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On a finite element method for measure-valued optimal control problems governed by the 1D generalized wave equation



Sur une méthode d'éléments finis pour résoudre des problèmes de contrôle optimal régis par l'équation d'onde unidimensionnelle généralisée

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ABSTRACT

The paper deals with the optimal control problems governed by the 1D wave equation with variable coefficients and the control spaces of either measure-valued functions $L^2_{W^*}(I, \mathcal{M}(\Omega))$ or vector measures $\mathcal{M}(\Omega, L^2(I))$. Bilinear finite element discretizations are constructed and their stability and error analysis is accomplished.

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RÉSUMÉ

Cet article traite des problèmes de contrôle optimal régis par l'équation d'onde 1D avec coefficients variables, les espaces de contrôle étant, soit des fonctions mesurées $L^2_{W^*}(I, \mathcal{M}(\Omega))$, soit des mesures vectorielles $\mathcal{M}(\Omega, L^2(I))$. On construit des discrétisations bilinéaires des éléments finis et on en analyse la stabilité et l'erreur.

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1. Introduction

Motivated by industrial applications as well as by applications in the natural sciences, in which one is interested in placing actuators in form of point sources in an optimal way or in the reconstruction of point sources from given measurements, measure valued optimal control problems involving PDEs attracted attention in the last years and have been analyzed from theoretical, numerical and algorithmic points of view, in particular, see [3–5,9,10] and references therein. In this paper, we

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consider a less studied optimal control problems governed by the initial–boundary value problem for the 1D wave equation having *variable coefficients* and with the control spaces \mathcal{M}_T of either measure-valued functions $L^2_{w^*}(I, \mathcal{M}(\Omega))$ or vector measures $\mathcal{M}(\Omega, L^2(I))$. Notice that the spaces contain time-dependent respectively moving or static point sources. In the case of constant coefficients, the regularity of the corresponding multidimensional problem was recently discussed in [10] by a different technique, which cannot be directly extended to variable coefficients.

We construct bilinear FEM discretizations for the state and adjoint state equations by the regularized Galerkin method, which is different from those applied in [10] (with no stability bounds and error estimates), study their stability, get auxiliary error estimates in several required norms, and finally present error estimates for both the optimal state variable and the cost functional. To the best of our knowledge, this is the first paper providing such numerical analysis for the considered control problems. Its difficulties come from a very low regularity of the solution to the 1D wave equation for so broad control spaces. This makes the error estimation a delicate matter, and a special technique is required to accomplish it [20].

The whole study comprises several different steps and contain a collection of results. The main of them are stated as theorems. This note is a short version of our paper [18], where full proofs of all the results presented here and much more references can be found.

2. The state equation, the control problem, and the adjoint state equation

1. We first define the initial–boundary value problem for the 1D generalized wave equation

$$\rho \partial_{tt} y - \partial_x(\kappa \partial_x y) = u \text{ in } I \times \Omega := (0, T) \times (0, L), \quad y|_{I \times \partial\Omega} = 0, \quad y|_{t=0} = y^0, \quad \partial_t y|_{t=0} = y^1, \quad (1)$$

considered as *the state equation*, with the coefficients $\rho, \kappa \in H^1(\Omega)$ satisfying $\rho(x) \geq \nu > 0$ and $\kappa(x) \geq \nu$ on Ω , the initial data $(y^0, y^1) \in V \times H := H^1_0(\Omega) \times L^2(\Omega)$ (in particular) and the control $u \in \mathcal{M}_T$; also $T > 0$ and $L > 0$.

Hereafter the control space \mathcal{M}_T is either the space $L^2_{w^*}(I, \mathcal{M}(\Omega))$ of weakly-star measurable, $\mathcal{M}(\Omega)$ -valued functions, where $\mathcal{M}(\Omega)$ is the space of the bounded Radon measures on Ω , or the space $\mathcal{M}(\Omega, L^2(I))$ of finite vector measures with values in $L^2(I)$, see [2,10] for precise definitions and more details. Let correspondingly \mathcal{C}_T be chosen as $L^2(I, \mathcal{C}_0(\Omega))$ or $\mathcal{C}_0(\Omega, L^2(I))$ where $\mathcal{C}_0(\Omega) = \{v \in \mathcal{C}(\bar{\Omega}) \mid v|_{x=0, L} = 0\}$. Then $\mathcal{M}_T = \mathcal{C}_T^*$, see [3,9] for more details. In particular, the following embeddings hold

$$\mathcal{M}(\Omega, L^2(I)) \hookrightarrow L^2_{w^*}(I, \mathcal{M}(\Omega)) \hookrightarrow L^2(I, V^*).$$

Here $V \subset H = H^* \subset V^*$ form a standard Gelfand triple of Hilbert spaces. Below all the results are valid for the both cases of the space \mathcal{M}_T if its choice is not specified.

Recall that for $(u, y^0, y^1) \in X \times V \times H$ with $X = L^2(I \times \Omega)$ or $H^1(I, V^*)$, problem (1) has a unique weak solution $y \in \mathcal{C}(\bar{I}, V) \cap \mathcal{C}^1(\bar{I}, H) \cap H^2(I, V^*)$ satisfying the integral identity

$$B(y, v) + (\rho \partial_t y(T), v(T))_H = \int_I (u, v)_\Omega dt + (\rho y^1, v(0))_H \text{ for any } v \in L^2(I, V) \cap H^1(I, H) \quad (2)$$

with the indefinite symmetric bilinear form

$$B(y, v) := -(\rho \partial_t y, \partial_t v)_{L^2(I \times \Omega)} + (\kappa \partial_x y, \partial_x v)_{L^2(I \times \Omega)}, \quad (3)$$

and the initial condition $y(0) = y^0$. The weak solution obeys the bound

$$\|y\|_{\mathcal{C}(\bar{I}, V)} + \|\partial_t y\|_{\mathcal{C}(\bar{I}, H)} + \|\partial_{tt} y\|_{L^2(I, V^*)} \leq c (\|u\|_X + \|(y^0, y^1)\|_{V \times H}). \quad (4)$$

For $(u, y^0, y^1) \in L^2(I, V^*) \times H \times V^*$, there exists a unique weaker solution $y \in \mathcal{C}(\bar{I}, H) \cap \mathcal{C}^1(\bar{I}, V^*)$ and it obeys the bound

$$\|y\|_{\mathcal{C}(\bar{I}, H)} + \|\mathcal{I}_t y\|_{\mathcal{C}(\bar{I}, V)} + \|\partial_t y\|_{\mathcal{C}(\bar{I}, V^*)} \leq c (\|u\|_{L^2(I, V^*)} + \|(y^0, y^1)\|_{H \times V^*}) \quad (5)$$

with $\mathcal{I}_t y(t) := \int_0^t y(s) ds$ on \bar{I} . Hereafter, c and c_i are independent of the data. See details, e.g., in [12,20].

We need to enlarge bound (4) for $u \in \mathcal{M}(\Omega, L^2(I))$ and first state two lemmas. Let $H^{(2)} := H^2(\Omega) \cap V$.

Lemma 1. *Let $(u, y^0, y^1) \in L^1(\Omega, L^2(I)) \times V \times H$ and $y \in \mathcal{C}(\bar{I}, H^{(2)}) \cap \mathcal{C}^1(\bar{I}, V) \cap H^2(I, H)$ be the corresponding strong solution to problem (1). Then y satisfies the following a priori bound*

$$\|y\|_{\mathcal{C}(\bar{I}, V)} + \|\partial_t y\|_{\mathcal{C}(\bar{I}, H)} + \|\kappa \partial_x y\|_{\mathcal{C}(\bar{\Omega}, L^2(I))} + \|\partial_t y\|_{\mathcal{C}_0(\Omega, L^2(I))} \leq c (\|u\|_{L^1(\Omega, L^2(I))} + \|(y^0, y^1)\|_{V \times H}).$$

The proof is based on a non-standard energy technique *in space* and not only in time, see [11, Ch. 2, Sections 4.1–4.3] and [7].

Lemma 2. Let $u \in \mathcal{M}(\Omega, L^2(I))$. Then there exists a sequence $\{u_n\} \subset L^2(I, V)$ such that $u_n \rightharpoonup^* u$ in $\mathcal{M}(\Omega, L^2(I))$ as $n \rightarrow \infty$ and $\|u_n\|_{\mathcal{M}(\Omega, L^2(I))} \leq \|u\|_{\mathcal{M}(\Omega, L^2(I))}$ for any $n \geq 1$.

The proof is based on the corollary of [8, Ch. III, Theorem 6] on the separation of convex sets.

Theorem 3. Let $(u, y^0, y^1) \in \mathcal{M}(\Omega, L^2(I)) \times V \times H$. Then there exists a unique weak solution y to problem (1), and it satisfies the bound

$$\|y\|_{C(\bar{I}, V)} + \|\partial_t y\|_{C(\bar{I}, H)} \leq c (\|u\|_{\mathcal{M}(\Omega, L^2(I))} + \|(y^0, y^1)\|_{V \times H}).$$

The proof follows mainly from Lemmas 1 and 2. In the case of the constant coefficients, the corresponding multidimensional result was proved by another techniques in [10], but it fails for variable coefficients.

2. Let $\mathcal{Y} := L^2(I \times \Omega) \times H \times V^*$. Owing to the above results, problem (1) is uniquely solvable for any $u \in \mathcal{M}_T$, and we can define the linear bounded operator

$$\hat{S}: \mathcal{M}_T \times H \times V^* \rightarrow \mathcal{Y}, \quad (u, y^0, y^1) \mapsto (y, y(T), \rho \partial_t y(T)).$$

Then, for fixed (y^0, y^1) , the control-to-state affine bounded mapping is given by

$$Su = \hat{S}(u, 0, 0) + \hat{S}(0, y^0, y^1).$$

Now we can formulate the control problem

$$j(u) = \frac{1}{2} \|Su - \mathbf{z}\|_{\mathcal{Y}}^2 + \alpha \|u\|_{\mathcal{M}_T} \rightarrow \min_{u \in \mathcal{M}_T} \tag{6}$$

with the given $\mathbf{z} := (z_1, z_2, z_3) \in \mathcal{Y}$ and the norms

$$\|\mathbf{z}\|_{\mathcal{Y}} = (\|z_1\|_{L^2(I, H_\rho)}^2 + \|z_2\|_{H_\rho}^2 + \|z_3\|_{V_k^*}^2)^{1/2}, \quad \|w\|_{H_\rho} = \|\sqrt{\rho}w\|_H, \quad \|w\|_{V_k^*} = \sup_{\|v\|_{V_k} \leq 1} \langle w, v \rangle_\Omega$$

where $\langle \cdot, \cdot \rangle_\Omega$ is the duality relation on $V^* \times V$ and $\|w\|_{V_k} = \|\sqrt{\kappa} \partial_x w\|_H$. The first term of (6) is the quadratic tracking functional, whereas the second term is the regularizing one, which favors point sources as solutions.

Proposition 4. The control problem (6) has a unique solution $\bar{u} \in \mathcal{M}_T$ and it satisfies the bound

$$\|\bar{u}\|_{\mathcal{M}_T} \leq C = C(\|(y^0, y^1)\|_{H \times V^*}, \|\mathbf{z}\|_{\mathcal{Y}}).$$

Hereafter $C > 0$ denotes increasing functions of the data norms. The proof is based on [9,10].

Now we can set $(\bar{y}, \bar{y}(T), \rho \partial_t \bar{y}(T)) := S\bar{u}$; here the function \bar{y} is the optimal state.

3. Next we discuss first order optimality conditions. To this end we define $p \in C(\bar{I}, V) \cap C^1(\bar{I}, H)$ as the weak solution to the adjoint problem for the 1D generalized wave equation

$$\rho \partial_{tt} p - \partial_x(\kappa \partial_x p) = \rho \phi \text{ in } I \times \Omega, \quad p|_{I \times \partial\Omega} = 0, \quad p|_{t=T} = p^0, \quad \partial_t p|_{t=T} = p^1 \tag{7}$$

for some given ϕ, p^0 and p^1 . We introduce the adjoint control-to-solution linear operator

$$S^*: \mathcal{Y} \rightarrow C(\bar{I}, V), \quad (\phi, p^0, p^1) \mapsto p \tag{8}$$

which is well defined and bounded according to bound (4).

We also need the operator $A^{-1}: V^* \rightarrow V, f \mapsto w$, where $w \in V$ solves the equation $-\partial_x(\kappa \partial_x w) = f$.

Proposition 5. Let $\bar{p} := S^*(\bar{y} - z_1, -(\bar{y}(T) - z_2), A^{-1}(\rho \partial_t \bar{y} - z_3))$ be the optimal adjoint state, see (7)–(8). An element $\bar{u} \in \mathcal{M}_T$ is an optimal control of (6) if and only if

$$\langle -\bar{p}, u - \bar{u} \rangle_{C_T, \mathcal{M}_T} + \alpha \|\bar{u}\|_{\mathcal{M}_T} \leq \alpha \|u\|_{\mathcal{M}_T} \text{ for any } u \in \mathcal{M}_T$$

where $\langle \cdot, \cdot \rangle_{C_T, \mathcal{M}_T}$ is the duality relation on $C_T \times \mathcal{M}_T$.

The proof follows [10].

We also introduce the Jordan decomposition $\mu = \mu^+ - \mu^-$ of a signed measure $\mu \in \mathcal{M}(\Omega)$ with uniquely defined elements $\mu^\pm \in \mathcal{M}(\Omega)^+$ [2]. Moreover, we recall the polar decomposition of a vector measure $\mu \in \mathcal{M}(\Omega, L^2(I))$: $d\mu = \mu' d|\mu|$, where μ' is the Radon–Nikodym-derivative of μ with respect to $|\mu|$.

Proposition 6. Let $\bar{u} \in \mathcal{M}_T$ be the optimal control of (6) and $\bar{p} \in C_T$ be the corresponding optimal adjoint state. Then $\|\bar{p}\|_{C_T} \leq \alpha$, and in the cases $\mathcal{M}_T = L^2_{w^*}(I, \mathcal{M}(\Omega))$ and $\mathcal{M}_T = \mathcal{M}(\Omega, L^2(I))$, the following properties respectively hold

$$\begin{aligned} \text{supp } \bar{u}^\pm(t) &\subset \{x \in \Omega \mid \bar{p}(t, x) = \mp \|\bar{p}(t, \cdot)\|_{C_0(\Omega)}\} \quad \text{for almost all } t \in I, \\ \text{supp } |\bar{u}| &\subset \{x \in \Omega \mid \|\bar{p}(\cdot, x)\|_{L^2(I)} = \alpha\}, \quad \bar{u}' = -\alpha^{-1} \bar{p} \text{ in } L^1(\Omega, |\bar{u}|, L^2(I)). \end{aligned} \tag{9}$$

A detailed discussion of the proof of such results based on Proposition 5 can be found in [3,9]. Theorem 3 implies a regularity of \bar{p} that next is applied to show improved regularity of \bar{u} .

Theorem 7. Let $\mathcal{M}_T = \mathcal{M}(\Omega, L^2(I))$, $\mathbf{z} \in \mathcal{Y}^1 := L^2(I, V) \times V \times H$ and \bar{u} be the optimal control of (6). Then $\bar{u} \in C^1(\bar{I}, \mathcal{M}(\Omega))$ and the following bound holds

$$\|\bar{u}\|_{C^1(\bar{I}, \mathcal{M}(\Omega))} \leq C = C(\|y^0, y^1\|_{V \times H}, \|\mathbf{z}\|_{\mathcal{Y}^1}).$$

The proof is based on the properties (9).

We redenote the several above spaces (recall that $H^{(2)} = H^2(\Omega) \cap V$) and define the corresponding interpolation spaces

$$H^{(-1)} = V^*, \quad H^{(0)} = H, \quad H^{(1)} = V, \quad H^{(\ell+1/2)} := (H^{(\ell)}, H^{(\ell+1)})_{1/2, \infty}, \quad \ell = -1, 0, 1,$$

using the real $K_{\lambda, q}$ -interpolation method of Banach spaces for $(\lambda, q) = (1/2, \infty)$ [1]. This method (in contrast to the complex interpolation method in [10]) finally allows one to derive error estimates without the term $-\varepsilon < 0$ in their orders. The spaces $H^{(\ell+1/2)}$ can be explicitly described in terms of the subspaces in the Nikolskii spaces $H^{\ell+1/2, 2}(\Omega)$, e.g., see [1,13,20]. In particular, recall that $H^{(1/2)} = \{w \in L^2(\Omega) \mid ow \in H^{1/2, 2}(\tilde{\Omega})\}$, where ow denotes the odd extension of w with respect to $x = 0$ and L from Ω to $\tilde{\Omega} := (-L, 2L)$, and thus $H^{(1/2)}$ contains discontinuous H^1 -piecewise functions.

Let also $\langle W \rangle_\Omega := L^{-1} \int_\Omega W \, dx$ and D_x be the distributional derivative. Define the space $H^{-1/2, 2}(\Omega)$ of distributions $w = D_x W$ with $W \in H^{1/2, 2}(\Omega)$ and $\langle W \rangle_\Omega = 0$ equipped with the norm $\|w\|_{H^{-1/2, 2}(\Omega)} = \|W\|_{H^{1/2, 2}(\Omega)}$. Then $H^{(-1/2)} = H^{-1/2, 2}(\Omega)$ up to the equivalence of norms. Note that, in particular, the Dirac delta-function $\delta_a(x) = D_x(H(x-a) - (1-a/L)) \in H^{(-1/2)}$ for any $a \in \Omega$, where $H(\xi) = 0$ for $\xi < 0$ and $H(\xi) = 1$ for $\xi > 0$ is the Heaviside function, see also [16] (Corollary after Lemma 1) for a corresponding result. There holds also that the space $H^{(-1/2)} = (V^*, H)_{1/2, \infty}$ is the dual space of $(H, V)_{1/2, 1} \hookrightarrow C_0(\Omega)$, see also [16, Theorem 1] for the last embedding. Therefore, $M(\Omega) \hookrightarrow H^{(-1/2)}$, see [18] for more details.

Let $Q = \Omega \times I$ and $\Delta_h W(x) = W(x+h) - W(x)$ be the forward difference in x . Define the spaces $H^{1/2, 0; 2}(Q)$ and $SHW^{1/2, 1; 2}(Q)$ of functions $W \in L^2(Q)$ such that respectively $|W|_{H^{1/2, 0; 2}(Q)} := \sup_{0 < h < L} h^{-1/2} \|\Delta_h W\|_{L^2((0, L-h) \times I)} < \infty$ and $\partial_t W \in H^{1/2, 0; 2}(Q)$ equipped with the norms

$$\|W\|_{H^{1/2, 0; 2}(Q)} = \|W\|_{L^2(Q)} + |W|_{H^{1/2, 0; 2}(Q)}, \quad \|W\|_{SHW^{1/2, 1; 2}(Q)} = \|W\|_{L^2(Q)} + \|\partial_t W\|_{H^{1/2, 0; 2}(Q)}.$$

Here $H^{1/2, 0; 2}(Q)$ is a particular anisotropic Nikolskii space (of the order 1/2 in x only) and $SHW^{1/2, 1; 2}(Q)$ is a particular space of functions having the dominating mixed smoothness (of the order 1/2 in x in the Nikolskii sense and 1 in t in the Sobolev sense). Note that $SHW^{1/2, 1; 2}(Q) \hookrightarrow H^{1/2, 0; 2}(Q)$.

For a Banach space $B(Q) \subset L^1(Q)$, let $B_\perp(Q)$ be the subspace of $W \in B(Q)$ such that $\langle W(\cdot, t) \rangle_\Omega = 0$ on I . Define the spaces $H^{-1/2, 0; 2}(Q)$ and $SHW^{-1/2, 1; 2}(Q)$ of distributions $w = D_x W$ with respectively $W \in H^{1/2, 0; 2}(Q)$ and $W \in SHW^{1/2, 1; 2}(Q)$ equipped with the norms

$$\|w\|_{H^{-1/2, 0; 2}(Q)} = \|W\|_{H^{1/2, 0; 2}(Q)}, \quad \|w\|_{SHW^{-1/2, 1; 2}(Q)} = \|W\|_{SHW^{1/2, 1; 2}(Q)}.$$

Note that all the spaces defined above in this section are Banach ones. Below we apply the following embeddings and equalities (with the equivalence of norms)

$$L^2_{w^*}(I, \mathcal{M}(\Omega)) \hookrightarrow (L^2(I, V^*), L^2(I, H))_{1/2, \infty} = H^{-1/2, 0; 2}(Q), \tag{10}$$

$$C^1(\bar{I}, \mathcal{M}(\Omega)) \hookrightarrow (H^1(I, V^*), H^1(I, H))_{1/2, \infty} = SHW^{-1/2, 1; 2}(Q). \tag{11}$$

The classical techniques of approximation by the Steklov averages in x can be used to justify them, in particular, see [19].

Finally, for $k = 0, 1$, define the anisotropic Nikolskii subspaces $\tilde{H}^{k+1/2, 0; 2}(Q)$ of functions $w \in L^2(Q)$ such that $\partial_x^k ow \in H^{1/2, 0; 2}(\tilde{Q})$ and (if $k = 1$) $w|_{\partial\Omega \times I} = 0$ equipped with the norm $\|w\|_{\tilde{H}^{k+1/2, 0; 2}(Q)} = \|\partial_x^k ow\|_{H^{1/2, 0; 2}(\tilde{Q})}$, where $\tilde{Q} = \tilde{\Omega} \times I$.

3. The discrete state equation, the discrete control problem, the discrete adjoint state equation and auxiliary stability bounds and error estimates

1. Now we construct the regularized finite element method to solve the state equation. We define the uniform grid $t_m = m\tau$ on \bar{I} with the step $\tau = T/M$ and a non-uniform grid $0 = x_0 < x_1 < \dots < x_N = L$ on $\bar{\Omega}$ with the steps $h_j = x_j - x_{j-1}$, where $M \geq 2$ and $N \geq 2$. Let also $h = \max_{1 \leq j \leq N} h_j$, $h_{\min} = \min_{1 \leq j \leq N} h_j$ and the space grid be *quasi-uniform*, i.e., $h \leq c_1 h_{\min}$; hereafter c, c_i and C are grid-independent. Let $V_\tau \subset H^1(I)$ and $V_h \subset V$ be the spaces of piecewise linear finite elements with respect to the defined grids on \bar{I} and $\bar{\Omega}$.

We approximate the state variable y by $\mathbf{y}_h \in V_h := V_\tau \otimes V_h$, $\mathbf{h} = (\tau, h)$, and additionally $\partial_t y(T)$ by $y_{Th}^1 \in V_h$. For $(u, y^0, y^1) \in \mathcal{M}_T \times H \times V^*$ the discrete state equation has the following weak form

$$B_\sigma(\mathbf{y}_h, v) + (\rho y_{Th}^1, v(T))_H = \langle u, v \rangle_{\mathcal{M}_T, C_T} + \langle \rho y^1, v(0) \rangle_\Omega \text{ for any } v \in V_h, \tag{12}$$

$$(\rho \mathbf{y}_h(0), \varphi)_H = (\rho y^0, \varphi)_H \text{ for any } \varphi \in V_h, \tag{13}$$

cp. (2), involving bilinear form (3) with the regularizing term

$$B_\sigma(y, v) := B(y, v) - \left(\sigma - \frac{1}{6}\right) \tau^2 (\kappa \partial_x \partial_t y, \partial_x \partial_t v)_{L^2(I \times \Omega)}, \tag{14}$$

where σ is the grid independent parameter. This follows [20] but to treat general v (not only with $v(T) = 0$), we have introduced $y_{Th}^1 \approx \partial_t y(T)$.

The regularizing term in (14) violates the Galerkin (i.e. the projection) principle, but allows one to guarantee below the unconditional stability for $\sigma > \frac{1}{4}$. On the other hand, to ensure stability also in the case $\sigma \leq \frac{1}{4}$ (in particular, for $\sigma = \frac{1}{6}$ when the regularizing term disappears), we impose CFL-type conditions from [20] linking the temporal and spatial grids

$$\tau^2 \alpha_h^2 \left(\frac{1}{4} - \sigma\right) \leq 1 - \varepsilon_0^2, \quad \tau^2 \alpha_h^2 \left(\frac{1 + \varepsilon_1^2}{4} - \sigma\right) \leq 1$$

for some (arbitrarily small) $0 < \varepsilon_0 < 1$ and $0 < \varepsilon_1 \leq 1$. Here α_h is the least constant in the well-known inverse inequality $\|\varphi\|_{\mathcal{Y}_\kappa} \leq \alpha_h \|\varphi\|_{H_\rho}$ for any $\varphi \in V_h$; it satisfies $c_1 h^{-1} \leq \alpha_h \leq c_2 h^{-1}$ with $c_1 > 0$.

The operator form of the discrete state equation is given at the end of the section.

We define the standard FEM projector $\pi_h^1: V \rightarrow V_h$ by

$$(\kappa \partial_x \pi_h^1 w, \partial_x \varphi)_H = (\kappa \partial_x w, \partial_x \varphi)_H \text{ for any } \varphi \in V_h.$$

Its approximation properties are well known [6]. Then we set $A_h^{-1} = \pi_h^1 A^{-1}$ and $\|f\|_{H_h^{-1}} := \|A_h^{-1} f\|_{\mathcal{Y}_\kappa}$.

Now we get a stability bound and error estimates in $\mathcal{C}(\bar{I}, H) \times H_h^{-1}$ for the discrete state equation.

Proposition 8. Let y and (\mathbf{y}_h, y_{Th}^1) be the solutions to problem (1) and its discrete version (12)–(13).

1. For $(u, y^0, y^1) \in L^2(I, V^*) \times V \times V^*$, the following stability bound holds

$$\|\mathbf{y}_h\|_{\mathcal{C}(\bar{I}, H)} + \|\rho y_{Th}^1\|_{H_h^{-1}} \leq c (\|u\|_{L^2(I, V^*)} + \|(y^0, y^1)\|_{V \times V^*}). \tag{15}$$

2. For $(u, y^0, y^1) \in H^{-1/2, 0; 2}(Q) \times V \times H^{(-1/2)}$, the following error estimate holds

$$\|y - \mathbf{y}_h\|_{\mathcal{C}(\bar{I}, H)} + \|\rho(\partial_t y(T) - y_{Th}^1)\|_{H_h^{-1}} \leq c (\tau + h)^{1/3} (\|u\|_{H^{-1/2, 0; 2}(Q)} + \|(y^0, y^1)\|_{V \times H^{(-1/2)}}). \tag{16}$$

3. For $(u, y^0, y^1) \in SHW^{-1/2, 1; 2}(Q) \times V \times H$, the following higher-order error estimate holds

$$\|y - \mathbf{y}_h\|_{\mathcal{C}(\bar{I}, H)} + \|\rho(\partial_t y(T) - y_{Th}^1)\|_{H_h^{-1}} \leq c (\tau + h)^{2/3} (\|u\|_{SHW^{-1/2, 1; 2}(Q)} + \|(y^0, y^1)\|_{V \times H}).$$

The proof exploits essentially [20, Theorems 2.1 (1) and 4.1]. Notice that a priori bound (15) implies the unique solvability of the discrete state equation (12)–(13) (and thus it is not a priori any more).

2. We introduce the linear operator

$$\hat{S}_h: \mathcal{M}_T \rightarrow \mathcal{Y}_h := V_h \times V_h \times (\rho \times V_h), \quad (u, y^0, y^1) \mapsto (\mathbf{y}_h, \mathbf{y}_h(T), \rho y_{Th}^1)$$

and then, for fixed (y^0, y^1) , the discrete control-to-state affine mapping defined by

$$S_h u = \hat{S}_h(u, 0, 0) + \hat{S}_h(0, y^0, y^1).$$

This allows us to consider the following semi-discrete optimal control problem

$$j_{\mathbf{h}}(u) = \frac{1}{2} \|S_{\mathbf{h}}u - \mathbf{z}\|_{\mathcal{Y}_{\mathbf{h}}}^2 + \alpha \|u\|_{\mathcal{M}_T} \rightarrow \min_{u \in \mathcal{M}_T} \tag{17}$$

with the squared semi-norm $\|\mathbf{z}\|_{\mathcal{Y}_{\mathbf{h}}}^2 := \|z_1\|_{L^2(I, H_{\rho})}^2 + \|z_2\|_{H_{\rho}}^2 + \|z_3\|_{H_{\mathbf{h}}^{-1}}^2$ for $\mathbf{z} \in \mathcal{Y}$.

Proposition 9.

1. Let $(y^0, y^1) \in V \times V^*$. The discrete optimal control problem (17) has a solution $\bar{u}_{\mathbf{h}} \in \mathcal{M}_T$ (not unique in general), and any solution satisfies the bound $\|\bar{u}_{\mathbf{h}}\|_{\mathcal{M}_T} \leq C$.
2. In the case $\mathcal{M}_T = \mathcal{M}(\Omega, L^2(I))$, the following equality holds

$$\min_{u \in \mathcal{M}_T} j_{\mathbf{h}}(u) = \min_{u \in \mathcal{M}_{\mathbf{h}}} j_{\mathbf{h}}(u), \text{ with } \mathcal{M}_{\mathbf{h}} := V_{\tau} \otimes \mathcal{M}_h, \mathcal{M}_h := \text{span}\{\delta_{x_1}, \dots, \delta_{x_{N-1}}\},$$

where δ_{x_i} is the Dirac δ -function concentrated at the node x_i .

Item 1 and its proof are similar to the continuous case, see Proposition 6. Item 2 allows one to pass from (17) to the fully discrete optimal control problem

$$j_{\mathbf{h}}(u_{\mathbf{h}}) = \frac{1}{2} \|S_{\mathbf{h}}u_{\mathbf{h}} - \mathbf{z}\|_{\mathcal{Y}_{\mathbf{h}}}^2 + \alpha \|u_{\mathbf{h}}\|_{\mathcal{M}_T} \rightarrow \min_{u_{\mathbf{h}} \in \mathcal{M}_{\mathbf{h}}} . \tag{18}$$

Theorem 10. Let $\mathbf{z} \in \mathcal{Y}$, $(y^0, y^1) \in V \times H^{(-1/2)}$ as well as \bar{u} and $\bar{u}_{\mathbf{h}} \in \mathcal{M}_T$ be the optimal controls of respectively problems (6) and (17). Then there holds $\bar{u}_{\mathbf{h}} \rightarrow^* \bar{u}$ in \mathcal{M}_T and $\|\bar{u}_{\mathbf{h}}\|_{\mathcal{M}_T} \rightarrow \|\bar{u}\|_{\mathcal{M}_T}$ as $\mathbf{h} \rightarrow 0$.

The proof utilizes the error estimate (16) and properties of the projector $\pi_{\mathbf{h}}^1$.

3. We define the general discrete adjoint state equation in the following weak form

$$B_{\sigma}(v, p_{\mathbf{h}}) - (\rho v(0), p_{0h}^1)_H = (\rho(y - z_1), v)_{L^2(I \times \Omega)} + (\rho(y(T) - z_2), v(T))_H \text{ for any } v \in V_{\mathbf{h}}, \tag{19}$$

$$p_{\mathbf{h}}(T) = A_{\mathbf{h}}^{-1}(\rho \partial_t y(T) - z_3) \tag{20}$$

where y is the solution to the state equation (1). Its operator form is given at the end of the section.

Next we describe the complete discrete optimality system.

Proposition 11. The solution to the discrete optimality system consists of:

- (1) the optimal discrete state $(\bar{y}_{\mathbf{h}}, \bar{y}_{T_h}^1)$ satisfying (12)–(13) for $u = \bar{u}_{\mathbf{h}}$,
- (2) the optimal discrete adjoint state $(\bar{p}, \bar{p}_{0h}^1)$ satisfying (19)–(20) with $(\bar{y}_{\mathbf{h}}, \bar{y}_{\mathbf{h}}(T), \rho \bar{y}_{T_h}^1)$ in the role of $(y, y(T), \rho \partial_t y(T))$,
- (3) the discrete optimal control $\bar{u}_{\mathbf{h}}$ satisfying the variational inequality

$$\langle -\bar{p}_{\mathbf{h}}, u - \bar{u}_{\mathbf{h}} \rangle_{\mathcal{C}_T, \mathcal{M}_T} + \|\bar{u}_{\mathbf{h}}\|_{\mathcal{M}_T} \leq \|u\|_{\mathcal{M}_T} \text{ for any } u \in \mathcal{M}_T.$$

The proof is based on the rather standard Lagrange techniques and mimics the continuous case.

Now we present a stability bound and error estimates in \mathcal{C}_T for the discrete adjoint state equation. We define the space $\mathcal{Y}^{k+1/2} = \tilde{H}^{k+1/2, 0; 2}(Q) \times H^{(k+1/2)} \times H^{(k-1/2)}$ for $k = 0, 1$.

Proposition 12. Let $p = S^*(y - z_1, -(y(T) - z_2), A^{-1}(\rho \partial_t y - z_3))$ and $(p_{\mathbf{h}}, p_{0h}^1)$ be its discrete counterpart solving (19)–(20).

1. If $y \in \mathcal{C}(\bar{I}, H) \cap C^1(\bar{I}, V^*)$ and $\mathbf{z} \in \mathcal{Y}$, then the following stability bound holds

$$\|p_{\mathbf{h}}\|_{\mathcal{C}(\bar{I}, V)} + \|\rho p_{0h}^1\|_{H_{\mathbf{h}}^{-1}} \leq c(\|y - z_1\|_{L^2(I \times \Omega)} + \|y(T) - z_2\|_H + \|\rho \partial_t y(T) - z_3\|_{V^*}). \tag{21}$$

Moreover, for $u \in L^2(I, V^*)$ and $(y^0, y^1) \in H \times V^*$ the following error estimate holds

$$\|p - p_{\mathbf{h}}\|_{\mathcal{C}(\bar{I}, H)} + \|\rho(\partial_t p(0) - p_{0h}^0)\|_{H_{\mathbf{h}}^{-1}} \leq c(\tau + h)^{2/3}(\|u\|_{L^2(I, V^*)} + \|(y^0, y^1)\|_{H \times V^*}).$$

2. If $u \in H^{-1/2, 0; 2}(Q)$, $\mathbf{z} \in \mathcal{Y}^{1/2}$ and $(y^0, y^1) \in H^{(1/2)} \times H^{(-1/2)}$, then the following error estimate in the uniform norm holds

$$\|p - p_{\mathbf{h}}\|_{\mathcal{C}(\bar{I} \times \bar{\Omega})} \leq c(\tau + h)^{2/3}(\|u\|_{H^{-1/2, 0; 2}(Q)} + \|\mathbf{z}\|_{\mathcal{Y}^{1/2}} + \|(y^0, y^1)\|_{H^{(1/2)} \times H^{(-1/2)}}).$$

3. If $u \in SHW^{-1/2,1;2}(Q)$, $\mathbf{z} \in \mathcal{Y}^{3/2}$ and $(y^0, y^1) \in H^{(3/2)} \times H^{(1/2)}$, then the following higher-order error estimate holds

$$\|p - p_h\|_{L^2(I, C_0(\Omega))} \leq c(\tau + h)^{4/3} (\|u\|_{SHW^{-1/2,1;2}(Q)} + \|\mathbf{z}\|_{\mathcal{Y}^{3/2}} + \|(y^0, y^1)\|_{H^{(3/2)} \times H^{(1/2)}}).$$

The proof follows mainly from [20, Theorems 2.1, 4.3 and 5.3]. Notice that a priori stability bound (21) (taken for $y = 0$) implies the unique solvability of the general discrete adjoint state equation (19)–(20).

4. To complete the section, we present the time-stepping formulation of the discrete state and adjoint state equations. We define the forward and backward difference quotients and the average operator in time

$$\delta_t v_m = \frac{v_{m+1} - v_m}{\tau}, \quad \bar{\delta}_t v_m = \frac{v_m - v_{m-1}}{\tau},$$

$$B^\tau v_0 = \frac{1}{3}v_0 + \frac{1}{6}v_1, \quad B^\tau v_m = \frac{1}{6}v_{m-1} + \frac{2}{3}v_m + \frac{1}{6}v_{m+1}, \quad 1 \leq m \leq M - 1, \quad B^\tau v_M = \frac{1}{6}v_{M-1} + \frac{1}{3}v_M.$$

We also define the self-adjoint positive-definite operators (the mass and stiffness matrices) B_h and L_h acting in V_h such that

$$(B_h \varphi_h, \psi_h)_{V_h} = (\rho \varphi_h, \psi_h)_H, \quad (L_h \varphi_h, \psi_h)_{V_h} = (\kappa \partial_x \varphi_h, \partial_x \psi_h)_H \quad \text{for any } \varphi, \psi \in V_h.$$

We recall the piecewise-linear ‘‘hat’’ basis functions in V^τ and V_h such that $e_m^\tau(t_k) = \delta_{m,k}$ for any $k, m = 0, \dots, M$ and $e_j^h(x_k) = \delta_{j,k}$ for any $j = 1, \dots, N - 1$ and $k = 0, \dots, N$; here $\delta_{m,k}$ is the Kronecker delta. For $w \in V^*$ and $u \in L^2(I, V^*)$, we introduce the vectors of averages $w^h = \{\langle w, e_j^h \rangle_\Omega\}_{j=1}^{N-1}$ and

$$u_m^h = \frac{1}{\tau} \{(u^h, e_m^\tau)_{L^2(I)}\}_{j=1}^{N-1}, \quad 1 \leq m \leq M - 1, \quad u_m^h = \frac{2}{\tau} \{(u^h, e_m^\tau)_{L^2(I)}\}_{j=1}^{N-1}, \quad m = 0, M.$$

Let $y_{h,m} = \{y_h(x_j, t_m)\}_{j=1}^{N-1}$. The forward time-stepping is implemented according to the operator equations that are equivalent to integral identities (12)–(13):

$$(B_h + \sigma \tau^2 L_h) \delta_t \bar{\delta}_t y_{h,m} + L_h y_{h,m} = u_m^h, \quad m = 2, \dots, M - 1, \tag{22}$$

$$(B_h + \sigma \tau^2 L_h) \delta_t y_{h,1} + \frac{\tau}{2} L_h y_{h,0} = (\rho y^1)^h + \frac{\tau}{2} u_0^h, \tag{23}$$

$$B_h y_{h,0} = (\rho y^0)^h \tag{24}$$

together with the counterpart of (23) at time T for y_{Th}^1 :

$$B_h y_{Th}^1 = (B_h + \sigma \tau^2 L_h) \bar{\delta}_t y_{h,M} - \frac{\tau}{2} L_h y_{h,M} + \frac{\tau}{2} u_M^h. \tag{25}$$

From equations (24), (23), (22), and (25), one sequentially finds $y_{h,0}$, $y_{h,1}$, $y_{h,m+1}$ and y_{Th}^1 .

Next the adjoint (backward) time-stepping is implemented according to the similar operator equations that are equivalent to the integral identity (19) and formula (20) with $(y_h, y_h(T), \rho y_{Th}^1)$ in the role of $(y, y(T), \rho \partial_t y(T))$:

$$(B_h + \sigma \tau^2 L_h) \delta_t \bar{\delta}_t p_{h,m} + L_h p_{h,m} = B_h B^\tau y_{h,m} - (\rho z_1)_m^h, \quad m = M - 1, \dots, 1, \tag{26}$$

$$-(B_h + \sigma \tau^2 L_h) \bar{\delta}_t p_{h,M} + \frac{\tau}{2} L_h p_{h,M} = B_h y_{h,M} - (\rho z_2)^h + \frac{\tau}{2} (B_h B^\tau y_{h,M} - (\rho z_1)_M^h), \tag{27}$$

$$L_h p_{h,M} = B_h y_{Th}^1 - z_3^h \tag{28}$$

together with the counterpart of (25) for p_{0h}^1 :

$$B_h p_{0h}^1 = (B_h + \sigma \tau^2 L_h) \delta_t p_{h,0} + \frac{\tau}{2} L_h p_{h,0} - \frac{\tau}{2} (B_h B^\tau y_{h,0} - (\rho z_1)_0^h). \tag{29}$$

From equations (28), (27), (26), and (29), one sequentially finds $p_{h,M}$, $p_{h,M-1}$, $p_{h,m-1}$ and p_{0h}^1 .

Notice that for $\sigma = 1/4$ the three-level in time method (22)–(25) is closely related to the well-known two-level Crank–Nicolson method applied to the 1D generalized wave equation rewritten formally as the first order in time system

$$\partial_t y = v, \quad \rho \partial_t v - \partial_x (\kappa \partial_x y) = u \quad \text{in } I \times \Omega,$$

see [20, Section 8] for details, as well as to the Petrov–Galerkin method like in [10]. In addition, after the mass lumping, for $\sigma = 0$, the method becomes explicit and is related to the leap-frog scheme; moreover, for any σ , it becomes close to three-level finite-difference schemes with such weight in time, e.g., see [15].

4. Error estimates for the state variable and the cost functional

Now we turn to the final results of the paper. First we give a general estimate for the error in the state variable $\bar{y} - \bar{y}_h$.

Proposition 13. *Let $\mathbf{z} \in \mathcal{Y}$ and $(y^0, y^1) \in V \times V^*$. Then the following inequality holds*

$$\|S\bar{u} - S_h\bar{u}_h\|_{\mathcal{Y}_h} \leq \|S\bar{u} - S_h\bar{u}\|_{\mathcal{Y}_h} + C\|\bar{p} - \hat{p}_h\|_{C_T}^{1/2},$$

where \hat{p}_h solves problem (19)–(20) for $(y, y(T), \rho \partial_t y(T)) = (\bar{y}, \bar{y}(T), \rho \partial_t \bar{y}(T)) \equiv S\bar{u}$.

The proof is similar to [9].

Now we state final error estimates for the state variable whose orders depend on the choice of \mathcal{M}_T and the data smoothness.

Theorem 14.

1. For $\mathcal{M}_T = L^2_{w^*}(I, \mathcal{M}(\Omega))$, $\mathbf{z} \in \mathcal{Y}^{1/2}$ and $(y^0, y^1) \in V \times H$, the following error estimate holds

$$\|\bar{y} - \bar{y}_h\|_{L^2(I \times \Omega)} + \|(\bar{y} - \bar{y}_h)(T)\|_H + \|\rho(\partial_t \bar{y}(T) - \bar{y}_{Th}^1)\|_{H_h^{-1}} \leq C(\tau + h)^{1/3}. \quad (30)$$

2. For $\mathcal{M}_T = \mathcal{M}(\Omega, L^2(I))$, $\mathbf{z} \in \mathcal{Y}^{3/2}$ and $(y^0, y^1) \in H^{(3/2)} \times H^{(1/2)}$, the following higher-order error estimate holds

$$\|\bar{y} - \bar{y}_h\|_{L^2(I \times \Omega)} + \|(\bar{y} - \bar{y}_h)(T)\|_H + \|\rho(\partial_t \bar{y}(T) - \bar{y}_{Th}^1)\|_{H_h^{-1}} \leq C(\tau + h)^{2/3}.$$

The proof follows from Proposition 13 due to Propositions 8 and 12 and embeddings (10)–(11). In Item 2, Theorem 7 is also essential.

Next we present a general error estimate for the cost functional.

Proposition 15. *Let $(y^0, y^1) \in V \times H$. Then for any $u \in \mathcal{M}_T$, the following inequality holds*

$$\begin{aligned} |j(u) - j_h(u)| &\leq c(\|Su - S_h u\|_{\mathcal{Y}_h}^2 \\ &+ (\|u\|_{\mathcal{M}_T} + \|(y^0, y^1)\|_{V \times H})(\|p - p_h\|_{C_T} + \|p(0) - p_h(0)\|_H + h\|\partial_t p(0)\|_H + \|\rho(\partial_t p(0) - p_{0h}^1)\|_{H_h^{-1}}) \\ &\quad + \|(A^{-1} - A_h^{-1})(\rho \partial_t y(T))\|_{V_\kappa}^2 + \|(A^{-1} - A_h^{-1})z_3\|_{V_\kappa}^2) \end{aligned}$$

with $(y, y(T), \rho \partial_t y(T)) = Su$ and the same p and (p_h, p_{0h}^1) as in Proposition 12.

Now we state the final error estimate for the cost functional of a higher order than (30) under the same assumptions.

Theorem 16. *For $\mathcal{M}_T = L^2_{w^*}(I, \mathcal{M}(\Omega))$, $\mathbf{z} \in \mathcal{Y}^{1/2}$ and $(y^0, y^1) \in V \times H$, the following error estimate holds*

$$|j(\bar{u}) - j_h(\bar{u}_h)| \leq C(\tau + h)^{2/3}.$$

The proof follows from Proposition 15 due to Propositions 8 and 12 and embedding (10).

The details of the implementation for the full numerical method involving a regularization of the fully discrete control problem (18) and a generalized Newton type method to solve it as well as successful computational results can be found in [14,17].

The full proofs of the results announced in this note and some computational results are given in [18].

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