T + 2\,\tilde{\lambda}\,\alpha\,\tilde{u}_{\max}^{\alpha-1}\,T + \frac{C}{R^2}\left(1+\sqrt{K}+\frac{1}{R}\right)T+1 \le C\left(K\,T+\frac{T}{R^2}+1+\tilde{\lambda}\,\alpha\,\tilde{u}_{\max}^{\alpha-1}\,T\right). \tag{2.28}$$

Therefore, we obtain from (2.28)

$$\left.\frac{|\nabla u|^2}{u^2}\right|_{(x,t)} \le C\left(K+\frac{1}{R^2}+\frac{1}{T}+\lambda\,\alpha\,u_{\max}^{\alpha-1}\right)\left(1+\log\frac{u_{\max}}{u}\right)^2. \tag{2.29}$$

This completes the proof of Theorem 1.1.

Proof of Corollary 1.2. If $\lambda < 0$ and $\alpha \in (1,+\infty)$ or $\lambda > 0$ and $\alpha \in (-\infty, 0)$, then from (1.5) and (1.6), we obtain

$$\frac{|\nabla u|^2}{u^2} \le C\left(K+\frac{1}{R^2}+\frac{1}{T}\right)\left(1+\log\frac{u_{\max}}{u}\right)^2, \tag{2.30}$$

which gives

$$\frac{|\nabla u|}{u} \le C\left(\sqrt{K}+\frac{1}{R}+\frac{1}{\sqrt{T}}\right)\left(1+\log\frac{u_{\max}}{u}\right). \tag{2.31}$$

By the assumption that the function $u+1$ satisfies $\log(u+1) = o(d(x)+\sqrt{|t|})$ near infinity, fixing a point (x_0,t_0) in space–time and using the estimate (2.31), we have

$$\frac{|\nabla u(x_0,t_0)|}{u(x_0,t_0)+1} \le \frac{C}{R}\cdot o(R).$$

Letting $R \to \infty$, it follows that $|\nabla u(x_0,t_0)| = 0$. Since (x_0,t_0) is arbitrary, we infer that u is a constant. This concludes the proof of Corollary 1.2. □

References

[1] S.-Y. Cheng, S.-T. Yau, Differential equations on Riemannian manifolds and their geometric applications, Commun. Pure Appl. Math. 28 (1975) 333–354.
[2] E.B. Davies, Heat Kernels and Spectral Theory, Cambridge Tracts in Math., vol. 92, Cambridge University Press, UK, 1989.
[3] R. Hamilton, A matrix Harnack estimate for the heat equation, Commun. Anal. Geom. 1 (1993) 113–125.
[4] G.Y. Huang, Z.J. Huang, H. Li, Gradient estimates for the porous medium equations on Riemannian manifolds, J. Geom. Anal. 23 (2013) 1851–1875.
[5] G.Y. Huang, Z.J. Huang, H. Li, Gradient estimates and differential Harnack inequalities for a nonlinear parabolic equation on Riemannian manifolds, Ann. Glob. Anal. Geom. 43 (2013) 209–232.
[6] G.Y. Huang, H. Li, Gradient estimates and entropy formulae of porous medium and fast diffusion equations for the Witten Laplacian, Pac. J. Math. 268 (2014) 47–78.
[7] P. Li, S.-T. Yau, On the parabolic kernel of the Schrödinger operator, Acta Math. 156 (1986) 153–201.
[8] J. Li, Gradient estimates and Harnack inequalities for nonlinear parabolic and nonlinear elliptic equations on Riemannian manifolds, J. Funct. Anal. 100 (1991) 233–256.
[9] J.F. Li, X.J. Xu, Differential Harnack inequalities on Riemannian manifolds I: linear heat equation, Adv. Math. 226 (2011) 4456–4491.
[10] L. Ma, Gradient estimates for a simple elliptic equation on complete non-compact Riemannian manifolds, J. Funct. Anal. 241 (2006) 374–382.
[11] P. Souplet, Q.S. Zhang, Sharp gradient estimate and Yau's Liouville theorem for the heat equation on noncompact manifolds, Bull. Lond. Math. Soc. 38 (2006) 1045–1053.
[12] X.J. Xu, Gradient estimates for $u_t = \Delta F(u)$ on manifolds and some Liouville-type theorems, J. Differ. Equ. 252 (2012) 1403–1420.
[13] Y.Y. Yang, Gradient estimates for a nonlinear parabolic equation on Riemannian manifolds, Proc. Amer. Math. Soc. 136 (2008) 4095–4102.
[14] Y.Y. Yang, Gradient estimates for the equation $\Delta u + cu^{-\alpha} = 0$ on Riemannian manifolds, Acta Math. Sin. Engl. Ser. 26 (2010) 1177–1182.
[15] J. Zhang, B.Q. Ma, Gradient estimates for a nonlinear equation $\Delta_f u + cu^{-\alpha} = 0$ on complete noncompact manifolds, Commun. Math. 19 (2011) 73–84.
[16] X.B. Zhu, Gradient estimates and Liouville theorems for nonlinear parabolic equations on noncompact Riemannian manifolds, Nonlinear Anal. 74 (2011) 5141–5146.