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Algebraic geometry

Parabolic subgroups and automorphism groups of Schubert varieties

Sous-groupes paraboliques et groupes d'automorphismes des variétés de Schubert

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A R T I C L E I N F O

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A B S T R A C T

Let G be a simple algebraic group of adjoint type over the field $\mathbb{C}$ of complex numbers, B be a Borel subgroup of G containing a maximal torus T of G. Let w be an element of the Weyl group W and $X(w)$ be the Schubert variety in G/B corresponding to w. In this article we show that given any parabolic subgroup P of G containing B properly, there is an element $w \in W$ such that P is the connected component, containing the identity element of the group of all algebraic automorphisms of $X(w)$.

R É S U M É

Soit G un groupe algébrique du type adjoint sur le corps des nombres complexes $\mathbb{C}$ et B un sous-groupe de Borel de G contenant un tore maximal T. Soit w un élément du groupe de Weil W et $X(w)$ la variété de Schubert dans G/B correspondant à w. Dans cet article, nous montrons que, pour tout sous-groupe parabolique P de G contenant B, il existe un élément w dans W tel que P est la composante connexe contenant l'élément identité du groupe des automorphismes algébriques de $X(w)$.

1. Introduction

Recall that if X is a projective variety over $\mathbb{C}$, the connected component containing the identity element of the group of all algebraic automorphisms of X is an algebraic group (see [12, Theorem 3.7, p. 17]). Let G be a simple algebraic group of adjoint type over $\mathbb{C}$. Let T be a maximal torus of G, and let R be the set of roots with respect to T. Let $R^+ \subset R$ be a set of positive roots. Let B^+ be the Borel subgroup of G containing T, corresponding to R^+. Let B be the Borel subgroup of G opposite to B^+ determined by T. For $w \in W$, let $X(w) := \overline{BwB/B}$ denote the Schubert variety in G/B

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corresponding to w. Let $Aut^0(X(w))$ denote the connected component containing the identity element of the group of all algebraic automorphisms of $X(w)$. Let α_0 denote the highest root of G with respect to T and B^+. For the left action of G on G/B, let P_w denote the stabiliser of $X(w)$ in G. If G is simply laced and $X(w)$ is smooth, then we have $P_w = Aut^0(X(w))$ if and only if $w^{-1}(\alpha_0) < 0$ (see [10, Theorem 4.2(2), p. 772]). Therefore, it is a natural question to ask whether, given any parabolic subgroup P of G containing B properly, there is an element $w \in W$ such that $P = Aut^0(X(w))$. In this article, we show that this question has an affirmative answer (see Theorem 2.1). If $P = B$, there is no such Schubert variety in G/B. We prove some partial results for Schubert varieties in partial flag varieties of type A_n. If P' is the maximal parabolic subgroup of $PSL(n+1,\mathbb{C})$ corresponding to the simple root α_1 or α_n, then G/P' is the projective space $\mathbb{P}^n$. The Schubert varieties in $\mathbb{P}^n$ are $\mathbb{P}^i$ $(0 \le i \le n)$. $\mathbb{P}^n$ is the only Schubert variety in $\mathbb{P}^n$ for which the action of B is faithful. Further, we have $Aut^0(\mathbb{P}^n) = PSL(n+1,\mathbb{C})$ (see Corollary 6.4). Therefore, the answer to the above question is negative if we consider partial flag varieties.

2. Notation and result

In this section, we set up some notation and preliminaries. We refer to [5], [7], [8], [9] for preliminaries in algebraic groups and Lie algebras.

Let G be a simple algebraic group of adjoint type over $\mathbb{C}$ and T be a maximal torus of G. Let $W = N_G(T)/T$ denote the Weyl group of G with respect to T and we denote the set of roots of G with respect to T by R. Let B^+ be a Borel subgroup of G containing T. Let B be the Borel subgroup of G opposite to B^+ determined by T. That is, $B = n_0 B^+ n_0^{-1}$, where n_0 is a representative in $N_G(T)$ of the longest element w_0 of W. Let $R^+ \subset R$ be the set of positive roots of G with respect to the Borel subgroup B^+. Note that the set of roots of B is equal to the set $R^- := -R^+$ of negative roots.

Let $S = \{\alpha_1, \ldots, \alpha_n\}$ denote the set of simple roots in R^+. For $\beta \in R^+$, we also use the notation $\beta > 0$. The simple reflection in W corresponding to α_i is denoted by s_{α_i}. Let $\mathfrak{g}$ be the Lie algebra of G. Let $\mathfrak{h} \subset \mathfrak{g}$ be the Lie algebra of T and $\mathfrak{b} \subset \mathfrak{g}$ be the Lie algebra of B. Let $X(T)$ denote the group of all characters of T. We have $X(T) \otimes \mathbb{R} = Hom_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}, \mathbb{R})$, the dual of the real form of $\mathfrak{h}$. The positive definite W-invariant form on $Hom_{\mathbb{R}}(\mathfrak{h}_{\mathbb{R}}, \mathbb{R})$ induced by the Killing form of $\mathfrak{g}$ is denoted by $(\ ,\ )$. We use the notation $\langle\ ,\ \rangle$ to denote $\langle \mu, \alpha \rangle = \frac{2(\mu,\alpha)}{(\alpha,\alpha)}$, for every $\mu \in X(T) \otimes \mathbb{R}$ and $\alpha \in R$. We denote by $X(T)^+$ the set of dominant characters of T with respect to B^+. Let ρ denote the half sum of all positive roots of G with respect to T and B^+. For any simple root α, we denote the fundamental weight corresponding to α by ω_α. For $1 \le i \le n$, let $h(\alpha_i) \in \mathfrak{h}$ be the fundamental coweight corresponding to α_i. That is, $\alpha_i(h(\alpha_j)) = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

For $w \in W$, let $l(w)$ denote the length of w. We define the dot action of W on $X(T) \otimes \mathbb{R}$ by

$$w \cdot \lambda = w(\lambda + \rho) - \rho, \text{ where } w \in W \text{ and } \lambda \in X(T) \otimes \mathbb{R}.$$

We set $R^+(w) := \{\beta \in R^+ : w(\beta) \in -R^+\}$. For $w \in W$, let $X(w) := \overline{BwB/B}$ denote the Schubert variety in G/B corresponding to w.

For a simple root α, we denote by P_α the minimal parabolic subgroup of G generated by B and n_α, where n_α is a representative of s_α in $N_G(T)$, and we denote by $P_{\hat{\alpha}}$ the maximal parabolic subgroup of G generated by B and $\{n_\beta : \beta \in S \setminus \{\alpha\}\}$, where n_β is a representative of s_β in $N_G(T)$. For a subset J of S, we denote by W_J the subgroup of W generated by $\{s_\alpha : \alpha \in J\}$. Let $W^J := \{w \in W : w(\alpha) \in R^+ \ for\ all\ \alpha \in J\}$. For each $w \in W_J$, choose a representative element $n_w \in N_G(T)$. Let $N_J := \{n_w : w \in W_J\}$. Let $P_J := BN_JB$.

Our main result in this article is the following.

Theorem 2.1. *Let G be a simple algebraic group of adjoint type over $\mathbb{C}$ and P be a parabolic subgroup of G containing B properly. Then there is an element $w \in W$ such that $P = Aut^0(X(w))$.*

Let $G = PSL(n+1,\mathbb{C})$. For $1 \le r \le n$ and $w \in W^{S \setminus \{\alpha_r\}}$, we denote the Schubert variety corresponding to w in the Grassmannian $G/P_{\hat{\alpha}_r}$, by $X_{P_{\hat{\alpha}_r}}(w)$.

Proposition 2.2. *Let $w = (s_{a_1} \cdots s_1)(s_{a_2} \cdots s_2) \cdots (s_{a_r} \cdots s_r) \in W(r)$. Let $J'(w) := \{i \in \{1, 2, \ldots, r-1\} : a_{i+1} - a_i \ge 2\}$, $J''(w) = \{1 + a_i : i \in J'(w)\}$ and $J(w) = \{\alpha_j : j \in \{1, \ldots, n\} \setminus J''(w)\}$. Then we have $P_{J(w)} = Aut^0(X_{P_{\hat{\alpha}_r}}(w))$.*

For more precise statement, see Proposition 6.2.

3. Proof of Theorem 2.1 except in three cases

In this section, we prove Theorem 2.1 in all cases except in three cases. The three cases left will be treated by Proposition 5.1.

Proof. Let P be a parabolic subgroup of G containing B properly. If $P = G$, then we take $w = w_0$, the longest element w_0 of W. In this case, we have the following:

$$Aut^0(X(w_0)) = Aut^0(G/B) = G \text{ (see [1, Theorem 2, p. 75])}.$$

Now we assume that P is any proper parabolic subgroup of G such that $B \subsetneq P \subsetneq G$. Since $B \subsetneq P \subsetneq G$, there is a subset $\emptyset \neq I \subsetneq S$ such that $P = P_I$. Consider $J = S \setminus I$. Hence, there exist unique elements $w_0^J \in W^J$ and $w_{0,J} \in W_J$ such that $w_0 = w_0^J \cdot w_{0,J}$. Consider the natural left action of G on G/B. Take $w = (w_0^J)^{-1}$. Then P is the stabiliser of $X(w)$, since $R^+(w^{-1}) \cap S = I$. The natural action of P on $X(w)$ induces a homomorphism,

$$\phi_w : P \longrightarrow Aut^0(X(w))$$

of algebraic groups.

We note that $\phi_w : P \longrightarrow Aut^0(X(w))$ is injective, since $w^{-1}(\alpha_0) < 0$ (see [10, Theorem 4.2(2), p. 772]).

Let $J' := -w_0(J)$, and $P' := P_{J'}$. Consider the natural morphism $\pi : G/B \longrightarrow G/P'$. We denote the restriction of π to $X(w)$ also by π. Then $\pi : X(w) \longrightarrow G/P'$ is a birational morphism. Therefore, by [5, Theorem 3.3.4(a), p. 96] and [5, Lemma 3.3.3(b), p. 95], we have:

$$\pi_*(\mathcal{O}_{X(w)}) = \mathcal{O}_{G/P'}.$$

Thus, from [4, Corollary 2.2., p. 45], π induces a homomorphism of algebraic groups:

$$\pi_* : Aut^0(X(w)) \longrightarrow Aut^0(G/P').$$

Since π is birational, $\pi_* : Aut^0(X(w)) \longrightarrow Aut^0(G/P')$ is injective.

If G is of type B_n, C_n or G_2, then $w_0 = -id$ (see [3, p. 216, p. 217, p. 233]). If G is of type B_n and $P = P_{\alpha_n}$, then $I = \{\alpha_n\}$. Therefore, $J' = -w_0(J) = J = S \setminus \{\alpha_n\}$ and $P' = P_{\hat{\alpha_n}}$. Thus, (G, P') is one of the three types as in the statement of [1, Theorem 2, p. 75]. If G is of the type C_n and $P = P_{\alpha_1}$, then $(G, P') = (G, P_{\hat{\alpha_1}})$ is one of the three types as in the statement of [1, Theorem 2, p. 75]. If G is of type G_2 and $P = P_{\alpha_1}$, then $(G, P') = (G, P_{\hat{\alpha_1}}) = (G, P_{\alpha_2})$ is one of the three types as in the statement of [1, Theorem 2, p. 75]. Similarly, we can see that if (G, P') is one of the three types as in [1, Theorem 2, p. 75], then (G, P) is one of the three types as in the statement of Proposition 5.1.

Case 1: G is not of type B_n, C_n and G_2. Then, for any parabolic subgroup P of G, (G, P) is not one of the three types as in Proposition 5.1. Therefore, (G, P') is not one of the three exceptional types as in the statement of [1, Theorem 2, p. 75].

Case 2: $G = B_n$ or C_n or G_2 and (G, P) is not one of the three types as in the statement of Proposition 5.1. In these cases, $w_0 = -id$ and $J' = -w_0(J) = J = S \setminus I$. Therefore, (G, P') is not one of the three exceptional types as in the statement of [1, Theorem 2, p. 75]. Thus (G, P) is not one of the three types as in the statement of Proposition 5.1 if and only if (G, P') is not one of the three exceptional types as in the statement of [1, Theorem 2, p. 75]. Hence, we have $Aut^0(G/P') = G$. Therefore, $Aut^0(X(w))$ is a parabolic subgroup of G containing P. Since P is the stabiliser of $X(w)$, we have $P = Aut^0(X(w))$. Now, the proof follows from Cases 1 and 2. □

4. Preliminaries for three left cases

Let V be a rational B-module. Let $\phi : B \longrightarrow GL(V)$ be the corresponding homomorphism of algebraic groups. The total space of the vector bundle $\mathcal{L}(V)$ on G/B is defined by the set of equivalence classes $\mathcal{L}(V) = G \times_B V$ corresponding to the following equivalence relation on $G \times V$:

$$(g, v) \sim (gb, \phi(b^{-1}) \cdot v) \text{ for } g \in G, b \in B, v \in V.$$

We denote the restriction of $\mathcal{L}(V)$ to $X(w)$ also by $\mathcal{L}(V)$. We denote the cohomology modules $H^i(X(w), \mathcal{L}(V))$ by $H^i(w, V)$ ($i \in \mathbb{Z}_{\geq 0}$). If $V = \mathbb{C}_\lambda$ is the one-dimensional representation $\lambda : B \longrightarrow \mathbb{C}^\times$ of B, then we denote $H^i(w, V)$ by $H^i(w, \lambda)$.

Let L_α denote the Levi subgroup of P_α containing T. Note that L_α is the product of T and the homomorphic image G_α of $SL(2, \mathbb{C})$ via a homomorphism $\psi : SL(2, \mathbb{C}) \longrightarrow L_\alpha$ (see [7, II, 1.3]). We denote the intersection of L_α and B by B_α. We note that the morphism $L_\alpha/B_\alpha \hookrightarrow P_\alpha/B$ induced by the inclusion $L_\alpha \hookrightarrow P_\alpha$ is an isomorphism. Therefore, to compute the cohomology modules $H^i(P_\alpha/B, \mathcal{L}(V))$ $(0 \leq i \leq 1)$ for any B-module V, we treat V as a B_α-module, and we compute $H^i(L_\alpha/B_\alpha, \mathcal{L}(V))$.

We use the following lemma to compute cohomology groups. The following lemma is due to Demazure (see [6, p. 1]). He used this lemma to prove Borel–Weil–Bott's theorem.

Lemma 4.1. *Let $w = \tau s_\alpha$, $l(w) = l(\tau) + 1$, and λ be a character of B. Then we have:*

(1) *if $\langle \lambda, \alpha \rangle \geq 0$, then $H^j(w, \lambda) = H^j(\tau, H^0(s_\alpha, \lambda))$ for all $j \geq 0$;*
(2) *if $\langle \lambda, \alpha \rangle \geq 0$, then $H^j(w, \lambda) = H^{j+1}(w, s_\alpha \cdot \lambda)$ for all $j \geq 0$;*
(3) *if $\langle \lambda, \alpha \rangle \leq -2$, then $H^{j+1}(w, \lambda) = H^j(w, s_\alpha \cdot \lambda)$ for all $j \geq 0$;*
(4) *if $\langle \lambda, \alpha \rangle = -1$, then $H^j(w, \lambda)$ vanishes for every $j \geq 0$.*

Let $\pi : \hat{G} \longrightarrow G$ be the simply connected covering of G. Let $\hat{L_\alpha}$ (respectively, $\hat{B_\alpha}$) be the inverse image of L_α (respectively, of B_α) in $\hat{G}$. Note that $\hat{L_\alpha}/\hat{B_\alpha}$ is isomorphic to L_α/B_α. We make use of this isomorphism to use the same notation for the vector bundle on L_α/B_α associated with a $\hat{B_\alpha}$-module. Let V be an irreducible $\hat{L_\alpha}$-module and λ be a character of $\hat{B_\alpha}$.

Then, we have the following lemma.

Lemma 4.2.

(1) *If* $\langle\lambda,\alpha\rangle \geq 0$*, then, the* $\hat{L_\alpha}$*-module* $H^0(L_\alpha/B_\alpha, V\otimes\mathbb{C}_\lambda)$ *is isomorphic to the tensor product of* V *and* $H^0(L_\alpha/B_\alpha,\mathbb{C}_\lambda)$*. Further, we have* $H^j(L_\alpha/B_\alpha, V\otimes\mathbb{C}_\lambda)=0$ *for every* $j\geq 1$*.*
(2) *If* $\langle\lambda,\alpha\rangle \leq -2$*, then, we have* $H^0(L_\alpha/B_\alpha, V\otimes\mathbb{C}_\lambda)=0$*. Further, the* $\hat{L_\alpha}$*-module* $H^1(L_\alpha/B_\alpha, V\otimes\mathbb{C}_\lambda)$ *is isomorphic to the tensor product of* V *and* $H^0(L_\alpha/B_\alpha,\mathbb{C}_{s_\alpha\cdot\lambda})$*.*
(3) *If* $\langle\lambda,\alpha\rangle = -1$*, then* $H^j(L_\alpha/B_\alpha, V\otimes\mathbb{C}_\lambda)=0$ *for every* $j\geq 0$*.*

Proof. By [9, I, Proposition 4.8, p. 53] and [9, I, Proposition 5.12, p. 77] for $j\geq 0$, we have the following isomorphism as $\hat{L_\alpha}$-modules:

$$H^j(L_\alpha/B_\alpha, V\otimes\mathbb{C}_\lambda)\simeq V\otimes H^j(L_\alpha/B_\alpha,\mathbb{C}_\lambda).$$

Now, the proof of the lemma follows from Lemma 4.1 by taking $w=s_\alpha$ and the fact that $L_\alpha/B_\alpha\simeq P_\alpha/B$. □

We now state the following Lemma on indecomposable $\hat{B_\alpha}$ (respectively, B_α) modules that will be used in computing the cohomology modules (see [2, Corollary 9.1, p. 30]).

Lemma 4.3.

(1) *Any finite-dimensional indecomposable* $\hat{B_\alpha}$*-module* V *is isomorphic to* $V'\otimes\mathbb{C}_\lambda$ *for some irreducible representation* V' *of* $\hat{L_\alpha}$*, and some character* λ *of* $\hat{B_\alpha}$*.*
(2) *Any finite dimensional indecomposable* B_α*-module* V *is isomorphic to* $V'\otimes\mathbb{C}_\lambda$ *for some irreducible representation* V' *of* $\hat{L_\alpha}$*, and some character* λ *of* $\hat{B_\alpha}$*.*

Proof. Proof of part (1) follows from [2, Corollary 9.1, p. 30].

Proof of part (2) follows from the fact that every B_α-module can be viewed as a $\hat{B_\alpha}$-module via the natural homomorphism. □

5. Proof of Theorem 2.1 in three left cases

To complete the proof of Theorem 2.1, it is sufficient to prove the following proposition. By (G,P), we mean that G is a simple algebraic group of adjoint type over $\mathbb{C}$ and P is a parabolic subgroup of G containing B.

Proposition 5.1. *Let* (G,P) *be one of the following types:*

(1) *G is of type* B_n *and* $P=P_{\alpha_n}$ *is the minimal parabolic subgroup of G corresponding to* α_n*;*
(2) *G is of type* C_n *and* $P=P_{\alpha_1}$ *is the minimal parabolic subgroup of G corresponding to* α_1*;*
(3) *G is of type* G_2 *and* $P=P_{\alpha_1}$ *is the minimal parabolic subgroup of G corresponding to* α_1*.*

Then, there exists an element $w\in W$ *such that* $P=Aut^0(X(w))$*.*

Proof. Let $T_{X(w)}$ be the tangent sheaf of $X(w)$. Let $T_{G/B}$ be the restriction of the tangent bundle to $X(w)$. Then $T_{X(w)}$ is a subsheaf of $T_{G/B}$ on $X(w)$. By [12, Lemma 3.4, p. 13], we have $Lie(Aut^0(X(w))=H^0(X(w),T_{X(w)})\subset H^0(X(w),T_{G/B})=H^0(w,\mathfrak{g}/\mathfrak{b})$.

As in the strategy of proof in Section 3, it is sufficient to prove that, for all the three types (G,P) as above, there is an element $w\in W$ such that

(i) P is the stabiliser of $X(w)$ in G;
(ii) $w^{-1}(\alpha_0)<0$;
(iii) $H^0(w,\mathfrak{g}/\mathfrak{b})=\mathfrak{g}$.

For instance, let $\phi_w: P\longrightarrow Aut^0(X(w))$ be the natural homomorphism induced by the action of P on $X(w)$.

Since $w^{-1}(\alpha_0)<0$, $\phi_w: P\longrightarrow Aut^0(X(w))$ is injective. Since $H^0(w,\mathfrak{g}/\mathfrak{b})=\mathfrak{g}$, we have $H^0(X(w),T_{X(w)})\subseteq\mathfrak{g}$. Therefore, $Aut^0(X(w))$ is a closed subgroup of G containing P. Since P is the stabiliser of $X(w)$ in G, we have $P=Aut^0(X(w))$.

We first make a note about statement (ii) and statement (iii). Let $w\in W$ be such that $w^{-1}(\alpha_0)<0$. To prove that $H^0(w,\mathfrak{g}/\mathfrak{b})=\mathfrak{g}$, it is sufficient to prove that for any negative root β, the dimension of the weight space $H^0(w,\mathfrak{g}/\mathfrak{b})_\beta$ is one.

The proof of this note is as follows.

The restriction of the natural map $\mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{b}$ to $\bigoplus_{\alpha\in R^+} \mathfrak{g}_\alpha$ is an isomorphism of T-modules and, hence, we have $\mathfrak{g}/\mathfrak{b} = \bigoplus_{\alpha\in R^+} \mathbb{C}_\alpha$. Since s_i permutes all positive roots other than α_i for every $1 \le i \le n$, every indecomposable B_{α_i}-summand V of $\mathfrak{g}/\mathfrak{b}$ with highest weight, a positive root different from α_i is indeed an $\hat{L}_{\alpha_i}$-module, and hence, for every $\alpha \in R^+ \setminus S$, the dimension of the weight space $H^0(s_i, \mathfrak{g}/\mathfrak{b})_\alpha$ is one. Using this argument and by induction on the length of w, we see that the dimension of the weight space $H^0(w, \mathfrak{g}/\mathfrak{b})_\alpha$ is one for every $\alpha \in R^+ \setminus S$. Further, since $(\mathfrak{g}/\mathfrak{b})_\alpha$ is one dimensional for every simple root α, each fundamental coweight $h(\alpha_i)$ $(1 \le i \le n)$ appears exactly once. Hence, it is sufficient to prove that, for any negative root β, the dimension of the weight space $H^0(w, \mathfrak{g}/\mathfrak{b})_\beta$ is one.

We prove the existence of an element $w \in W$ satisfying the first two conditions and that the dimension of the weight space $H^0(w, \mathfrak{g}/\mathfrak{b})_\beta$ is one for any negative root β in all the three cases, separately.

Case 1: assume that G is of type B_n and $P = P_n$. For every $1 \le r \le n-1$, let $v_r = s_n s_{n-1} \cdots s_r$. Take $w = v_1 v_2 \cdots v_{n-1}$. It is easy to see that P_n is the stabiliser of $X(w)$.

In this case, $\alpha_0 = \omega_2$. So, we have $v_1^{-1}(\alpha_0) = \alpha_2 + 2(\sum_{i=3}^{n} \alpha_i)$. This is the highest root of type B_{n-1} corresponding to the root system whose set of simple roots is $S \setminus \{\alpha_1\}$. By induction on the rank of G, we have $w^{-1}(\alpha_0) = (v_2 \cdots v_{n-1})^{-1}(\alpha_2 + 2(\sum_{i=3}^{n} \alpha_i)) < 0$.

Now, if $v \in W$ is of minimal length such that the dimension of $H^0(v, \mathfrak{g}/\mathfrak{b})_\beta$ is at least two for some negative root β, then $\beta = -(\sum_{j=i}^{n} \alpha_j)$ for some $1 \le i \le n-1$.

The justification of the above statement is as follows. Clearly, for any such v, $l(v) > 1$. Choose $\gamma \in S$ such that $l(s_\gamma v) = l(v) - 1$. Let $u = s_\gamma v$.

Then, we have $\dim H^0(s_\gamma, H^0(u, \mathfrak{g}/\mathfrak{b}))_\beta \ge 2$.

If $\langle \beta, \gamma \rangle = 1$, then there exists an indecomposable B_γ-summand V of $H^0(u, \mathfrak{g}/\mathfrak{b})$ such that $H^0(u, V)_\beta \ne 0$. In this case, either $V = \mathbb{C}_\beta \oplus \mathbb{C}_{\beta-\gamma}$ or $V = \mathbb{C}_\beta$.

So we have $\dim H^0(s_\gamma, H^0(u, \mathfrak{g}/\mathfrak{b}))_\beta = 1$.

If $\langle \beta, \gamma \rangle = -1$, we have either $V = \mathbb{C}_\beta \oplus \mathbb{C}_{\beta+\gamma}$ or $V = \mathbb{C}_{\beta+\gamma}$.

So we have $\dim H^0(s_\gamma, H^0(u, \mathfrak{g}/\mathfrak{b}))_\beta = 1$.

If $\langle \beta, \gamma \rangle = 2$, then there exists a unique indecomposable B_γ-summand V of $H^0(u, \mathfrak{g}/\mathfrak{b})$ with highest weight β.

Therefore, $\dim H^0(s_\gamma, H^0(u, \mathfrak{g}/\mathfrak{b}))_\beta = 1$.

If $\langle \beta, \gamma \rangle = -2$, then there exists a unique indecomposable B_γ-summand of $H^0(u, \mathfrak{g}/\mathfrak{b})$ with highest weight $\beta + 2\gamma$. Therefore, $\dim H^0(s_\gamma, H^0(u, \mathfrak{g}/\mathfrak{b}))_\beta = 1$.

Following the case-by-case analysis as above, we conclude that $\langle \beta, \gamma \rangle = 0$ and that there is a unique indecomposable B_γ-summand V of $H^0(u, \mathfrak{g}/\mathfrak{b})$ such that $V = \mathbb{C}_{\beta+\gamma} \oplus \mathbb{C}_\beta$. In particular, we have $\beta + \gamma \in R^-$. Since G is of type B_n, we have $\gamma = \alpha_n$ and $\beta = -(\sum_{j=i}^{n} \alpha_j)$ for some $1 \le i \le n-1$.

By induction on the rank of G, we may assume that $H^0(v_2 v_3 \cdots v_{n-1}, \mathfrak{g}/\mathfrak{b})_{-(\sum_{j=i}^{n} \alpha_j)}$ is one dimensional for every $2 \le i \le n-1$. Also $H^0(v_2 v_3 \cdots v_{n-1}, \mathfrak{g}/\mathfrak{b})_{-(\sum_{j=1}^{n} \alpha_j)} = 0$.

Since $\langle \sum_{j=i}^{n} \alpha_j, \alpha_1 \rangle = 0$ for every $3 \le i \le n-1$, the restriction of the evaluation map

$$H^0(s_1 v_2 v_3 \cdots v_{n-1}, \mathfrak{g}/\mathfrak{b})_{-(\sum_{j=i}^{n} \alpha_j)} \longrightarrow H^0(v_2 v_3 \cdots v_{n-1}, \mathfrak{g}/\mathfrak{b})_{-(\sum_{j=i}^{n} \alpha_j)}$$

is an isomorphism for every $3 \le i \le n-1$ (see Lemma 4.1 and Lemma 4.2).

Since $\langle -(\sum_{j=2}^{n} \alpha_j), \alpha_1 \rangle = 1$, we have

$$H^0(s_1 v_2 v_3 \cdots v_{n-1}, \mathfrak{g}/\mathfrak{b})_{-(\sum_{j=i}^{n} \alpha_j)} = H^0(s_1, H^0(v_2 v_3 \cdots v_{n-1}, \mathfrak{g}/\mathfrak{b}))_{-(\sum_{j=i}^{n} \alpha_j)}$$

is one dimensional for every $i = 1, 2$ (see Lemma 4.1 and Lemma 4.2).

Now, it is easy to see that, for every $2 \le r \le n$, the evaluation map

$$H^0(s_r s_{r-1} \cdots s_2, H^0(s_1 v_2 v_3 \cdots v_{n-1}, \mathfrak{g}/\mathfrak{b}))_{-(\sum_{j=i}^{n} \alpha_j)} \longrightarrow H^0(s_1 v_2 v_3 \cdots v_{n-1}, \mathfrak{g}/\mathfrak{b})_{-(\sum_{j=i}^{n} \alpha_j)}$$

is an isomorphism for every $1 \leq i \leq n$ by induction on r and using Lemma 4.1, Lemma 4.2. Thus, the space $H^0(w, \mathfrak{g}/\mathfrak{b})_\alpha$ is one dimensional for every negative root α.

Case 2: assume that G is of type C_n $(n \geq 3)$ and $P = P_1$. Take $w = s_1 s_2 \cdots s_n$. In this case we have $\alpha_0 = 2\omega_1$, and $w^{-1}(\alpha_0) = -\alpha_n$. Further, the stabiliser of $X(w)$ in G is P_1.

First, note that

$$H^0(s_n, \mathfrak{g}/\mathfrak{b}) = (\bigoplus_{\alpha \in R^+} \mathbb{C}_\alpha) \oplus \mathbb{C}_{h(\alpha_n)} \oplus \mathbb{C}_{-\alpha_n} \text{ (see Lemma 4.1 and Lemma 4.2).}$$

Further, we have:

$$\begin{aligned} H^0(s_{n-1}s_n, \mathfrak{g}/\mathfrak{b}) &= H^0(s_{n-1}, H^0(s_n, \mathfrak{g}/\mathfrak{b})) \\ &= (\bigoplus_{\alpha \in R^+} \mathbb{C}_\alpha) \oplus \mathbb{C}_{h(\alpha_n)} \oplus \mathbb{C}_{-\alpha_n} \oplus \mathbb{C}_{h(\alpha_{n-1})} \oplus \mathbb{C}_{-\alpha_{n-1}} \\ &\quad \oplus \mathbb{C}_{-(\alpha_{n-1}+\alpha_n)} \oplus \mathbb{C}_{-(2\alpha_{n-1}+\alpha_n)} \text{ (see Lemma 4.1 and Lemma 4.2).} \end{aligned}$$

By using Lemma 4.1, Lemma 4.2 and the descending induction on $1 \leq r \leq n-1$, we see that

$$H^0(s_r \cdots s_{n-1}s_n, \mathfrak{g}/\mathfrak{b}) = (\bigoplus_{\alpha \in R^+} \mathbb{C}_\alpha) \oplus (\bigoplus_{i=r}^{n} \mathbb{C}_{h(\alpha_i)}) \oplus \mathbb{C}_{-\mu},$$

where μ runs over all positive roots in $\sum_{i=r}^n \mathbb{Z}_{\geq 0}\alpha_i$. Thus, we have $H^0(w, \mathfrak{g}/\mathfrak{b}) = \mathfrak{g}$.

Case 3: assume that G is of type G_2 and $P = P_1$. Take $w = s_1 s_2 s_1 s_2$. Here, we follow the convention in [7]. In this case, we have $\alpha_0 = 3\alpha_1 + 2\alpha_2$. Further, $w^{-1}(\alpha_0) = -\alpha_2$.

First note that

$$H^0(s_2, \mathfrak{g}/\mathfrak{b}) = (\bigoplus_{\alpha \in R^+} \mathbb{C}_\alpha) \oplus \mathbb{C}_{h(\alpha_2)} \oplus \mathbb{C}_{-\alpha_2} \text{ (see Lemma 4.1 and Lemma 4.2),}$$

$$H^0(s_1, H^0(s_2, \mathfrak{g}/\mathfrak{b})) = (\bigoplus_{\alpha \in R^+} \mathbb{C}_\alpha) \oplus \mathbb{C}_{h(\alpha_2)} \oplus \mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{h(\alpha_1)} \oplus \mathbb{C}_{-\alpha_1} \oplus (\bigoplus_{i=1}^{3} \mathbb{C}_{-(\alpha_2+i\alpha_1)})$$

(see Lemma 4.1 and Lemma 4.2).

Therefore, we have:

$$H^0(s_1s_2, \mathfrak{g}/\mathfrak{b}) = (\bigoplus_{\alpha \in R^+} \mathbb{C}_\alpha) \oplus \mathbb{C}_{h(\alpha_2)} \oplus \mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{h(\alpha_1)} \oplus \mathbb{C}_{-\alpha_1} \oplus (\bigoplus_{i=1}^{3} \mathbb{C}_{-(\alpha_2+i\alpha_1)}),$$

$$H^0(s_2, H^0(s_1s_2, \mathfrak{g}/\mathfrak{b})) = (\bigoplus_{\alpha \in R^+} \mathbb{C}_\alpha) \oplus \mathbb{C}_{h(\alpha_2)} \oplus \mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{h(\alpha_1)} \oplus \mathbb{C}_{-\alpha_1} \oplus (\bigoplus_{i=1}^{3} \mathbb{C}_{-(\alpha_2+i\alpha_1)}) \oplus \mathbb{C}_{-(3\alpha_1+2\alpha_2)} = \mathfrak{g}$$

(see Lemma 4.1 and Lemma 4.2).

Therefore, we have:

$$H^0(s_2s_1s_2, \mathfrak{g}/\mathfrak{b}) = (\bigoplus_{\alpha \in R^+} \mathbb{C}_\alpha) \oplus \mathbb{C}_{h(\alpha_2)} \oplus \mathbb{C}_{-\alpha_2} \oplus \mathbb{C}_{h(\alpha_1)} \oplus \mathbb{C}_{-\alpha_1} \oplus (\bigoplus_{i=1}^{3} \mathbb{C}_{-(\alpha_2+i\alpha_1)}) \oplus \mathbb{C}_{-(3\alpha_1+2\alpha_2)}.$$

Thus, we have $H^0(w, \mathfrak{g}/\mathfrak{b}) = H^0(s_1, \mathfrak{g}) = \mathfrak{g}$. □

Example 5.2. Let $G = PSL(3, \mathbb{C})$. In this case, B is the set of invertible lower triangular matrices, $P_{\alpha_1} = Aut^0(X(s_1s_2))$ and $X(s_1s_2)$ is smooth.

Remark 5.3. In Theorem 2.1, for a given parabolic subgroup P of G containing B properly, the Schubert variety $X(w)$ for which $P = Aut^0(X(w))$ is not necessarily smooth. For example, take $G = PSL(4, \mathbb{C})$, and $P_{\alpha_2} = Aut^0(X(s_2s_1s_3s_2))$. Note that $X(s_2s_1s_3s_2)$ is not smooth (see [11, Theorem 2.2, p. 48]).

6. Automorphism groups of Schubert varieties in partial flag varieties of type A_n

In this section, we discuss about parabolic subgroups of $G = PSL(n+1, \mathbb{C})$ and the connected component containing the identity element of the group of all algebraic automorphisms of Schubert varieties in the Grassmannian $G/P_{\hat{\alpha}_r}$, where $1 \le r \le n$ and $P_{\hat{\alpha}_r} = P_{S\setminus\{\alpha_r\}}$.

Lemma 6.1. *Let $G = PSL(n+1, \mathbb{C})$. Let $1 \le r \le n$ and $w \in W^{S\setminus\{\alpha_r\}}$. Then $w^{-1}(\alpha_0) < 0$ if and only if there exists an increasing sequence $1 \le a_1 < a_2 < \cdots < a_r = n$ of positive integers such that $w = (s_{a_1} \cdots s_1)(s_{a_2} \cdots s_2) \cdots (s_{a_r} \cdots s_r)$.*

Proof. Note that $\alpha_0 = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Let $w \in W^{S\setminus\{\alpha_r\}}$ be such that $w \neq id$. Then there exists an integer $1 \le i \le r$ and an increasing sequence of positive integers $i \le a_i < a_{i+1} < \cdots < a_r \le n$ such that $w = (s_{a_i} \cdots s_i)(s_{a_{i+1}} \cdots s_{i+1}) \cdots (s_{a_r} \cdots s_r)$. Now, it is easy to see that $w^{-1}(\alpha_0) < 0$ if and only if $i = 1$ and $a_r = n$. $\square$

Let $W(r) = \{w \in W^{S\setminus\{\alpha_r\}} : w = (s_{a_1} \cdots s_1)(s_{a_2} \cdots s_2) \cdots (s_{a_r} \cdots s_r)$, where $1 \le a_1 < a_2 < \cdots < a_r = n\}$. For $w \in W^{S\setminus\{\alpha_r\}}$, we denote the Schubert variety in the Grassmannian $G/P_{\hat{\alpha}_r}$ corresponding to w by $X_{P_{\hat{\alpha}_r}}(w)$.

Proposition 6.2. *Let $w = (s_{a_1} \cdots s_1)(s_{a_2} \cdots s_2) \cdots (s_{a_r} \cdots s_r) \in W(r)$. Let $J'(w) := \{i \in \{1, 2, \ldots, r-1\} : a_{i+1} - a_i \ge 2\}$, $J''(w) = \{1 + a_i : i \in J'(w)\}$ and $J(w) = \{\alpha_j : j \in \{1, \ldots, n\} \setminus J''(w)\}$. Then we have $P_{J(w)} = Aut^0(X_{P_{\hat{\alpha}_r}}(w))$.*

Proof. Let P_w be the stabiliser of $X_{P_{\hat{\alpha}_r}}(w)$ in G. First, we show that $P_w = P_{J(w)}$. If $a_{i+1} - a_i \ge 2$ for some $1 \le i \le r-1$ then $s_{1+a_i} w > w$, and $s_{1+a_i} w \in W^{S\setminus\{\alpha_r\}}$. Hence, s_{1+a_i} is not in the Weyl group of P_w. Therefore, P_w is a subgroup of $P_{J(w)}$. Let $R(P_{\hat{\alpha}_r}) = R \cap (\sum_{\alpha \in S\setminus\{\alpha_r\}} \mathbb{Z}\alpha)$. Further, it is easy to see that, for $\alpha \in J(w)$, we have either $w^{-1}(\alpha) < 0$ or $w^{-1}(\alpha) \in R(P_{\hat{\alpha}_r})$. Therefore, $P_{J(w)} \subseteq P_w$.

Let $\psi_w : P_{J(w)} \longrightarrow Aut^0(X_{P_{\hat{\alpha}_r}}(w))$ be the natural homomorphism induced by the action of $P_{J(w)}$ on $X_{P_{\hat{\alpha}_r}}(w)$.

Since $w \in W(r)$, $w^{-1}(\alpha_0) < 0$ (see Lemma 6.1). Therefore, $\psi_w : P_{J(w)} \longrightarrow Aut^0(X_{P_{\hat{\alpha}_r}}(w))$ is injective.

Let $\mathfrak{p}_{\hat{\alpha}_r}$ be the Lie algebra of $P_{\hat{\alpha}_r}$. Since G is simply laced, the restriction map $H^0(w_{0,r}, \mathfrak{g}/\mathfrak{p}_{\hat{\alpha}_r}) \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{p}_{\hat{\alpha}_r})$ is surjective, where $w_{0,r} \in W^{S\setminus\{\alpha_r\}}$ is the minimal representative of w_0 (see [10, Lemma 3.5(3), p. 770]).

Further, since $w^{-1}(\alpha_0) < 0$, $H^0(w_{0,r}, \mathfrak{g}/\mathfrak{p}_{\hat{\alpha}_r}) = \mathfrak{g} \longrightarrow H^0(w, \mathfrak{g}/\mathfrak{p}_{\hat{\alpha}_r})$ is an isomorphism.

Therefore, we have $H^0(X_{P_{\hat{\alpha}_r}}(w), T_{X_{P_{\hat{\alpha}_r}}(w)}) \subseteq \mathfrak{g}$. Hence $Aut^0(X_{P_{\hat{\alpha}_r}}(w))$ is a closed subgroup of G containing $P_{J(w)}$. Thus we have $P_{J(w)} = Aut^0(X_{P_{\hat{\alpha}_r}}(w))$. $\square$

Corollary 6.3. *Let $B \subsetneq P$ be a parabolic subgroup of G and $w \in W^{S\setminus\{\alpha_r\}}$ such that $P = Aut^0(X_{P_{\hat{\alpha}_r}}(w))$. Then we have $P = P_{J(w)}$.*

Corollary 6.4.

(1) *If $P \neq G$, then there is no element $w \in W^{S\setminus\{\alpha_1\}}$ such that $P = Aut^0(X_{P_{\hat{\alpha}_1}}(w))$.*
(2) *If $P \neq G$, then there is no element $w \in W^{S\setminus\{\alpha_n\}}$ such that $P = Aut^0(X_{P_{\hat{\alpha}_n}}(w))$.*

Proof. Proof of (1): the Schubert varieties in $G/P_{\hat{\alpha}_1}$ are projective space $\mathbb{P}^i$ $(0 \le i \le n)$. Therefore the automorphism groups of these Schubert varieties are $PSL(i+1, \mathbb{C})$ $(0 \le i \le n)$. Further, the map ϕ_w is injective for only one w.

Proof of (2): it is similar to that of (1). $\square$

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