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Deformation cohomology of Lie algebroids and Morita equivalence

Cohomologie de déformation d'un algèbroïde de Lie et équivalence de Morita

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ABSTRACT

Let $A \Rightarrow M$ be a Lie algebroid. In this short note, we prove that a pull-back of A along a fibration with homologically m-connected fibers shares the same deformation cohomology of A up to degree m.

RÉSUMÉ

Soit $A \Rightarrow M$ un algébroïde de Lie. Dans cette note, nous prouvons qu'un *pull-back* de A lelong d'une fibration ayant des fibres homologiquement m-connexes possède la même cohomologie de déformation que A jusqu'au degré m.

0. Introduction

Lie groupoids can be understood as atlases on certain singular spaces, specifically, *differentiable stacks*, and (by the very definition of stack) two Lie groupoids are Morita equivalent if they give rise to the same differentiable stack [2]. This means that, when using Lie groupoids to model differentiable stacks, Morita invariants describe the intrinsic geometry of the stack. For instance, Lie groupoid cohomology, and the deformation cohomology of a Lie groupoid are Morita invariants, but there are more many examples. The terminology is motivated by the fact that the relationship between a Lie groupoid and its stack is analogous to the relationship between a ring and its category of modules.

Lie algebroids are infinitesimal counterparts of Lie groupoids. However, the former are more general than the latter in the sense that, while all Lie groupoids *differentiate* to a Lie algebroid, not all Lie algebroids *integrate* to a Lie groupoid. A consequence of this is that there is not a notion of Morita equivalence of Lie algebroids which is universally good, but there are several non-equivalent alternatives. The weakest (but reasonable) possible one is the *weak Morita equivalence* introduced

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by Ginzburg in [9]. For Poisson manifolds, this notion is weaker than Xu's Morita equivalence [14], but it makes sense for non-integrable Poisson manifolds.

In a similar way as for Lie groupoids, it is important to identify as many Morita invariants of Lie algebroids as possible. In [4], Crainic proves (a statement equivalent to the fact) that if two Lie algebroids are Morita equivalent in a suitable sense, then they share the same de Rham cohomology in low degree. In this note, we prove the analogous result for the deformation cohomology of Lie algebroids. Notice that, for Lie groupoids, the Morita invariance of Lie groupoid cohomology has been proved by Crainic himself in [4], while the deformation cohomology has been introduced, and its Morita invariance has been proved, only very recently, by Crainic, Mestre, and Struchiner in [5].

We assume that the reader is familiar with Lie algebroids and their description in terms of graded manifolds. We only recall that a degree k $\mathbb{N}$-manifold is a graded manifold whose coordinates are concentrated in non-negative degree up to degree k, and an $\mathbb{N}Q$-manifold is an $\mathbb{N}$-manifold equipped with an homological vector field. For instance, if $A \Rightarrow M$ is a Lie algebroid, then shifting by one the degree of the fibers of the vector bundle $A \to M$, we get a degree 1 $\mathbb{N}Q$-manifold whose homological vector field is the de Rham differential d_A of A. Correspondence $A \rightsquigarrow A[1]$ establishes an equivalence between the category of Lie algebroids and the category of degree-1 $\mathbb{N}Q$-manifolds.

1. The deformation complex of a Lie algebroid

Let $A \Rightarrow M$ be a Lie algebroid. In degree k, the deformation complex of A, denoted by $(C_{\mathrm{def}}(A), \delta)$, consists of $(k+1)$-multiderivations of A, i.e. $\mathbb{R}$-$(k+1)$-linear maps

$$c : \Gamma(A) \times \cdots \times \Gamma(A) \to \Gamma(A)$$

such that there exists a (necessarily unique) vector bundle map $s_c : \wedge^k A \to TM$ with c and s_c satisfying the following Leibniz rule

$$c(\alpha_1, \dots, \alpha_k, f\alpha_{k+1}) = s_c(\alpha_1, \dots, \alpha_k)(f)\alpha_{k+1} + f c(\alpha_1, \dots, \alpha_k, \alpha_{k+1}),$$

for all $\alpha_1, \dots, \alpha_{k+1} \in \Gamma(A)$, and $f \in C^\infty(M)$. The differential δ is then defined as

$$\begin{aligned}\delta c(\alpha_0, \dots, \alpha_{k+1}) = &\sum_i (-)^i [\alpha_i, c(\alpha_0, \dots, \widehat{\alpha_i}, \dots, \alpha_{k+1})] \\ &+ \sum_{i<j} (-)^{i+j} c([\alpha_i, \alpha_j], \alpha_0, \dots, \widehat{\alpha_i}, \dots, \widehat{\alpha_j}, \dots, \alpha_{k+1}),\end{aligned}$$

for all $\alpha_0, \dots, \alpha_{k+1} \in \Gamma(A)$. The cohomology of $C_{\mathrm{def}}(A)$ is called the *deformation cohomology* of A and it is denoted by $H_{\mathrm{def}}(A)$ [7].

Actually, $C_{\mathrm{def}}(A)$ is not just a complex, but it is the DG Lie algebra (even more a DG Lie–Rinehart algebra over the de Rham algebra of A) controlling deformations of A, in the sense that

- ▹ Lie algebroid structures on A corresponds bijectively to Maurer–Cartan elements in $C_{\mathrm{def}}(A)$, and
- ▹ if two Lie algebroid structures are isotopic, the corresponding Maurer–Cartan elements are gauge equivalent, and the converse is also true when M is compact.

There is a simple alternative description of $C_{\mathrm{def}}(A)$ as the DG Lie algebra of graded derivations of the de Rham algebra $(C(A), \mathrm{d}_A)$, where $C(A) = \Gamma(\wedge^\bullet A^*)$, and d_A is the usual Lie algebroid de Rham differential. A cochain $c \in C^k_{\mathrm{def}}(A)$ corresponds to the degree k derivation D_c mapping $\omega \in C^l(A)$ to $D_c\omega \in C^{k+l}(A)$, with

$$\begin{aligned}D_c\omega(\alpha_1, \dots, \alpha_{k+l}) = &\sum_{\sigma \in S_{k,l}} (-)^\sigma s_c(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(k)})(\omega(\alpha_{\sigma(k+1)}, \dots \alpha_{\sigma(k+l)})) \\ &- \sum_{\sigma \in S_{k+1,l-1}} (-)^\sigma \omega(c(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(k+1)}), \alpha_{\sigma(k+2)}, \dots \alpha_{\sigma(k+l)}).\end{aligned}$$

When taking this point of view, the Lie bracket in $C_{\mathrm{def}}(A)$ is just the graded commutator $[-,-]$ of derivations and δ is $[\mathrm{d}_A, -]$. Finally, we can interpret $C(A)$ as the DG algebra of smooth functions on the degree 1 $\mathbb{N}Q$-manifold $A[1]$, and then cochains in $C_{\mathrm{def}}(A)$ are just vector fields on $A[1]$:

$$C(A) = C^\infty(A[1]), \quad \text{and} \quad C_{\mathrm{def}}(A) = \mathfrak{X}(A[1]).$$

In the following, we will mostly take this point of view.

Given two Lie algebroids $A \Rightarrow M$ and $B \Rightarrow N$ and a Lie algebroid map $F : A \to B$ covering a smooth map $M \to N$, there is a DG algebra map $F^* : C(B) \to C(A)$. One can also connect the deformation complexes as follows. Apply the shift functor to

F to get a map of $\mathbb{N}Q$-manifolds: $F[1]: A[1] \to B[1]$, and denote by $C(F)$ the space of $F[1]$-relative vector fields, i.e. graded $\mathbb{R}$-linear maps $Z: C(B) \to C(A)$ satisfying the following Leibniz rule

$$Z(\omega_1 \wedge \omega_2) = Z(\omega_1) \wedge F^*(\omega) + (-)^{|Z||\omega_1|} F^*(\omega_1) \wedge Z(\omega_2).$$

In other words, $C(F) = C(A) \otimes C_{\mathrm{def}}(B)$, where the tensor product is over $C(B)$, and we changed the scalars via $F^*: C(B) \to C(A)$. Yet in other (more geometric) terms, $C(F)$ consists of sections of the graded vector bundle $F^*(TB[1]) \to A[1]$. Clearly, $C(F)$ is a DG $(C(A), C(B))$-bimodule, whose differential $\delta: C(F) \to C(F)$ is given by

$$\delta Z = \mathrm{d}_A \circ Z - (-)^Z Z \circ \mathrm{d}_B, \quad Z \in C(F).$$

Additionally, there are DG module maps:

$$C_{\mathrm{def}}(A) \xrightarrow{F_\star} C(F) \xleftarrow{F^\star} C_{\mathrm{def}}(B)$$

given by

$$F_\star : X \mapsto X \circ F^*, \quad \text{and} \quad F^\star : Y \mapsto F^* \circ Y.$$

2. Morita equivalence of Lie algebroids

There is no *universally good* notion of Morita equivalence for Lie algebroids. Actually, there are several morally similar but inequivalent definitions, all of which involve the notion of *pull-back Lie algebroid*. Let $A \Rightarrow M$ be a Lie algebroid with anchor $\rho: A \to TM$, and let $\pi: P \to M$ be a surjective submersion. Put

$$\pi^! A := TP \,{}_{\mathrm{d}\pi}\!\times_\rho A = \{(v, a) \in TP \times A : \mathrm{d}\pi(v) = \rho(a)\}.$$

Then $\pi^! A$ is a Lie algebroid over P in the following way. First of all, sections of $\pi^! A \to P$ are pairs (X, α), where X is a vector field on P and α is a section of the pull-back bundle $\pi^* A \to P$. Additionally $\mathrm{d}\pi(X_p) = \rho(\alpha_p)$ for all $p \in P$. It is easy to see that there exists a unique Lie algebroid structure $\pi^! A \Rightarrow P$ such that the anchor $\pi^! A \to TP$ is the projection $(X, \alpha) \mapsto X$, and the bracket is

$$\big[(X, \pi^*\alpha), (Y, \pi^*\beta)\big] = \big([X, Y], \pi^*[\alpha, \beta]\big)$$

on sections of the special form $(X, \pi^*\alpha), (Y, \pi^*\beta)$, with $\alpha, \beta \in \Gamma(A)$. It follows that the natural projection $\Pi: \pi^! A \to A$ is a Lie algebroid map (covering $\pi: P \to M$). In particular, there are DG module maps

$$C_{\mathrm{def}}(\pi^! A) \xrightarrow{\Pi_\star} C(\Pi) \xleftarrow{\Pi^\star} C_{\mathrm{def}}(A).$$

It is worth remarking, for later applications, that there is a short exact sequence of vector bundles over P,

$$0 \longrightarrow VP \longrightarrow \pi^! A \longrightarrow \pi^* A \longrightarrow 0, \tag{1}$$

where VP is the vertical tangent bundle of $P \to M$, and the inclusion $VP \hookrightarrow \pi^! A$ maps a vertical vector field X to $(X, 0)$. Clearly, the projection $\pi^! A \to \pi^* A$ maps (X, α) to α.

Definition 2.1 *(Ginzburg [9]).* Two Lie algebroids $A \Rightarrow M$ and $B \Rightarrow N$ are (*weak*) *Morita equivalent* if there exist surjective submersions

$$M \xleftarrow{\pi} P \xrightarrow{\tau} N$$

with simply connected fibers, such that the pull-back Lie algebroids $\pi^! A$ and $\tau^! B$ are isomorphic.

Remark 1. Let $A \Rightarrow M$ be a Lie algebroid, $P \to M$ be a surjective submersion and let $E \to M$ be a vector bundle carrying a representation of A. Then $\pi^* E \to P$ carries a representation of the pull-back Lie algebroid $\pi^! A$. Definition 2.1, originally due to Ginzburg, is then motivated by the fact that, if π has simply connected fibers, correspondence $E \rightsquigarrow \pi^* E$ establishes an equivalence between the categories of A-representations and of $\pi^! A$-representations. However, other reasonable definitions of Morita equivalence are possible. For instance, one could require submersions π, τ to have fibers with specific, higher connectedness (or even cohomological connectedness) properties.

Remark 2. Let $A \Rightarrow M$ and $B \Rightarrow N$ be weak Morita equivalent Lie algebroids, and let $A \leftarrow P \to B$ be surjective submersions realizing the equivalence. Then A and B share several properties. For instance, there is a bijection between their leaf spaces, and corresponding leaves have the same fundamental group. Additionally, the Lie algebroid structures *transverse* to corresponding leaves are isomorphic and so are the stabilizer of A at $x \in M$ and the stabilizer of B at $y \in N$ whenever x, y are projections of the same point in P [9]. Finally, A and B share the same 0-th and 1-st de Rham cohomology (see also Theorem 3.1 below).

We conclude this section by describing the de Rham complex of a pull-back Lie algebroid $\pi^! A \Rightarrow P$. To do this, we will interpret $C(A)$ as a DG algebra over differential forms on M via the pull-back $\rho^* : \Omega(M) \to C(A)$ along the anchor.

Lemma 2.2. *There are canonical isomorphisms of DG modules*

$$C(\pi^! A) = \Omega(P) \underset{\Omega(M)}{\otimes} C(A) \quad \textit{and} \quad C(\Pi) = \Omega(P) \underset{\Omega(M)}{\otimes} C_{\mathrm{def}}(A). \tag{2}$$

Proof. The simplest proof is via graded geometry. Consider the pull-back diagram

$$\begin{array}{ccc} \pi^! A & \xrightarrow{\rho} & TP \\ {\scriptstyle \Pi}\downarrow & & \downarrow{\scriptstyle \mathrm{d}\pi} \\ A & \xrightarrow{\rho} & TM \end{array} .$$

The top row consists of Lie algebroids over P, while the bottom row consists of Lie algebroids over M. Shifting by 1 the degree in the fibers of all of them, we get a diagram of $\mathbb{N}Q$-manifolds

$$\begin{array}{ccc} \pi^! A[1] & \longrightarrow & T[1]P \\ \downarrow & & \downarrow \\ A[1] & \longrightarrow & T[1]M \end{array} . \tag{3}$$

It follows from the functorial properties of the shift that (3) is a pull-back diagram as well. Hence functions on $\pi^! A[1]$ are the tensor product over functions on $T[1]M$ of the functions on $T[1]P$ and the functions on $A[1]$. This is precisely the content of the first isomorphism in (2). The second isomorphism immediately follows from the first one and the fact that $\Pi[1]$-relative vector fields are the tensor product over functions on $A[1]$ of vector fields on $A[1]$ and functions on $\pi^! A[1]$.

3. Morita invariance of the deformation cohomology

The de Rham cohomologies of Lie algebroids are Morita invariant. More precisely, we have the following theorem due to Crainic.

Theorem 3.1 *(Crainic [4]). Let $A \Rightarrow M$ be a Lie algebroid and let $\pi : P \to M$ be a surjective submersion with homologically m-connected fibers, then A and the pull-back Lie algebroid $\pi^! A$ share the same de Rham cohomology up to degree m. Specifically, the graded vector space map $\Pi^* : H_{\mathrm{dR}}(A) \to H_{\mathrm{dR}}(\pi^! A)$ is an isomorphism in degree $q \leq m$.*

Proof. We briefly recall Crainic's proof. This will be useful in the following. Begin noticing that the inclusion $VP \hookrightarrow \pi^! A$ is the inclusion of a Lie subalgebroid. Accordingly, there is a distinguished subcomplex, and an ideal, $F_1 C \subset C(\pi^! A)$ consisting of cochains vanishing when applied to sections of VP. Hence $C(\pi^! A)$ is canonically equipped with a filtration

$$C(\pi^! A) = F_0 C \supset F_1 C \supset \cdots \supset F_p C \supset \cdots$$

where $F_p C$ is the p-th exterior power of $F_1 C$. We denote by E the associated (first quadrant) spectral sequence which computes $H_{\mathrm{dR}}(\pi^! A)$. It follows from the short exact sequence (1) that

$$E_0^{p,q} = F_p C^{p+q} / F_{p+1} C^{p+q} = V\Omega^q \otimes C^p(A)$$

where $V\Omega = \Gamma(\wedge^\bullet V^* P)$ are vertical differential forms on P, and the tensor product is over $C^\infty(M)$. Now, it is easy to see that differential

$$\mathrm{d}_0^{p,\bullet} : V\Omega^\bullet \otimes C^p(A) \to V\Omega^{\bullet+1} \otimes C^p(A)$$

is just the vertical de Rham differential d^V (up to tensoring by $C^p(A)$), and, from the connectedness hypothesis, we have $H^0(V\Omega, \mathrm{d}^V) = C^\infty(M)$, and $H^q(V\Omega, \mathrm{d}^V) = 0$ for $0 < q \leq m$. Hence $E_1^{\bullet,0} = C(A)$, and

$$E_1^{\bullet,q} = H^q(V\Omega^\bullet \otimes C(A), \mathrm{d}^V) = 0 \quad \text{for } 0 < q \leq m.$$

Additionally $\mathrm{d}_1^{\bullet,0} : C(A) \to C(A)$ is precisely the de Rham differential d_A. We conclude that

$$E_2^{p,q} = E_\infty^{p,q} = 0 \quad \text{for } 1 < p + q \leq m,$$

and

$$H^q_{\mathrm{dR}}(A) = E_2^{0,q} = E_\infty^{0,q} = H^q_{\mathrm{dR}}(\pi^! A) \quad \text{for } q \leq m.$$

We now come to our main result. The following theorem (and its corollary) is our version of the Morita invariance of the deformation cohomology.

Theorem 3.2. *Let $A \Rightarrow M$ be a Lie algebroid and let $\pi : P \to M$ be a surjective submersion with homologically m-connected fibers, then A and the pull-back Lie algebroid $\pi^! A$ share the same deformation cohomology up to degree m.*

Proof. The present proof is inspired by the Crainic, Mestre, and Struchiner proof of the Morita invariance of the deformation cohomology of Lie groupoids [5]. However, notice that, in our statement, the Lie algebroid A needs not to be integrable. Consider the DG module maps

$$C_{\mathrm{def}}(\pi^! A) \xrightarrow{\Pi_\star} C(\Pi) \xleftarrow{\Pi^\star} C_{\mathrm{def}}(A).$$

Our strategy consists in proving that

(i) $\Pi_\star$ is a quasi-isomorphism (regardless the connectedness properties of the fibers of π);
(ii) $\Pi^\star$ induces an isomorphism in cohomology up to degree m.

We begin with (i). As π is a surjective submersion, then $\Pi_\star$ is surjective. Actually it consists in restricting a derivation of $C(\pi^! A)$ to the DG subalgebra $C(A) \hookrightarrow C(\pi^! A)$. Geometrically, it consists in composing a vector field on $\pi^! A[1]$ with projection $d\Pi[1] : T\pi^! A[1] \to TA[1]$. Denote $\mathcal{K} := \ker \Pi_\star$ and consider the short exact sequence of DG modules

$$0 \longrightarrow \mathcal{K} \longrightarrow C_{\mathrm{def}}(\pi^! A) \xrightarrow{\Pi_\star} C(\Pi) \longrightarrow 0.$$

It is enough to show that $\mathcal{K}$ is acyclic. To do this, we construct a contracting homotopy $h : \mathcal{K} \to \mathcal{K}$ for $(\mathcal{K}, \delta)$. Notice that $\mathcal{K}$ consists of derivations of $C(\pi^! A)$ vanishing on $C(A)$, equivalently it consists of vector fields on $\pi^! A[1]$ that are the vertical wrt projection $\Pi[1] : \pi^! A[1] \to A[1]$. As (3) is a pull-back diagram, the vector fields in $\mathcal{K}$ are completely determined by their composition with $T\pi^! A[1] \to TT[1]P$. This shows that there is a DG module isomorphism

$$\mathcal{K} \xrightarrow{\simeq} C(\pi^! A) \underset{\Omega(P)}{\otimes} \mathcal{V} = C(A) \underset{\Omega(M)}{\otimes} \mathcal{V}$$

where $\mathcal{V}$ is the DG $\Omega(P)$-module of vector fields on $T[1]P$ that are vertical with respect to projection $T[1]P \to T[1]M$. Now we construct a contracting homotopy $h_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$ for $\mathcal{V}$. First recall that the differential in $\mathcal{V}$ is the commutator with the de Rham differential $d : \Omega(P) \to \Omega(P)$. Now, every vector field V on $T[1]P$ is a derivation of $\Omega(P)$. Hence, it can be uniquely written in the form $V = \mathcal{L}_J + i_K$, where J, K are form valued vector fields on P. Additionally, V is vertical with respect to $T[1]P \to T[1]M$ if and only if J, K are both vertical wrt $P \to M$. Put $h_{\mathcal{V}}(V) := (-)^{|V|} i_J$, so that $V = [d, h_{\mathcal{V}}(V)] + h_{\mathcal{V}}([d, V])$, showing that $h_{\mathcal{V}}$ is indeed a contracting homotopy. It is easy to see that $h_{\mathcal{V}}$ is $\Omega(M)$-linear and we define

$$h : C(A) \underset{\Omega(M)}{\otimes} \mathcal{V} \to C(A) \underset{\Omega(M)}{\otimes} \mathcal{V}, \quad \omega \otimes V \mapsto (-)^{|\omega|} \omega \otimes h_{\mathcal{V}}(V).$$

A straightforward computation shows that h is a contracting homotopy. Hence $\mathcal{K}$ is acyclic.

Now we prove (ii). The proof is analogous to that of Theorem 3.1. As $C(\Pi)$ is a DG $C(\pi^! A)$-module, there is a filtration

$$C(\Pi) = F_0 C \cdot C(\Pi) \supset F_1 C \cdot C(\Pi) \supset \cdots \supset F_p C \cdot C(\Pi) \supset \cdots$$

hence a spectral sequence computing the cohomology of $C(\Pi)$, that we denote again by E. From Lemma 2.2 and from (1) again, we have

$$E_0^{p,q} = F_p C^{p+q} \cdot C(\Pi) / F_{p+1} C^{p+q} \cdot C(\Pi) = V\Omega^q \otimes C^p_{\mathrm{def}}(A),$$

where the tensor product is over $C^\infty(M)$, and the differential $d_0 : E_0 \to E_0$ is the vertical de Rham differential $d^V : V\Omega \to V\Omega$ (tensorized by $C^\bullet_{\mathrm{def}}(A)$). Now the proof proceeds exactly as the proof of Theorem 3.1, and we leave the details to the reader.

Corollary 3.3. *Let $A \Rightarrow M$ and $B \Rightarrow N$ be (weak) Morita equivalent Lie algebroids. Then A and B share the same* 0*-th and* 1*-st deformation cohomology. If, additionally, the Morita equivalence is realized by surjective submersions $M \leftarrow P \to N$ with homologically m-connected fibers, then A and B share the same deformation cohomology up to degree m.*

4. An illustrative example: deformations of weak Morita equivalent foliations

Let M be a manifold and let $\mathcal{F}$ be a foliation of M. The deformations of $\mathcal{F}$ are controlled by $\Omega(\mathcal{F}, TM/T\mathcal{F})$: leafwise differential forms with values in the Bott representation [11]. Cochain complex $\Omega(\mathcal{F}, TM/T\mathcal{F})$ is a deformation retract of the deformation complex $C_{\mathrm{def}}(T\mathcal{F})$ [7,13]. In particular, $\Omega(\mathcal{F}, TM/T\mathcal{F})$ and $C_{\mathrm{def}}(T\mathcal{F})$ share the same cohomology. This means that, morally, deforming a foliation $\mathcal{F}$ or its tangent algebroid $T\mathcal{F}$ is the same. This should be expected from the fact that small deformations of $T\mathcal{F}$ preserve the injectivity of the anchor.

Now let $\mathcal{V} \subset \mathcal{H}$ be a flag of foliations of a manifold P. In other words, $\mathcal{V}$ and $\mathcal{H}$ are foliations, and the leaves of $\mathcal{V}$ are contained into leaves of $\mathcal{H}$. Yet in other terms $T\mathcal{V} \subset T\mathcal{H}$. Assume that $\mathcal{V}$ is simple, i.e. its leaf space M is a manifold and the projection $\pi : P \to M$ is a surjective submersion. In other words, $\mathcal{V} = VP$: the vertical bundle of P with respect to π. From involutivity $\pi_*(T\mathcal{H}) = T\mathcal{F}$ for a, necessarily unique, foliation $\mathcal{F}$ of M, and $T\mathcal{H} = (d\pi)^{-1}(T\mathcal{F})$. It is then immediate to see that $T\mathcal{H} = \pi^! T\mathcal{F}$: the pull-back Lie algebroid. So, if P has homologically m-connected fibers, $T\mathcal{F}$ and $T\mathcal{H}$ share the same deformation cohomology up to degree m. Now, using that $\Omega(\mathcal{F}, TM/T\mathcal{F})$ is a deformation retract of $C_{\mathrm{def}}(T\mathcal{F})$ (and similarly for $\mathcal{H}$) we immediately get the following theorem.

Theorem 4.1. *Let $\mathcal{V} \subset \mathcal{H}$ be a flag of foliations on P. Assume that $\mathcal{V}$ is simple and let $\mathcal{F}$ be the foliation induced by $\mathcal{H}$ on the leaf space of $\mathcal{V}$ via projection. If the leaves of $\mathcal{V}$ are m-simply connected, then $\mathcal{F}$ and $\mathcal{H}$ share the same deformation cohomology up to degree m.*

It is now natural to define a weak Morita equivalence for foliated manifolds. Namely, two foliated manifolds $(M, \mathcal{F})$ and $(N, \mathcal{G})$ are *weak Morita equivalent* if the tangent algebroids of $\mathcal{F}$ and $\mathcal{G}$ are weak Morita equivalent. Theorem 4.1 then reveals that if $(M, \mathcal{F})$ and $(N, \mathcal{G})$ are Morita equivalent, then $\mathcal{F}$ and $\mathcal{G}$ share the same 0-th and 1-st deformation cohomology. If, additionally, the Morita equivalence is realized by surjective submersions with homologically m-connected fibers, then $\mathcal{F}$ and $\mathcal{G}$ share the same deformation cohomology up to degree m.

Finally, notice that the tangent algebroid of a foliation $\mathcal{F}$ is always integrable. Any integration of $\mathcal{F}$ is called a foliation groupoid [6]. It would be interesting to explore the relationship between the weak Morita equivalence of foliations and the Morita equivalence of their foliation groupoids (particularly the monodromy and the holonomy groupoids). However, this goes beyond the scopes of the present note.

5. Final remarks

There is another approach to the deformation cohomology of a Lie algebroid. Namely, the deformation complex of a Lie algebroid $A \Rightarrow M$ can be seen as the *linear de Rham complex* of the cotangent VB-algebroid $(T^*A \Rightarrow A^*) \to (A \Rightarrow M)$ [10,1]. Recall that a VB-algebroid is a vector bundle object in the category of Lie algebroids, and its linear de Rham complex is the subcomplex in the de Rham complex consisting of cochains that are linear with respect to the vector bundle structure. It is possible to define a notion of (weak) Morita equivalence for VB-algebroids respecting the vector bundle structure. It is then natural to expect that (1) if two Lie algebroids are (weak) Morita equivalent, then their cotangent VB-algebroids are (weak) Morita equivalent, and (2) if two VB-algebroids are (weak) Morita equivalent, then their linear de Rham cohomologies are the same in low degree. If so, then Theorem 3.2 would be an immediate corollary. This alternative approach to the Morita invariance of the deformation cohomology of Lie algebroids is actually being investigated in a separate work [12]. We remark that Morita equivalence for VB-groupoids, i.e. vector bundle objects in the category of Lie groupoids, is defined and discussed in [8], where the authors prove the Morita invariance of the VB-groupoid cohomology. Finally, VB-groupoid cohomology is related to VB-algebroid cohomology by a Van-Est type map [3].

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References

[1] C. Arias Abad, M. Crainic, Representations up to homotopy of Lie algebroids, J. Reine Angew. Math. 663 (2012) 91–126, e-print, arXiv:0901.0319.
[2] K. Behrend, P. Xu, Differentiable stacks and gerbes, J. Symplectic Geom. 9 (2011) 285–341, e-print, arXiv:math/0605694.
[3] A. Cabrera, T. Drummond, Van Est isomorphism for homogeneous cochains, Pac. J. Math. 287 (2017) 297–336, e-print, arXiv:1602.06887.
[4] M. Crainic, Differentiable and algebroid cohomology, Van Est isomorphisms, and characteristic classes, Comment. Math. Helv. 78 (2003) 681–721, e-print, arXiv:math/0008064.
[5] M. Crainic, J.N. Mestre, I. Struchiner, Deformations of Lie groupoids, e-print, arXiv:1510.02530, 2015.
[6] M. Crainic, I. Moerdijk, Foliation groupoids and their cyclic homology, Adv. Math. 1157 (2001) 177–197, e-print, arXiv:math/0003119.
[7] M. Crainic, I. Moerdijk, Deformations of Lie brackets: cohomological aspects, J. Eur. Math. Soc. 10 (2008) 1037–1059, e-print, arXiv:math/0403434.
[8] M. del Hoyo, C. Ortiz, Morita equivalences of vector bundles, e-print, arXiv:1612.09289, 2016.
[9] V.L. Ginzburg, Grothendieck groups of Poisson vector bundles, J. Symplectic Geom. 1 (2001) 121–170, e-print, arXiv:math.DG/0009124.
[10] A. Gracia-Saz, R.A. Mehta, Lie algebroid structures on double vector bundles and representation theory of Lie algebroids, Adv. Math. 223 (2010) 1236–1275, e-print, arXiv:0810.0066.
[11] J.L. Heitsch, A cohomology for foliated manifolds, Bull. Amer. Math. Soc. 79 (1973) 1283–1285.
[12] P.P. La Pastina, L. Vitagliano, Deformations of linear Lie brackets, in preparation.
[13] L. Vitagliano, On the strong homotopy Lie–Rinehart algebra of foliation, Commun. Contemp. Math. 16 (2014) 1450007, e-print, arXiv:1204.2467.
[14] P. Xu, Morita equivalence of Poisson manifolds, Commun. Math. Phys. 142 (1991) 493–509.