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On the T°-slices of a finite group ☆

Sur les T°-tranches d'un groupe fini

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ABSTRACT

A slice (G, S) of finite groups is a pair consisting of a finite group G and a subgroup S of G. In this paper, we show that some properties of finite groups extend to slices of finite groups. In particular, by analogy with B-groups, we introduce the notion of T°-slice, and show that any slice of finite groups admits a largest quotient T°-slice.

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RÉSUMÉ

Une tranche (G, S) de groupes finis est un couple formé d'un groupe fini G et d'un sous-groupe S de G. Dans cet article, nous démontrons que certaines propriétés des groupes finis s'étendent aux tranches de groupes finis. En particulier, par analogie avec les B-groupes, nous introduisons la notion de T°-tranche, et nous montrons que toute tranche de groupes finis admet un plus grand quotient qui soit une T°-tranche.

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1. Introduction

In this note, we extend to slices of finite groups some notions relative to finite groups. Throughout the paper, G will denote a finite group.

In order to study the lattice of biset subfunctors of the Burnside functor $\mathbb{Q} \otimes B$, Serge Bouc (see [2]) studies the effect of the elementary biset operations on the primitive idempotents of the Burnside algebra $\mathbb{Q} \otimes B(G)$: these idempotents e_H^G are indexed by the subgroups H of G, up to conjugation and given (see [6], [8]) by

$$e_H^G = \frac{1}{|N_G(H)|} \sum_{K \leq H} |K| \mu(K, H) [G/K]$$

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where $[G/K]$ denotes the isomorphism class of the G-set G/K, and μ is the Möbius function of the poset of subgroups of G.

Serge Bouc shows in particular that, for each normal subgroup N of G, there is a rational number $\mathrm{m}_{G,N}$ such that

$$\mathrm{Def}^G_{G/N} e^G_G = \mathrm{m}_{G,N} e^{G/N}_{G/N}.$$

This constant $\mathrm{m}_{G,N}$ is given by

$$\mathrm{m}_{G,N} = \frac{1}{|G|} \sum_{\substack{X \leq G \\ XN=G}} |X| \mu(X, G).$$

This leads to the introduction of a special class of finite groups, called B-groups: a group G is a B-group if $\mathrm{m}_{G,N} = 0$ for any non-trivial normal subgroup N of G.

When N and M are maximal subgroups of G such that $\mathrm{m}_{G,N} \neq 0$ and $\mathrm{m}_{G,M} \neq 0$, then the quotients G/N and G/M are isomorphic. Such a quotient G/N does not depend on N, up to isomorphism. It is denoted by $\beta(G)$. This quotient $\beta(G)$ of G is a B-group, and moreover, any B-group H that is a quotient of G is actually a quotient of $\beta(G)$. Further results on this invariant $\beta(G)$ and B-groups can be found in [1] and [4].

Recall that a slice of finite groups is a pair (T, S) consisting of a finite group T and a subgroup S of T. We say that the slice (V, U) is a quotient of the slice (T, S) if there exists a surjective group homomorphism $\varphi : T \to V$ such that $\varphi(S) = U$. When ϕ is an isomorphism, we say that (T, S) and (V, U) are isomorphic.

A slice of a finite group G is a slice (T, S) consisting of subgroups of G. The group G acts by conjugation on the set $\Pi(G)$ of its slices.

In [3], the slice Burnside ring $\Xi(G)$ is introduced: it is a commutative ring, which has a $\mathbb{Z}$-basis $\langle T, S\rangle_G$ indexed by the conjugacy classes of slices (T, S) of G. It is shown that $\mathbb{Q} \otimes \Xi(G)$ is a split semisimple $\mathbb{Q}$-algebra, whose primitive idempotents are also indexed by conjugacy classes of slices of G, and given by

$$\xi^G_{T,S} = \frac{1}{|N_G(T, S)|} \sum_{U \leq S \leq V \leq T} |U| \mu(U, S) \mu(V, T) \langle V, U\rangle_G.$$

It is also shown in [3] that the assignments $G \to \Xi(G)$ and $G \to \mathbb{Q} \otimes \Xi(G)$ are Green biset functors. On can try to describe the lattice of subfunctors of $\mathbb{Q} \otimes \Xi$, or its lattice of ideals.

In [7], it is shown that the idempotents $\xi^G_{G,S}$ play a similar role for $\mathbb{Q} \otimes \Xi$ as the idempotents e^G_G do for $\mathbb{Q} \otimes B$: namely, for any $N \trianglelefteq G$, there exists a rational number $\mathrm{m}_{G,S,N}$ such that

$$\mathrm{Def}^G_{G/N} \xi^G_{G,S} = \mathrm{m}_{G,S,N} \xi^{G/N}_{G/N,SN/N}.$$

This constant $\mathrm{m}_{G,S,N}$ is given by

$$\mathrm{m}_{G,S,N} = \frac{|N_G(SN) : SN|}{|N_G(S)|} \sum_{\substack{U \leq S \leq V \leq G \\ VN=G \\ UN=SN}} |U| \mu(U, S) \mu(V, G).$$

By Proposition 5.5 in [7] we have

$$\mathrm{m}_{G,S,N} = \frac{|N_G(SN) : SN|}{|N_G(S) : S|} \mathrm{m}_{S,S\cap N} \mathrm{m}^{\circ}_{G,S,N}$$

where $\mathrm{m}^{\circ}_{G,S,N} = \sum_{\substack{S \leq X \leq G \\ XN=G}} \mu(X, G)$. We observe that $\mathrm{m}_{G,S,1} = \mathrm{m}^{\circ}_{G,S,1} = 1$ for any slice (G, S). Slices (G, S) such that $\mathrm{m}_{G,S,N} = 0$ for any non-trivial normal subgroup N of G have been called T-slices in [7]. By analogy, we say that a slice (G, S) is a T°-slice if $\mathrm{m}^{\circ}_{G,S,N} = 0$ for any non-trivial normal subgroup N of G.

As a first step towards a description of subfunctors of $\mathbb{Q} \otimes \Xi$, we state the following property of T°-slices.

Theorem 1.1. *Let (G, S) be a slice. Then there exists a slice (H, U) such that*

- *(H, U) is a T°-slice and $(G, S) \twoheadrightarrow (H, U)$.*
- *if (K, V) is a T°-slice such that $(G, S) \twoheadrightarrow (K, V)$, then $(H, U) \twoheadrightarrow (K, V)$.*
  Moreover, any two slices (H, U) with these properties are isomorphic. We set $\tau^{\circ}(G, S) = (H, U)$.

2. Proof of Theorem 1.1

We start with a proposition:

Proposition 2.1.

1. *Let X be a subgroup of G, and M be a normal subgroup of G. Then*

$$\mu(X,G)=\sum_{\substack{YM=G\\ Y\geq X\\ Y\cap M=X\cap M}}\mu(X,Y)\mu(Y,G).$$

2. *Let S be a subgroup of G, and M, N be normal subgroups of G. Then*

$$\mathrm{m}^{\circ}_{G,S,N}=\sum_{\substack{YN=YM=G\\ Y\geq S}}\mu(Y,G)\mathrm{m}^{\circ}_{G/M,SM/M,(Y\cap N)M/M}.$$

In particular, if $N\geq M$, then $\mathrm{m}^{\circ}_{G,S,N}=\mathrm{m}^{\circ}_{G,S,M}\mathrm{m}^{\circ}_{G/M,SM/M,N/M}$.

3. *If M is a normal subgroup of G, maximal such that $\mathrm{m}^{\circ}_{G,S,M}\neq 0$, then $(G/M,SM/M)$ is a T°-slice.*

Proof. 1. Let $[X,G]$ be the lattice of subgroups of G containing X. In this lattice, a complement of XM is a subgroup Y of G that contains X such that $\langle Y,XM\rangle=G$ and $Y\cap XM=X$.
Since $X\leq Y$, the first condition is equivalent to $YM=G$, and the second one is equivalent to $X(Y\cap M)=X$, i.e. $Y\cap M=X\cap M$. Hence $|Y||M|=|G||X\cap M|$, so $|Y|$ depends only on X and M. Hence there is no strict inclusion between complements of XM, and Crapo's formula (see [5]) gives the following result:

$$\mu(X,G)=\sum_{\substack{YM=G\\ Y\geq X\\ Y\cap M=X\cap M}}\mu(X,Y)\mu(Y,G)$$

for any normal subgroup M of G.

2. By 1, we have

$$\mathrm{m}^{\circ}_{G,S,N}=\sum_{X,Y}\mu(X,Y)\mu(Y,G)$$

where the sum is over all pair (X,Y) with

$$X\leq Y,\ XN=G,\ YM=G,\ Y\cap M=X\cap M,\ S\leq X\leq G$$

i.e.

$$YN=YM=G,\ X(Y\cap N)=Y,\ S\leq X\leq Y,\ X\geq (Y\cap M).$$

For a fixed Y, summing over X is equivalent to summing over a subgroup R $(=X/(Y\cap M))$ of $Y/(Y\cap M)$ such that

$$R.(Y\cap N)(Y\cap M)/(Y\cap N)=Y/(Y\cap N),\ S.(Y\cap M)/(Y\cap M)\leq R\leq Y/(Y\cap M) \tag{*}$$

Hence, the sum on X is equal to

$$\sum_{R}\mu(R,Y/(Y\cap M))=\mathrm{m}^{\circ}_{Y/(Y\cap M),S.(Y\cap M)/(Y\cap M),(Y\cap N)(Y\cap M)/(Y\cap N)}=\mathrm{m}^{\circ}_{G/M,SM/M,(Y\cap N)M/M},$$

where R runs through subgroups of $Y/(Y\cap M)$ fulfilling Condition (*). The second equality here follows from the fact that since $YM=G$, there is a group isomorphism

$$Y/(Y\cap M)\cong G/M,$$

and this isomorphism sends $S(Y\cap M)/(Y\cap M)$ to SM/M and $(Y\cap N)(Y\cap M)/(Y\cap M)$ to $(Y\cap N)M/M$.
Hence,

$$\mathrm{m}^{\circ}_{G,S,N}=\sum_{\substack{YN=YM=G\\ Y\geq S}}\mu(Y,G)\mathrm{m}^{\circ}_{G/M,SM/M,(Y\cap N)M/M}.$$

If $N \geq M$ and if $YM = G$, then $N = (Y \cap N)M$ and

$$\mathrm{m}^{\circ}_{G,S,N} = \sum_{\substack{YM=G\\ Y\geq S}} \mu(Y,G)\mathrm{m}^{\circ}_{G/M,SM/M,N/M} = \mathrm{m}^{\circ}_{G,S,M}\mathrm{m}^{\circ}_{G/M,SM/M,N/M}.$$

3. Straightforward. □

Let us give a proof of Theorem 1.1.

Let S be a subgroup of G. Let moreover N be a normal subgroup of G such that $m^{0}_{G,S,N} \neq 0$, and let (K, V) be a T°-slice that is a quotient of (G, S). Then there exists a surjective group homomorphism $\varphi : G \twoheadrightarrow K$ such that $\varphi(S) = V$. We denote by M the kernel of φ, so (K, V) is isomorphic to $(G/M, SM/M)$. Since $(G/M, SM/M)$ is a T°-slice, Proposition 2.1 gives

$$\mathrm{m}^{\circ}_{G,S,N} = \sum_{\substack{YN=YM=G\\ Y\cap N\leq M\\ Y\geq S}} \mu(Y,G).$$

Since $\mathrm{m}^{\circ}_{G,S,N} \neq 0$, there exists a subgroup Y of G that satisfies:

$$YN = YM = G,\ (Y \cap N) \leq M,\ S \leq Y.$$

Then there exists a surjective homomorphism $Y/(Y \cap N) \twoheadrightarrow Y/(Y \cap M)$ that sends the group $S(Y \cap N)/(Y \cap N)$ to the group $S(Y \cap M)/(Y \cap M)$.

Since $Y/(Y \cap N) \cong G/N$, $Y/(Y \cap M) \cong G/M$, $S(Y \cap N)/(Y \cap N) \cong SN/N$, $S(Y \cap M)/(Y \cap M) \cong SM/M$, we have a surjective group homomorphism $G/N \twoheadrightarrow G/M$ sending SN/N to SM/M. In other words,

$$(G/N, SN/N) \twoheadrightarrow (G/M, SM/M) \cong (K, V),$$

so (K, V) is a quotient of $(G/N, SN/N)$.

If M' is a normal subgroup of G, maximal such that $\mathrm{m}^{\circ}_{G,S,M'} \neq 0$, then by Proposition 2.1 the slice $(H, U) := (G/M', SM'/M')$ is a T°-slice, so

$$(G/N, SN/N) \twoheadrightarrow (H, U).$$

Moreover, since $\mathrm{m}^{\circ}_{G,S,M'} \neq 0$, the slice (K, V) is a quotient of $(G/M', SM'/M')$ i.e.

$$(H, U) \twoheadrightarrow (K, V).$$

Hence (H, U) has the properties required in Theorem 1.1. If (H', U') is another slice with these properties, then (H, U) and (H', U') are quotients of each other, so they are isomorphic. This completes the proof.

Remark 2.2. It is natural to ask if the analogue of Theorem 1.1 holds for T-slices instead of T°-slices. Namely, for any slice (G, S), does there exist a slice $\tau(G, S) = (H, U)$ such that

- (H, U) is a T-slice and $(G, S) \twoheadrightarrow (H, U)$,
- if (K, V) is a T-slice and $(G, S) \twoheadrightarrow (K, V)$, then $(H, U) \twoheadrightarrow (K, V)$?

The following example shows that the answer is no: consider the direct product $G = C_2 \times D_8$ of a group of order 2 and a dihedral group of order 8. Let a denote the generator of C_2, and let $\{b, c\}$ be a set of generators of D_8, where b has order 2 and c has order 4. Set moreover $d = c^2$. Let S be the subgroup of G generated by a and b. Thus $S \cong C_2 \times C_2$.

We denote by N the subgroup of G generated by ad, and M the subgroup of G generated by d.

Hence $|N| = |M| = 2$ and the subgroups N and M are central in G. We have $G/N \cong D_8$ and $G/M \cong (C_2)^3$.

The slices $(G/N, SN/N)$ and $(G/M, SM/M)$ are both quotients of (G, S). On can check that they are both T-slices. If there exists a T-slice (H, U) with the required properties, then in particular H is a quotient of G, and both G/N and G/M are quotients of H. It follows that $H \cong G$, and then (H, U) is isomorphic to (G, S).

This is a contradiction, since (G, S) is not a T-slice.

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