C. R. Acad. Sci. Paris, Ser. I 356 (2018) 420–432

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Algebraic geometry/Differential geometry

Singularities and semistable degenerations for symplectic topology

Singularités et dégénérescences semi-stables en topologie symplectique

Mohammad F. Tehrani [a], Mark McLean [b], Aleksey Zinger [b,1]

[a] *Simons Center for Geometry and Physics, Stony Brook University, Stony Brook, NY 11794, USA*
[b] *Department of Mathematics, Stony Brook University, Stony Brook, NY 11794, USA*

ARTICLE INFO

Article history:
Received 6 July 2017
Accepted after revision 20 February 2018
Available online 23 February 2018

Presented by Claire Voisin

ABSTRACT

We overview our work [7–11,6] defining and studying normal crossings varieties and subvarieties in symplectic topology. This work answers a question of Gromov on the feasibility of introducing singular (sub)varieties into symplectic topology in the case of normal crossings singularities. It also provides a necessary and sufficient condition for smoothing normal crossings symplectic varieties. In addition, we explain some connections with other areas of mathematics and discuss a few directions for further research.

RÉSUMÉ

Nous résumons nos travaux [7–11,6], où l'on définit et étudie les variétés et sous-variétés à croisements normaux en géométrie symplectique. Ils répondent à une question de Gromov sur la possibilité d'introduire de telles (sous-)variétés singuliéres en topologie symplectique, dans le cas de singularités à croisements normaux. Nous donnons également une condition nécessaire et suffisante pour lisser ces variétés symplectiques à croisements normaux. De plus, nous expliquons les liens avec d'autres domaines mathématiques et discutons quelques directions pour de futures recherches.

1. Introduction

Singularities are ubiquitous in algebraic geometry. Even if one wishes to study only smooth varieties, one is often forced to investigate certain singular varieties as well. This is especially true in enumerative geometry, moduli theory, and the minimal model program. Degeneration techniques are particular instances where such singularities arise. Suppose, for ex-

E-mail addresses: mtehrani@scgp.stonybrook.edu (M.F. Tehrani), markmclean@math.stonybrook.edu (M. McLean), azinger@math.stonybrook.edu (A. Zinger).

[1] Partially supported by NSF grant 1500875.

https://doi.org/10.1016/j.crma.2018.02.009
1631-073X/

ample, that one wishes to study a particular deformation-invariant property of a smooth variety X such as Gromov–Witten invariants. One might then try to degenerate X through a flat one-parameter family to a simpler, but possibly singular, variety X_0 and study X_0 instead. For instance, when one wishes to compute Gromov–Witten invariants, the degeneration formulas of [19,31] can be particularly useful. In mirror symmetry, such ideas have been suggested in [27] and are central to the Gross–Siebert program [15].

Gromov [14, p. 343] asked if there was an appropriate notion of singular variety in symplectic topology and when such varieties could be smoothed. Introducing singularities and degeneration techniques into symplectic topology would be extremely useful for the following reasons. First, symplectic manifolds are significantly more flexible than algebraic varieties, since one can apply Moser style arguments and sometimes even an h-principle [14]. Smoothing or degenerating such symplectic varieties should then become a problem of a topological nature; smoothing algebraic varieties is in contrast a notoriously difficult problem. Therefore one would not need to rely on subtle analytic invariants. Second, these techniques could be used to study a much larger class of symplectic manifolds (not just those coming from Kähler geometry). For instance, they could be useful in mirror symmetry for non-Kähler symplectic manifolds; see for example [37, Section 2.3]. They could also be used to construct interesting examples of symplectic manifolds by smoothing singular ones, as has been done to great effect in [13].

This note is an overview of our work [7–11,6], which introduces and studies symplectic topology notions of normal crossings (or NC) divisor and variety. In Section 2, we explain what these notions are. In Section 3, we describe geometric notions of regularization for NC symplectic divisors and varieties, which is basically a "nice" neighborhood of the singular locus. Every NC symplectic divisor/variety is deformation equivalent to one with a regularization. In fact, we propose to

view a symplectic (sub)variety as a deformation equivalence class of objects, not as an individual object.

In Section 4, we explain our result on the smoothability of NC symplectic varieties. It provides a purely topological necessary *and* sufficient condition for such a variety to be smoothable. In Section 5, we explain how to use certain local Hamiltonian torus actions to degenerate a smooth symplectic manifold into an NC symplectic variety. Finally, in Section 6, we discuss connections with other areas of mathematics and directions for further developments.

We would like to thank A. Cannas da Silva, K. Fukaya, E. Ionel, E. Lerman, B. Parker, A. Pires, H. Ruddat, D. Stapleton, J. Starr, and D. Sullivan for enlightening discussions on various aspects of our work [7–11,6]. We are also grateful to the referee for reading our initial submission very thoroughly, pointing out a number of typos and inaccurate wordings, and suggesting several improvements in the exposition.

2. Normal crossings (sub)varieties

A symplectic manifold is a manifold X of an even real dimension $2n$ together with a closed nondegenerate 2-form ω, i.e. $\mathrm{d}\omega = 0$ and $\omega^n|_x \neq 0$ for every $x \in X$. In particular, ω^n is a volume/orientation form on X. An almost complex structure J on X is a vector bundle endomorphism J of TX covering id_X such that $J^2 = -\mathrm{id}_{TX}$. Every complex (holomorphic) structure on X determines an almost complex structure on X, but the converse is not true if $n > 1$. The Nijenhuis $(2,1)$-tensor,

$$N_J(\xi,\zeta) = \frac{1}{4}\big([\xi,\zeta] + J[\xi, J\zeta] + J[J\xi,\zeta] - [J\xi, J\zeta]\big) \in \Gamma(X;TX) \qquad \forall \xi,\zeta \in \Gamma(X;TX),$$

is the obstruction to the integrability of J, i.e. J arises from a holomorphic structure on X if and only if $N_J \equiv 0$; see [28].

Every symplectic manifold (X,ω) can be equipped with an ω-compatible almost complex structure in the sense that $\omega(\cdot, J\cdot)$ is a metric. The space $\mathcal{J}_\omega(X)$ of ω-compatible almost complex structures is infinite-dimensional and contractible. Such a triple (X,ω,J) is called an almost Kähler manifold. A Kähler manifold is an almost Kähler manifold such that the almost complex structure is integrable. The category of Kähler manifolds includes smooth complex projective varieties, i.e. smooth varieties cut out by polynomial equations in a complex projective space $\mathbb{P}^N$.

In contrast to algebraic/Kähler geometry, symplectic topology is significantly more "flexible". For instance, symplectic manifolds have no local invariants and smooth families of symplectic manifolds with cohomologous symplectic forms are symplectomorphic. We are interested in finding out which algebraic structures can be generalized to useful structures in the symplectic topology category.

A symplectic submanifold of a symplectic manifold (X,ω) is a submanifold V of X such that $\omega|_V$ is a symplectic form. Symplectic submanifolds are the analogues of smooth subvarieties in (complex) algebraic geometry. For example, a smooth symplectic divisor is a symplectic submanifold of real codimension 2 (or complex codimension 1) and a smooth symplectic curve is a symplectic submanifold of real dimension 2. The normal bundle

$$\pi_{\mathcal{N}_X V}: \mathcal{N}_X V \equiv \frac{TX|_V}{TV} \approx TV^{\omega} \equiv \big\{ v \in T_xX : x \in V,\ \omega(v,w) = 0\ \forall\, w \in T_xV \big\} \longrightarrow V \tag{1}$$

of a symplectic submanifold V of (X,ω) inherits a fiberwise symplectic form $\omega|_{\mathcal{N}_X V}$ from ω. An $\omega|_{\mathcal{N}_X V}$-compatible Hermitian structure on $\mathcal{N}_X V$ is a triple $(\mathfrak{i},\rho,\nabla)$, where $\mathfrak{i}$ is an $\omega|_{\mathcal{N}_X V}$-compatible complex structure on $\mathcal{N}_X V$, ρ is a Hermitian metric with

$$\rho_{\mathbb{R}}(\cdot,\cdot)=\omega|_{\mathcal{N}_X V}(\cdot,\mathfrak{i}\cdot),$$

and ∇ is a Hermitian connection compatible with $(\mathfrak{i},\rho)$. The space of $\omega|_{\mathcal{N}_X V}$-compatible Hermitian structures is non-empty and contractible.

Let $V\subset X$ be a smooth symplectic divisor. An $\omega|_{\mathcal{N}_X V}$-compatible Hermitian structure $(\mathfrak{i},\rho,\nabla)$ as above determines a 1-form α_∇ on $\mathcal{N}_X V-V$ whose restriction to each fiber $\mathcal{N}_X V|_x-\{x\}\cong\mathbb{C}^*$ is the 1-form $\mathrm{d}\theta$ with respect to the polar coordinates (r,θ) on $\mathbb{C}$. We also denote by ρ the square-of-the-norm function on $\mathcal{N}_X V$. If $\widetilde{J}$ is the almost complex structure on $\mathcal{N}_X V$ induced by $\mathfrak{i}$ and an almost complex structure J_V on V via ∇, then

$$\alpha_\nabla=d^c\ln\rho\equiv-\frac{1}{4\pi}\frac{\mathrm{d}\rho}{\rho}\circ\widetilde{J}_V\,.$$

The closed 2-form

$$\widehat{\omega}\equiv\pi^*(\omega|_V)+\frac{1}{2}\mathrm{d}(\rho\alpha_\nabla)\in\Omega^2(\mathcal{N}_X V) \tag{2}$$

is well defined, is nondegenerate in a small neighborhood of V, and restricts to the standard symplectic form $\mathrm{d}(r^2\mathrm{d}\theta)$ on each fiber. By the Symplectic Neighborhood Theorem [23, Theorem 3.30], there exists an identification (called an ω-regularization, in what follows)

$$\Psi:\mathcal{N}'_X V\longrightarrow X,\qquad \Psi|_V=\mathrm{id}_V,\qquad \mathrm{d}\Psi|_V=\mathrm{id}, \tag{3}$$

of a small neighborhood $\mathcal{N}'_X V$ of V in $\mathcal{N}_X V$ with a neighborhood of V in X such that $\Psi^*\omega=\widehat{\omega}$. Regularizations are useful for applications, such as the symplectic sum construction of [13,22]. They also ensure the existence of almost complex structures J on X that are “nice” along V. These are in turn useful for constructing relative Gromov–Witten invariants of (X,V), for example.

In the 1980s, Gromov combined the rigidity of algebraic geometry with the flexibility of the smooth category and initiated the use of J-holomorphic maps from Riemann surfaces $(\Sigma,\mathfrak{j})$ into (X,J),

$$u:(\Sigma,\mathfrak{j})\longrightarrow(X,J),\qquad \bar\partial u\equiv\frac{1}{2}\big(\mathrm{d}u+J\mathrm{d}u\circ\mathfrak{j}\big)=0, \tag{4}$$

as a generalization of holomorphic maps. The singularities of the image of a J-holomorphic map u are locally the same as the algebraic ones; see [24, Appendix E]. If J is ω-compatible, then the smooth locus of u is a symplectic submanifold of (X,ω). A singular symplectic variety in complex dimension 1 can thus be defined as a subset of X that can be realized as the image of a J-holomorphic map. The spaces of J-holomorphic maps have a nice deformation theory, which makes it possible to study these objects in families; see [24, Chapter 3] and [21, Section 3], for example. The idea of studying J-holomorphic maps from higher-dimensional domains is not as promising because the Cauchy–Riemann equation (4) is over-determined if the dimension of Σ is greater than 2.

In parallel with his introduction of J-holomorphic curve techniques into symplectic topology, Gromov asked about the feasibility of introducing notions of singular (sub-)varieties of higher dimension suitable for this field; see [14, p. 343]. By the last paragraph, the idea of defining such singular objects as images of J-holomorphic maps is not promising and we should consider an intrinsic approach. Nevertheless, we still require that for such a singular object V in X, the space (or a nice subspace) of almost complex structures on X compatible with V to be non-empty and “manageable”.

In algebraic geometry, divisors, i.e. subvarieties of codimension 1, are dual objects to curves and have long been of particular importance. On the symplectic side, smooth symplectic divisors appear in different contexts such as in relation with complex line bundles [5], symplectic sum constructions [13,22], relative Gromov–Witten theory and degeneration formulas for Gromov–Witten invariants [36,20,17,30], symplectic geometry of affine varieties [25,26], and homological mirror symmetry [33]. The following question is thus one of the most important specializations of Gromov’s inquiry:

Can one define a soft notion of (singular) symplectic divisor that only involves soft intrinsic symplectic data, but at the same time is compatible with rigid auxiliary almost Kähler data needed for making such a notion useful?

NC divisors/varieties are the most basic and important classes of singular objects in complex algebraic (or Kähler) geometry. An NC divisor in a smooth variety X is a subvariety V locally defined by an equation of the form

$$z_1\cdots z_k=0 \tag{5}$$

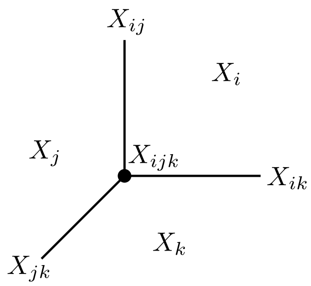

Fig. 1. A 3-fold SC variety.

in a holomorphic coordinate chart $(z_1, \dots, z_n)$ on X. A simple normal crossings (or SC) divisor is a global transverse union of smooth divisors, i.e.

$$V = \bigcup_{i \in S} V_i \subset X.$$

An NC variety of complex dimension n is a variety $X_\emptyset$ that can be locally embedded as an NC divisor in $\mathbb{C}^{n+1}$. In other words, every sufficiently small open set U in $X_\emptyset$ can be written as

$$U = \Big(\bigsqcup_{i \in S} U_i \Big) \Big/ \sim, \qquad U_{ij} \approx U_{ji} \quad \forall\, i, j \in S,\ i \neq j,$$

where $\{U_{ij}\}_{j \in S-i}$ is an SC divisor in a smooth component U_i of U. A simple normal crossings (or SC) variety is a global transverse union of smooth varieties $\{X_i\}_{i \in S}$ along SC divisors $\{X_{ij}\}_{j \in S-i}$ in X_i, i.e.

$$X_\emptyset = \Big(\bigsqcup_{i \in S} X_i \Big) \Big/ \sim, \qquad X_{ij} \approx X_{ji} \quad \forall\, i, j \in S,\ i \neq j.$$

A 3-fold SC variety is shown in Fig. 1.

NC varieties often emerge as nice limits of smooth algebraic varieties. A semistable degeneration is a one-parameter family $\pi : \mathcal{Z} \longrightarrow \Delta$, where Δ is a disk around the origin in $\mathbb{C}$ and $\mathcal{Z}$ is a smooth variety, such that the central fiber $\mathcal{Z}_0 \equiv \pi^{-1}(0)$ is an NC variety and the fibers over $\Delta^* \equiv \Delta - \{0\}$ are smooth. Semistable degenerations play a central role in algebraic geometry and mirror symmetry. They appear in compactification of moduli spaces, Hodge theory, Gromov–Witten theory, etc.

Example 1 *([2, Section 6.2]).* Let P be a homogeneous cubic polynomial in $x_0, \dots, x_3$ and

$$\mathcal{Z}' = \big\{ \big(t, [x_0, x_1, x_2, x_3]\big) \in \mathbb{C} \times \mathbb{P}^3 : x_1 x_2 x_3 = tP(x_0, x_1, x_2, x_3) \big\} \subset \mathbb{C} \times \mathbb{P}^3.$$

Let $\pi' : \mathcal{Z}' \longrightarrow \mathbb{C}$ be the projection map to the first factor. If P is generic and $\Delta \subset \mathbb{C}$ is a sufficiently small disk around the origin, then $\pi'^{-1}(t)$ is a smooth cubic hypersurface (divisor) in $\mathbb{P}^3$ for every $t \in \Delta^*$. For $t = 0$, $\pi'^{-1}(t)$ is the SC variety

$$X'_\emptyset = \{0\} \times \big(X'_1 \cup X'_2 \cup X'_3 \big) \subset \{0\} \times \mathbb{P}^3 \qquad \text{with}$$
$$X'_i \equiv (x_i = 0) \approx \mathbb{P}^2 \ \ \forall\, i \in \{1, 2, 3\}, \quad X'_{ij} \equiv X'_i \cap X'_j \approx \mathbb{P}^1 \ \ \forall\, i, j \in \{1, 2, 3\},\ i \neq j.$$

However, the total space $\mathcal{Z}'$ of π' is not smooth at the 9 points of

$$\mathcal{Z}'^{\,\mathrm{sing}} \equiv \{0\} \times \big(X'_\partial \cap (P = 0) \big) \subset X_\emptyset, \qquad \text{where} \quad X'_\partial = X'_{12} \cup X'_{13} \cup X'_{23} \subset \mathbb{P}^3.$$

A small resolution $\mathcal{Z}$ of $\mathcal{Z}'|_\Delta \equiv \pi'^{-1}(\Delta)$ can be obtained by blowing up each singular point on X'_{ij} in either X'_i or X'_j. The map π' then induces a projection $\pi : \mathcal{Z} \longrightarrow \Delta$ and defines a semistable degeneration. Every fiber of π over Δ^* is a smooth cubic surface. The central fiber $\pi^{-1}(0)$ is the SC variety $X_\emptyset \equiv X_1 \cup X_2 \cup X_3$ with 3 smooth components, each a blowup of $\mathbb{P}^2$ at some number of points. If each singular point on X'_{ij} is blown up in X'_i with $i < j$, then $\mathcal{Z}$ is obtained from $\mathcal{Z}'$ through two global blowups of $\mathbb{C} \times \mathbb{P}^3$ and is thus projective.

As a first step to answer Gromov's inquiry, we introduce topological notions of NC symplectic divisor and variety in [7,9].

Definition 2 *([7, Definition 2.1]).* An SC symplectic divisor in a symplectic manifold (X,ω) is a finite transverse union $V\equiv\bigcup_{i\in S}V_i$ of smooth symplectic divisors $\{V_i\}_{i\in S}$ such that for every $I\subset S$ the submanifold

$$V_I\equiv\bigcap_{i\in I}V_i\subset X$$

is symplectic and its symplectic and intersection orientations are the same.

Such a collection $\{V_i\}_{i\in S}$ of symplectic submanifolds of (X,ω) is sometimes called positively intersecting. For example, if the real dimension of X is 4, V is an SC symplectic divisor if and only if the V_i's intersect transversely and every point of a pairwise intersection $V_i\cap V_j$ is positive. By [7, Example 2.7], the compatibility-of-orientation condition along all strata cannot be replaced with a condition on a smaller subset of such strata.

Definition 3 *([7, Definition 2.5]).* An SC symplectic variety is a pair $(X_\emptyset,(\omega_i)_{i\in S})$, where

$$X_\emptyset=\Big(\bigsqcup_{i\in S}X_i\Big)\Big/\sim,\quad X_{ij}\approx X_{ji}\ \ \forall\, i,j\in S,\ i\neq j,$$

for a finite collection $(X_i,\omega_i)_{i\in S}$ of symplectic manifolds, some SC symplectic divisor $\{X_{ij}\}_{j\in S-\{i\}}$ in X_i for each $i\in S$, and symplectic identifications $X_{ij}\approx X_{ji}$ for all $i,j\in S$ distinct.

An NC symplectic divisor is a subset of a symplectic manifold (X,ω) locally equal to an SC symplectic divisor. An NC symplectic variety is a space locally equal to an SC symplectic variety. In other words, it is a topological space together with "charts" mapping homeomorphically to SC symplectic varieties and with structure-preserving overlap maps. There are global descriptions of NC symplectic divisors and varieties in terms of transverse immersions that respect certain permutation symmetries; see [9] for details.

3. Regularizations

In order to show that Definitions 2 and 3 are appropriate analogues of the corresponding notions in algebraic geometry and are suitable for applications in symplectic topology, we have to show that the SC symplectic divisors and varieties of Definitions 2 and 3 admit (possibly after deformations) a contractible space of compatible almost complex structures. In the case of a smooth divisor, one needs a regularization (3) to construct "nice" almost complex structures. We need a similar thing for NC symplectic divisors and varieties. In other words, we need to construct "nice" neighborhoods of an NC symplectic divisor V in the ambient manifold X and of the singular locus X_∂ of an SC symplectic variety $X_\emptyset$ (after possibly deforming these objects). We describe below what regularizations for SC/NC divisors/varieties are. The precise definitions are contained in [7,9].

Let V be an SC symplectic divisor in a symplectic manifold (X,ω) as in Definition 2. By the transversality assumption, the homomorphisms

$$\mathcal{N}_XV_I\longrightarrow\bigoplus_{i\in I}\mathcal{N}_XV_i\big|_{V_I},\qquad I\subset S,$$

induced by the inclusions $TV_I\subset TV_i|_{V_I}$ are isomorphisms. For $I'\subset I\subset S$, define

$$\mathcal{N}_{I;I'}=\bigoplus_{i\in I-I'}\mathcal{N}_XV_i|_{V_I}\subset\mathcal{N}_XV_I\,.$$

We denote by

$$\pi_I:\mathcal{N}_XV_I\longrightarrow V_I,\quad \Pi_I:TX|_{V_I}\longrightarrow\mathcal{N}_XV_I,\quad \pi_{I;I'}:\mathcal{N}_XV_I=\mathcal{N}_{I;I'}\oplus\mathcal{N}_{I;I-I'}\longrightarrow\mathcal{N}_{I;I'}$$

the natural projection maps.

A system of regularizations for $\{V_i\}_{i\in S}$ in X is a collection of smooth embeddings

$$\Psi_I:\mathcal{N}'_XV_I\longrightarrow X,\quad I\subset S,$$

from open neighborhoods $\mathcal{N}'_XV_I\subset\mathcal{N}_XV_I$ of V_I so that $\Psi_I|_{V_I}=\mathrm{id}_{V_I}$, $\mathrm{d}\Psi_I$ induces the identity map on $\mathcal{N}_XV_I$, and

$$\Psi_I\big(\mathcal{N}_{I;I'}\cap\mathrm{Dom}(\Psi_I)\big)=V_{I'}\cap\mathrm{Im}(\Psi_I)\qquad\forall\,I'\subset I\subset S\,.$$

This implies that $\mathrm{d}\Psi_I$ induces an isomorphism

$$\mathfrak{D}\Psi_{I;I'}:\pi^*_{I;I'}\mathcal{N}_{I;I-I'}\big|_{\mathcal{N}_{I;I'}\cap\mathrm{Dom}(\Psi_I)}\longrightarrow\mathcal{N}_XV_{I'}\big|_{V_{I'}\cap\mathrm{Im}(\Psi_I)};$$

see [7, Section 2.2]. It is given by the derivative

$$\mathfrak{D}\Psi_{I;I'}\big(v_{I'},v_{I-I'}\big)\equiv\Pi_{I'}\bigg(\frac{\mathrm{d}}{\mathrm{d}t}\Psi_I(v_{I'},tv_{I-I'})\Big|_{t=0}\bigg)\quad\forall(v_{I'},v_{I-I'})\in\pi_{I;I'}^{-1}(\mathrm{Dom}(\Psi_I)),\tag{6}$$

of Ψ_I in the direction "normal" to $V_{I'}$. In the $I=I'$ case, this derivative is the identity map. If $|S|=1$, i.e. V is a smooth divisor, a system of regularizations is a single map Ψ as in (3) and Definition 4 below imposes no additional condition.

Definition 4. A regularization for V in X is a system of regularizations for $\{V_i\}_{i\in S}$ in X as above such that

$$\mathrm{Dom}(\Psi_I)=\mathfrak{D}\Psi_{I;I'}^{-1}\big(\mathrm{Dom}(\Psi_{I'})\big),\quad \Psi_I=\Psi_{I'}\circ\mathfrak{D}\Psi_{I;I'}|_{\mathrm{Dom}(\Psi_I)}\qquad\forall\, I'\subset I\subset S\,.$$

Definition 5 *([7, Definition 2.9]).* An ω-regularization for V in X consists of a choice of Hermitian structure $(\mathfrak{i}_{I;i},\rho_{I;i},\nabla^{(I;i)})$ on $\mathcal{N}_X V_i|_{V_I}$ for all $i\in I\subset S$ together with a regularization for V in X as in Definition 4, so that

$$\Psi_I^*\omega=\pi^*(\omega|_{V_I})+\frac{1}{2}\sum_{i\in I}\mathrm{d}(\rho_{I;i}\alpha_{\nabla^{(I;i)}})\quad\forall\, I\subset S,$$

and (6) is an isomorphism of split Hermitian vector bundles for all $I'\subset I\subset S$.

Since the intersection of the singular locus of an SC symplectic variety with each irreducible component X_i is an SC symplectic divisor inside X_i, one gets the following straightforward definition of a regularization for these objects.

Definition 6 *([7, Definition 2.15]).* Let $(X_\emptyset,(\omega_i)_{i\in S})$ be an SC symplectic variety as in Definition 3. An $(\omega_i)_{i\in S}$-regularization is an ω_i-regularization $\mathcal{R}_i$ for the SC symplectic divisor $\{X_{ij}\}_{j\in S-\{i\}}$ in X_i for each $i\in S$ so that the restrictions of $\mathcal{R}_i$ and $\mathcal{R}_j$ to X_{ij} give identical $\omega_i|_{X_{ij}}$-regularizations of the SC symplectic divisor $\{X_{ijk}\}_{k\in S-\{i,j\}}$ in X_{ij} for all distinct $i,j\in S$.

Since an NC symplectic divisor $V\subset X$ is equal to an SC symplectic divisor $V_p\subset X$ near each point $p\in V$, we can define an ω-regularization for V to be an ω-regularization for V_p for each $p\in V$ so that any two such ω-regularizations associated with $p,q\in V$ agree on $V_p\cap V_q$. Regularizations for NC symplectic varieties are defined similarly; see [9] for details.

Unlike a smooth symplectic divisor, an NC symplectic divisor $V\subset X$ need not admit an ω-regularization. If an SC symplectic divisor V as in Definition 2 admits an ω-regularization, then its smooth components V_i are in fact ω-orthogonal. On the other hand, many applications (such as symplectic constructions and Gromov–Witten theory) care only about the deformation equivalence classes of the symplectic structure. Therefore, the alternative philosophy proposed in [7] is to

study NC symplectic divisors/varieties up to deformation equivalence and show that each deformation equivalence class has a subspace of sufficiently "nice" representatives.

More concretely, for a transverse union $V\equiv\bigcup_{i\in S}V_i$ of closed real codimension 2 submanifolds of a manifold X, let $\mathrm{Symp}^+(X,V)$ be the space of all symplectic forms ω on X such that V is an SC symplectic divisor in (X,ω). We also define a space of auxiliary data $\mathrm{Aux}(X,V)$ to be the space of pairs $(\omega,\mathcal{R})$, where $\omega\in\mathrm{Symp}^+(X,V)$ and $\mathcal{R}$ is an ω-regularization of V in X. Let

$$\Pi:\mathrm{Symp}^+(X,V)\longrightarrow H^2(M;\mathbb{R})$$

be the map sending ω to its de Rham equivalence class $[\omega]$. The following is a weaker version of the main result of [7] for SC symplectic divisors.

Theorem 7 *([7, Theorem 2.13]). Let $V\equiv\bigcup_{i\in S}V_i$ be a transverse union of closed real codimension 2 submanifolds of a manifold X. Then the projection maps*

$$\pi:\mathrm{Aux}(X,V)\longrightarrow\mathrm{Symp}^+(X,V),\qquad \pi\big|_{\Pi^{-1}(\alpha)}:\{\Pi\circ\pi\}^{-1}(\alpha)\longrightarrow\Pi^{-1}(\alpha),\ \ \alpha\in H^2_{\mathrm{dR}}(M),$$

are weak homotopy equivalences.

There is a direct analogue of Theorem 7 for SC symplectic varieties; see [7, Theorem 2.17]. There are also similar results for NC symplectic divisors and varieties; see [9].

For many applications, the most important consequences of Theorem 7 are the following. First, for each $\omega\in\mathrm{Symp}^+(X,V)$, there exists a path $(\omega_t)_{t\in[0,1]}$ of cohomologous symplectic forms in $\mathrm{Symp}^+(X,V)$ such that $\omega_0=\omega$ and V admits an ω_1-regularization in X. By the Moser Isotopy Theorem [23, Section 3.2], this is equivalent to saying that every SC symplectic divisor in (X,ω) is isotopic through SC symplectic divisors inside (X,ω) to one that admits an ω-regularization. Second, for two pairs

$$(\omega_0,\mathcal{R}_0),(\omega_1,\mathcal{R}_1)\in \mathrm{Aux}(X,V)$$

and a path $(\omega_t)_{t\in[0,1]}$ of symplectic forms $\mathrm{Symp}^+(X,V)$ connecting ω_0 and ω_1, there exists a deformation $(\omega_{t,\tau})_{t,\tau\in[0,1]}$ of this path in $\mathrm{Symp}^+(X,V)$ fixing the end points, i.e.

$$\big(\omega_{t,0}\big)_{t\in[0,1]}=\big(\omega_t\big)_{t\in[0,1]},\qquad \omega_{0,\tau}=\omega_0,\ \omega_{1,\tau}=\omega_1\ \ \forall\,\tau\in[0,1],$$

such that the ending path $(\omega_{t,1})_{t\in[0,1]}$ can be lifted to a path

$$\big(\omega_{t,1},\widetilde{\mathcal{R}}_t\big)_{t\in[0,1]}\in \mathrm{Aux}(X,V)\quad\text{with }\widetilde{\mathcal{R}}_0\cong\mathcal{R}_0,\ \ \widetilde{\mathcal{R}}_1\cong\mathcal{R}_1.$$

If $(\omega_t)_{t\in[0,1]}$ is a path of cohomologous forms, then its deformation $(\omega_{t,\tau})_{t,\tau\in[0,1]}$ can be chosen to consist of cohomologous forms as well.

Theorem 7 implies that every SC symplectic divisor is isotopic to one admitting a "nice" compatible almost complex structure. We define the space of almost Kähler data $\mathrm{AK}(X,V)$ to be the space of triples $(\omega,\mathcal{R},J)$, where $\omega\in\mathrm{Symp}^+(X,V)$ is a symplectic structure, $\mathcal{R}$ is an ω-regularization of V as in Definition 6, and $J\in\mathcal{J}_\omega(X)$ is such that for each $I\subset S$:

- V_I is J-holomorphic,
- Ψ_I^*J restricted to each fiber F of $\pi_I|_{\mathrm{Dom}(\Psi_I)}$ is equal to $\bigoplus_{i\in I}\mathfrak{i}_{I;i}|_F$,
- $\pi_I\circ\Psi_I^{-1}$ is a $(J|_{V_I},J|_{\mathrm{Im}(\Psi_I)})$-holomorphic map.

By Theorem 7 and a straightforward induction argument, the projection maps

$$\pi':\mathrm{AK}(X,V)\longrightarrow\mathrm{Aux}(X,V),$$
$$\pi'|_{\{\Pi\circ\pi\}^{-1}(\alpha)}:\{\Pi\circ\pi\circ\pi'\}^{-1}(\alpha)\longrightarrow\{\Pi\circ\pi\}^{-1}(\alpha),\ \ \alpha\in H^2_{\mathrm{dR}}(M),$$

are also weak homotopy equivalences. Therefore, the above conclusions concerning lifts from $\mathrm{Symp}^+(X,V)$ hold with $\mathrm{Aux}(X,V)$ replaced by $\mathrm{AK}(X,V)$ as well.

If $(\omega,\mathcal{R},J)\in\mathrm{AK}(X,V)$, J is very regular around V. In particular, it respects the $\mathbb{C}^*$-action on the components $\mathcal{N}_XV_i|_{V_I}$ of the normal bundle $\mathcal{N}_XV_I$ of V_I in X and the image of its Nijenhuis tensor on $TX|_{V_I}$ lies in TV_I (i.e. it vanishes in the normal direction to V_I). Regularizations for NC divisors and varieties similarly ensure the existence of almost complex structures with analogous properties on symplectic manifolds containing NC divisors and on NC varieties themselves. The above properties of J are very desirable for applications involving J-holomorphic curve techniques; works such as [20,17,19,30,2] make use of these properties in contexts involving various specializations of NC symplectic divisors and varieties introduced in [7,9].

In algebraic geometry, one can associate a holomorphic line bundle with any Cartier divisor. It is straightforward to extend this to the symplectic topology category in the case of a smooth symplectic divisor V in a symplectic manifold (X,ω). Fix an identification Ψ as in (3) and an $\omega|_{\mathcal{N}_XV}$-compatible complex structure $\mathfrak{i}$ on $\mathcal{N}_XV$ so that $\mathcal{N}_XV$ becomes a complex line bundle over V. Then,

$$\mathcal{O}_X(V)\equiv\big(\Psi^{-1\,*}\pi^*_{\mathcal{N}_XV}\mathcal{N}_XV|_{\Psi(\mathcal{N}'_XV)}\sqcup(X-V)\times\mathbb{C}\big)/\!\sim\ \longrightarrow X,$$
$$\Psi^{-1\,*}\pi^*_{\mathcal{N}_XV}\mathcal{N}_XV|_{\Psi(\mathcal{N}'_XV)}\ni\big(\Psi(v),v,cv\big)\sim\big(\Psi(v),c\big)\in(X-V)\times\mathbb{C},$$

is a complex line bundle over X with

$$c_1(\mathcal{O}_X(V))=\mathrm{PD}_X([V]_X)\in H^2(X;\mathbb{Z}),\tag{7}$$

where $[V]_X$ is the homology class in X represented by V. The space of pairs $(\Psi,\mathfrak{i})$ involved in explicitly constructing this line bundle is contractible. Therefore, a family B in

$$\mathrm{Symp}^+(X,V)=\mathrm{Symp}(X,V)$$

determines a complex line bundle $\mathcal{O}_{B\times X}(V)$ over $B\times X$.

Regularizations for NC symplectic divisors and varieties extend the construction of the previous paragraph to these spaces. In particular, an NC symplectic divisor V in a symplectic manifold (X,ω) determines a complex line bundle $\mathcal{O}_X(V)$ over X satisfying (7). It also determines a complex vector bundle $TX(-\log V)$ of rank equal to half the real dimension of X satisfying

$$c\big(TX(-\log V)\big)=c(TX,\omega)/\big(1+\mathrm{PD}_X([V^{(1)}]_X)+\mathrm{PD}_X([V^{(2)}]_X)+\ldots\big)\in H^2(X;\mathbb{Q}),\tag{8}$$

where $V^{(r)} \subset V$ is the r-fold locus (locally intersection of at least r branches of V); see [10]. This vector bundle extends the notion of logarithmic tangent bundle, which plays a central role in the Gross–Siebert program [15,16], to symplectic topology. The deformation equivalence classes of both bundles depend only on the deformation equivalence class of ω in $\mathrm{Symp}^+(X,V)$. If V is an SC symplectic divisor as in Definition 2, then

$$\mathcal{O}_X(V) = \bigotimes_{i\in S} \mathcal{O}_X(V_i) \longrightarrow X$$

and the equality in (8) holds in $H^2(X;\mathbb{Z})$. An NC symplectic variety $(X_\emptyset, \omega_\emptyset)$ determines a complex line bundle $\mathcal{O}_{X_\partial}(X_\emptyset)$ over the singular locus X_∂ of $X_\emptyset$, which we call the normal bundle of X_∂. The deformation equivalence class of $\mathcal{O}_{X_\partial}(X_\emptyset)$ similarly depends only on the deformation equivalence class of $\omega_\emptyset$ in the space $\mathrm{Symp}^+(X_\emptyset)$ of all NC-symplectic-variety structures on $X_\emptyset$. If $(X_\emptyset, \omega_\emptyset)$ is an SC symplectic variety as in Definition 3, then a regularization for $(X_\emptyset, \omega_\emptyset)$ determines line bundles

$$\mathcal{O}_{X_i^c}(X_i) \longrightarrow X_i^c \equiv \bigcup_{j\in S-\{i\}} X_j \subset X_\emptyset, \qquad i\in S,$$

obtained by canonically identifying the line bundles

$$\mathcal{O}_{X_j}(X_{ij})\big|_{X_{jk}} = \mathcal{O}_{X_{jk}}(X_{ijk}) = \mathcal{O}_{X_k}(X_{ik})\big|_{X_{jk}}$$

over X_{jk}; see [8, Section 2.1]. In this case,

$$\mathcal{O}_{X_\partial}(X_\emptyset) = \bigotimes_{i\in S} \mathcal{O}_{X_i^c}(X_i)\big|_{X_\partial} \longrightarrow X_\partial \equiv \bigcup_{\substack{i,j\in S\\ i\neq j}} X_{ij}\,.$$

4. Smoothings of NC symplectic varieties

Having introduced analogues of NC divisors/varieties into symplectic topology, we next present an analogue of the algebro-geometric notion of semistable degeneration.

Definition 8 *([8, Definition 2.6], [10]).* If $(\mathcal{Z}, \omega_{\mathcal{Z}})$ is a symplectic manifold and $\Delta \subset \mathbb{C}$ is a disk around the origin, a smooth surjective map $\pi : \mathcal{Z} \longrightarrow \Delta$ is a nearly regular symplectic fibration if

- $X_\emptyset \equiv \mathcal{Z}_0 \equiv \pi^{-1}(0)$ is an NC symplectic divisor in $(\mathcal{Z}, \omega_{\mathcal{Z}})$,
- π is a submersion outside of the singular locus X_∂ of $X_\emptyset$,
- for every $\lambda \in \Delta - \{0\}$, $\mathcal{Z}_\lambda \equiv \pi^{-1}(\lambda)$ is a symplectic submanifold of $(\mathcal{Z}, \omega_{\mathcal{Z}})$.

The restriction of $\omega_{\mathcal{Z}}$ to $X_\emptyset$ above determines an NC symplectic variety $(X_\emptyset, \omega_\emptyset)$. For each $I \subset [N]$, the derivatives of π along the normal bundles $\mathcal{N}_{\mathcal{Z}} X_i$ of X_i in $\mathcal{Z}$ induce a homomorphism

$$D_I\pi : \bigotimes_{i\in I} \mathcal{N}_{\mathcal{Z}} X_i\big|_{X_I} \longrightarrow \mathbb{C}$$

that vanishes along X_J with $I \subsetneq J \subset [N]$. We call the nearly regular symplectic fibration of Definition 8 a one-parameter family of smoothings of $(X_\emptyset, \omega_\emptyset)$ if the homomorphism

$$\mathcal{N}_{\mathcal{Z}} X_i\big|_x \longrightarrow \mathbb{C}, \qquad v_i \longrightarrow D_I\pi\big((v_j)_{j\in I}\big),$$

is an orientation-preserving isomorphism for all $i \in I \subset [N]$, $x \in X_I$ with $x \notin X_J$ if $I \subsetneq J \subset [N]$, and $v_j \in \mathcal{N}_{\mathcal{Z}} X_j|_x - \{0\}$ for $j \in I - \{i\}$.

From the complex geometry point of view, such a family replaces the nodal singularity $z_1 \dots z_N = 0$ in $\mathbb{C}^n$, i.e. a union of N coordinate hyperplanes in the central fiber $\pi^{-1}(0)$, by a smoothing $z_1 \dots z_N = \lambda$ with $\lambda \in \mathbb{C}^*$ in a generic fiber. A regularization for $(X_\emptyset, \omega_\emptyset)$ determines a complex vector bundle $\log_{\mathcal{Z}} TX_\emptyset$ over $X_\emptyset$ of rank equal to half the real dimension of $X_\emptyset$ satisfying

$$c\big(\log_{\mathcal{Z}} TX_\emptyset\big) = c\big(T\mathcal{Z}(-\log X_\emptyset)\big)\big|_{X_\emptyset}\,.$$

This bundle is the analogue of the logarithmic tangent bundle of a smoothable NC variety in algebraic geometry. Its deformation equivalence class again depends only on the deformation equivalence class of $\omega_{\mathcal{Z}}$ in $\mathrm{Symp}^+(\mathcal{Z}, X_\emptyset)$.

In light of the deformation equivalence philosophy stated on page 425, we say that an NC symplectic variety $(X_\emptyset, \omega_\emptyset)$ is smoothable if some NC symplectic variety $(X_\emptyset, \omega'_\emptyset)$ deformation equivalent to $(X_\emptyset, \omega_\emptyset)$ admits a one-parameter family of smoothings. In this section, we answer the following question:

Which SC symplectic varieties are smoothable?

The d-semistability condition of [12, Definition (1.13)] is well known to be an obstruction to the smoothability of an NC variety in a one-parameter family with a smooth total space in the algebraic geometry category. As shown in [32], the d-semistability condition is not the only obstruction in the algebraic category, even in the simplest non-trivial case discussed below.

Let (X_1,ω_1) and (X_2,ω_2) be smooth symplectic manifolds with identical copies of a smooth symplectic divisor $X_{12}\subset X_1, X_2$. Then

$$\big(X_\emptyset = X_1\cup_{X_{12}} X_2, \omega_\emptyset=(\omega_1,\omega_2)\big) \tag{9}$$

is an SC symplectic variety. If (X_1,ω_1), (X_2,ω_2), and X_{12} are smooth projective varieties, the d-semistability condition of [12] in this case is the existence of an isomorphism

$$\mathcal{O}_{X_\partial}(X_\emptyset)\equiv \mathcal{N}_{X_1}X_{12}\otimes_{\mathbb{C}}\mathcal{N}_{X_2}X_{12}\approx \mathcal{O}_{X_{12}}\longrightarrow X_{12} \tag{10}$$

in the category of *holomorphic* line bundles. By the now classical symplectic sum construction, suggested in [14, p. 343] and carried out in [13,22], the smoothability of the 2-fold SC symplectic variety (9) in the sense of Definition 8 is *equivalent* to the existence of an isomorphism (10) in the category of *complex* line bundles.

The topological type of smooth fibers $(X_\#,\omega_\#)$ of the one-parameter family of smoothings produced by the construction of [13] depends only on the homotopy class of isomorphisms (10). With such a choice fixed, this construction involves choosing an $\omega_1|_{\mathcal{N}_{X_1}X_{12}}$-compatible almost complex structure on $\mathcal{N}_{X_1}X_{12}$, an $\omega_2|_{\mathcal{N}_{X_2}X_{12}}$-compatible almost complex structure on $\mathcal{N}_{X_2}X_{12}$, and a representative for the above homotopy class. Because of these choices, the resulting symplectic manifold $(X_\#,\omega_\#)$ is determined by (X_1,ω_1), (X_2,ω_2), and the choice of the homotopy class only up to symplectic deformation equivalence. Since the symplectic deformations of the SC symplectic variety (9) do not affect the deformation equivalence class of $(X_\#,\omega_\#)$, it would have been sufficient to carry out the symplectic sum construction of [13] only on a path-connected set of representatives for each deformation equivalence class of the SC symplectic variety (9).

The above change in perspective turns out to be very useful for smoothing out arbitrary NC symplectic varieties in [8, 10] and thus answering another question of [14, p. 343]. By the next theorem, the direct analogue of the d-semistability condition of [12] is the only obstruction for the smoothability of an arbitrary NC symplectic variety. Regularizations are the essential auxiliary data in the proof of this result.

Theorem 9 *([8, Theorem 2.7], [10]). An NC symplectic variety $(X_\emptyset,\omega_\emptyset)$ is smoothable if and only if the associated line bundle $\mathcal{O}_{X_\partial}(X_\emptyset)$ is trivializable. Furthermore, the germ at the zero fiber of the deformation equivalence class of the nearly regular symplectic fibration $(\mathcal{Z},\omega_{\mathcal{Z}},\pi)$ provided by the proof of this statement is determined by a homotopy class of trivializations of $\mathcal{O}_{X_\partial}(X_\emptyset)$. If, in addition, X_∂ is compact, the deformation equivalence class of a smooth fiber $(\mathcal{Z}_\lambda,\omega_\lambda)$ is also determined by a homotopy class of these trivializations.*

If $(X_\emptyset,\omega_\emptyset)$ is an SC symplectic variety as in Definition 3 and $X_\emptyset$ is compact, we call (the deformation equivalence class of) a generic fiber of the resulting one-parameter family the multifold or $|S|$-fold symplectic sum of $(X_i,\omega_i)_{i\in S}$.

If $(X_\emptyset,\omega_\emptyset)$ is an SC symplectic variety as in Definition 3,

$$\mathcal{O}_{X_\partial}(X_\emptyset)\big|_{X_{ij}} = \mathcal{N}_{X_i}X_{ij}\otimes\mathcal{N}_{X_j}X_{ij}\otimes\bigotimes_{\substack{k\in S\\ k\neq i,j}}\mathcal{O}_{X_{ij}}(X_{ijk}) \qquad \forall\, i,j\in S,\ i\neq j. \tag{11}$$

By [8, Example 2.10], the triviality of $\mathcal{O}_{X_\partial}(X_\emptyset)$ is in general stronger than the triviality of the restrictions (11); the latter is known as the triple point condition in the algebraic geometric literature. If the natural homomorphism

$$H^2(X_\partial;\mathbb{Z})\longrightarrow\bigoplus_{\substack{i,j\in S\\ i\neq j}}H^2(X_{ij};\mathbb{Z}) \tag{12}$$

is injective or at most one of the homomorphisms

$$H^1(X_{ij};\mathbb{Z})\longrightarrow\bigoplus_{\substack{k\in S\\ k\neq i,j}}H^1(X_{ijk};\mathbb{Z}), \qquad i,j\in S,\ i\neq j, \tag{13}$$

is not surjective, then $\mathcal{O}_{X_\partial}(X_\emptyset)$ is trivial if and only if all restrictions in (11) are trivial.

In Example 1, the line bundle (11) corresponding to $X'_{ij}\approx\mathbb{P}^1$ is equal to $\mathcal{O}_{\mathbb{P}^1}(3)$. Therefore, $X'_\emptyset$ is not smoothable. After blowing up the 3 singular points on each X'_{ij}, we get:

$$\mathcal{N}_{X_i}X_{ij}\otimes\mathcal{N}_{X_j}X_{ij}\otimes\mathcal{O}_{X_{ij}}(X_{123})\cong\mathcal{O}_{\mathbb{P}^1} \qquad \forall\, i,j=1,2,3,\ i\neq j.$$

In this case, the homomorphism (12) is injective and all homomorphisms (13) are surjective. By Theorem 9, the NC symplectic variety $X_\emptyset$ of Example 1 is thus smoothable. In this case, there is only one homotopy class of trivializations of $\mathcal{O}_{X_\partial}(X_\emptyset)$. The smoothing provided by the proof of Theorem 9 is conjecturally equivalent to the one of Example 1, provided the latter is projective.

The proof of the SC case of Theorem 9 in [8] explicitly constructs $(\mathcal{Z}, \omega_{\mathcal{Z}}, \pi)$ by gluing together local charts $\mathcal{Z}_I$ with $I \subset S$ non-empty; these charts are equipped with symplectic forms and local smoothings. We deform $(\omega_i)_{i\in S}$ in $\mathrm{Symp}^+(X_\emptyset)$ so that $X_\emptyset$ admits an $(\omega_i)_{i\in S}$-regularization $\mathcal{R}$ and choose a trivialization Φ of $\mathcal{O}_{X_\partial}(X_\emptyset)$ so that it is compatible with $\mathcal{R}$ in a suitable sense. If $|I| \ge 2$, $\mathcal{Z}_I$ is a neighborhood of a large open subset X_I° of X_I in

$$\mathcal{N}X_I|_{X_I^\circ} \equiv \bigoplus_{i\in I} \mathcal{N}_{X_{I-i}} X_I|_{X_I^\circ};$$

the closure of X_I° is "slightly" disjoint from all X_J with $J \supsetneq I$. If $I = \{i\}$ with $i \in S$, $\mathcal{Z}_I$ is a neighborhood of $X_i^\circ \times \{0\}$ in $X_i^\circ \times \mathbb{C}$. The regularization $\mathcal{R}$ and the trivialization Φ are used to glue the charts $\mathcal{Z}_{I'}$ and $\mathcal{Z}_I$ for $I' \subset I$ with $|I'| \ge 2$ and $|I'| = 1$, respectively.

If $|I| \ge 2$, the restriction of π to $\mathcal{Z}_I$ is a positive multiple of the composition

$$\mathcal{Z}_I \hookrightarrow \bigoplus_{i\in I} \mathcal{N}_{X_{I-i}} X_I\big|_{X_I^\circ} \longrightarrow \bigotimes_{i\in I} \mathcal{N}_{X_{I-i}} X_I\big|_{X_I^\circ} = \mathcal{O}_{X_\partial}(X_\emptyset)|_{X_I^\circ} \xrightarrow{\Phi} X_I^\circ \times \mathbb{C} \longrightarrow \mathbb{C}.$$

If $|I| = 1$, the restriction of π to $\mathcal{Z}_I$ is a positive multiple of the projection to the second component. The symplectic form $\omega_{\mathcal{Z}}$ on $\mathcal{Z}$ is built by interpolating between the symplectic forms (2) determined by the product Hermitian structures $(\mathfrak{i}_{I;i}, \rho_{I;i}, \nabla^{(I;i)})_{i\in I}$ on $\mathcal{Z}_I$ with $|I| \ge 2$ and certain product symplectic forms on $\mathcal{Z}_I$ with $|I| = 1$.

The proof of the SC case of Theorem 9 outlined above is extended to the general case in [10]. By [8, Proposition 5.1] and its extension to the NC case in [10], every one-parameter family of smoothings of an NC symplectic variety $(X_\emptyset, \omega_\emptyset)$ determines a homotopy class of trivializations of $\mathcal{O}_{X_\partial}(X_\emptyset)$. By [8, Proposition 5.5] and its extension to the NC case in [10], the homotopy class determined by the family provided by the proof of Theorem 9 is the input homotopy class. We believe that the equivalence classes of smoothings of a compact NC symplectic variety $(X_\emptyset, \omega_\emptyset)$ correspond to the homotopy classes of trivializations of $\mathcal{O}_{X_\partial}(X_\emptyset)$. This is equivalent to the following.

Conjecture 10. *Let $(\mathcal{Z}, \omega_{\mathcal{Z}}, \pi)$ be a one-parameter family of smoothings of a compact NC symplectic variety $(X_\emptyset, \omega_\emptyset)$. Then $(\mathcal{Z}, \omega_{\mathcal{Z}}, \pi)$ is deformation equivalent to a smoothing of $(X_\emptyset, \omega_\emptyset)$ provided by the proof of Theorem 9.*

Remark 11. The surgery construction of [34,35] on 4-dimensional symplectic manifolds along pairwise positively intersecting immersed surfaces, also called N-fold symplectic sum construction, agrees with ours (which is consistent with algebraic geometry and [14, p. 343]) only for $N = 3$. In particular, the setting of [35, Theorem 2.7] is essentially the $\dim_{\mathbb{R}} X = 4$ case of the setting of [8, Theorem 2.7]. The output of [35, Theorem 2.7] is then symplectically deformation equivalent to the smooth fibers of the one-parameter family provided by [8, Theorem 2.7]. The perspectives taken in [35] and in [8] are fundamentally different as well. The viewpoint in [35] is that of surgery on 4-dimensional manifolds; the viewpoint in [8] is that of smoothing a variety in a one-dimensional family with a smooth total space. The configurations in [35] with $N \ge 4$ correspond to varieties, such as

$$\{(x, y, z, w) \in \mathbb{C}^4 : \ xy = 0, \ zw = 0\}, \tag{14}$$

that do not even admit such smoothings. The total space of the natural one-parameter smoothing of (14), i.e. with 0 replaced by $\lambda \in \mathbb{C}$, is singular at the origin.

5. SC degenerations of symplectic manifolds

We next discuss the potential for reversing the construction of Theorem 9.

Can one degenerate a symplectic manifold (X, ω) into some SC symplectic variety $(X_i, \omega_i)_{i\in S}$ in a one-parameter family?

The $|S| = 2$ case of this question is the now classical symplectic cut construction of [18]. Given a free Hamiltonian S^1-action generated by Hamiltonian h on an open subset W of X so that $\widetilde{V} \equiv h^{-1}(0)$ is a separating hypersurface, this construction decomposes (X, ω) into two symplectic manifolds, (X_-, ω_-) and (X_+, ω_+). It cuts X into closed subsets $U^{\le}$ and $U^{\ge}$ along $\widetilde{V}$ and collapses their boundary $\widetilde{V}$ to a smooth symplectic divisor $V \equiv \widetilde{V}/S^1$ inside (X_-, ω_-) and (X_+, ω_+). The associated "wedge",

$$X_\emptyset \equiv X_- \cup_V X_+,$$

is a 2-fold SC symplectic variety as in (9). If we assume instead that $\widetilde{V}$ is non-separating, the result would be an NC symplectic variety.

In [11], we generalize and enhance the symplectic cut construction of [18] to produce a nearly regular symplectic fibration with regular fibers deformation equivalent to (X,ω). The central singular fiber of this fibration is what we call a multifold (or N-fold) symplectic cut of (X,ω). The input of this construction is a multifold cutting configuration $\mathscr{C}$ defined below.

For a finite non-empty set S, let

$$(S^1)^S_\bullet=\bigg\{(\mathrm{e}^{\mathrm{i}\theta_i})_{i\in S}\in(S^1)^S:\prod_{i\in S}\mathrm{e}^{\mathrm{i}\theta_i}=1\bigg\}.$$

For $I\subset S$, we identify $(S^1)^I$ with the subgroup

$$\big\{(\mathrm{e}^{\mathrm{i}\theta_i})_{i\in S}\in(S^1)^S:\mathrm{e}^{\mathrm{i}\theta_i}=1\;\forall\,i\in S-I\big\}$$

of $(S^1)^S$ in the natural way and let

$$(S^1)^I_\bullet\equiv(S^1)^S_\bullet\cap(S^1)^I\,.$$

Denote by $\mathfrak{t}_{I;\bullet}\subset\mathfrak{t}_{S;\bullet}$ the Lie algebra of $(S^1)^I_\bullet$ and by $\mathfrak{t}^*_{I;\bullet}$ its dual. For $i,j\in I\subset S$, the homomorphism

$$\mathfrak{t}^*_{I;\bullet}=\mathbb{R}^I/\big\{a^I\in\mathbb{R}^I:a\in\mathbb{R}\big\}\longrightarrow\mathbb{R},\qquad\eta\equiv\big[(a_k)_{k\in I}\big]\longrightarrow\eta_{ij}\equiv a_j-a_i\,,$$

is well-defined. We write $(\eta)_i<(\eta)_j$ (resp. $(\eta)_i\le(\eta)_j$, $(\eta)_i=(\eta)_j$) if $0<\eta_{ij}$ (resp. $0\le\eta_{ij}$, $0=\eta_{ij}$).

Definition 12. A multifold Hamiltonian configuration for a symplectic manifold (X,ω) is a tuple

$$\mathscr{C}\equiv\big(U_I,\phi_I,\mu_I:U_I\longrightarrow\mathfrak{t}^*_{I;\bullet}\big)_{\emptyset\neq I\subset S},\tag{15}$$

where S is a finite non-empty set, $(U_I)_{\emptyset\neq I\subset S}$ is an open cover of X, and ϕ_I is a Hamiltonian $(S^1)^I_\bullet$-action on U_I with moment map μ_I, such that

(a) $U_I\cap U_{I'}=\emptyset$ unless $I\subset I'$ or $I'\subset I$;
(b) $\mu_I(x)|_{\mathfrak{t}_{I';\bullet}}=\mu_{I'}(x)$ for all $x\in U_I\cap U_{I'}$ and $I'\subset I\subset S$;
(c) $(\mu_I(x))_i<(\mu_I(x))_j$ for all $x\in U_I\cap U_{I'}$, $i\in I'\subset I\subset S$, and $j\in I-I'$.

Definition 13. A multifold cutting configuration for (X,ω) is a multifold Hamiltonian configuration as in (15) such that the restriction of the $(S^1)^I_\bullet$-action ϕ_I to $(S^1)^{I'}_\bullet$ is free on the preimage of $0\in\mathfrak{t}^*_{I';\bullet}$ under the moment map

$$\mu_{I';I}:\big\{x\in U_I:\big(\mu_I(x)\big)_i<\big(\mu_I(x)\big)_j\;\forall\,i\in I',\;j\in I-I'\big\}\longrightarrow\mathfrak{t}^*_{I';\bullet},\quad\mu_{I';I}(x)=\mu_I(x)\big|_{\mathfrak{t}_{I';\bullet}},$$

for all $I'\subset I\subset S$ with $I'\neq\emptyset$.

We use a multifold cutting configuration $\mathscr{C}$ in [11] to decompose (X,ω) into $|S|$ symplectic manifolds (X_i,ω_i) at once. We first cut X into the closed subspaces

$$U_i^{\le}\equiv\bigcup_{i\in I\subset S}\big\{x\in U_I:(\mu_I(x))_i\le(\mu_I(x))_j\;\forall\,j\in I\big\},\qquad i\in S.$$

The subspace $U_i^{\le}$ has boundary and corners

$$U_I^{\le}\equiv\bigcup_{I\subset J\subset S}\big\{x\in U_J:(\mu_J(x))_i\le(\mu_J(x))_j\;\forall\,i\in I,\;j\in J\big\},\qquad i\in I\subset S,\;|I|\ge2.$$

We collapse each $U_I^{\le}$ by the $(S^1)^I_\bullet$-action ϕ_I to obtain symplectic manifolds (X_i,ω_i) with $i\in S$ and symplectic submanifolds $X_I\equiv U_I^{\le}/(S^1)^I_\bullet$ of real codimension $2(|I|-1)$ in (X_i,ω_i) with $i\in I\subset S$. For each $i\in S$, $\{X_{ij}\}_{i\in S-i}$ is an SC symplectic divisor in (X_i,ω_i). The entire collection $\{X_I\}$ determines an SC symplectic variety $(X_\emptyset,\omega_\emptyset)$, which we call a multifold (or $|S|$-fold) symplectic cut of (X,ω).

We also construct a symplectic manifold $(\mathcal{Z},\omega_{\mathcal{Z}})$ containing $X_\emptyset$ as an SC symplectic divisor and a smooth map $\pi:\mathcal{Z}\longrightarrow\mathbb{C}$ so that $X_\emptyset=\pi^{-1}(0)$ and $\omega_{\mathcal{Z}}|_{X_\emptyset}=\omega_\emptyset$. The restriction of π to a neighborhood $\mathcal{Z}'$ of $X_\emptyset$ in $\mathcal{Z}$ is a nearly regular symplectic fibration and thus determines a one-parameter family of smoothings of the SC symplectic variety $(X_\emptyset,\omega_\emptyset)$. If X is compact, then a generic fiber of $\pi|_{\mathcal{Z}'}$ is deformation equivalent to (X,ω). In such a case, we call $(\mathcal{Z}',\omega_{\mathcal{Z}}|_{\mathcal{Z}'},\pi|_{\mathcal{Z}'})$ a multifold (or $|S|$-fold) SC symplectic degeneration of (X,ω). This is a symplectic topology analogue of the algebro-geometric

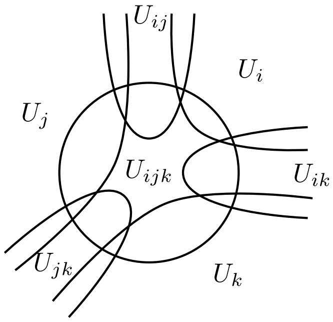

Fig. 2. A 3-fold cutting configuration.

notion of semistable degeneration (i.e. smooth one-parameter family of degenerations of a smooth algebraic variety to an NC algebraic variety).

The symplectic manifold $(\mathcal{Z}, \omega_{\mathcal{Z}})$ is obtained by gluing together charts $(\mathcal{Z}_I^\circ, \varpi_I)$ with $I \subset S$ non-empty. Each $(\mathcal{Z}_I^\circ, \varpi_I)$ is the Symplectic Reduction [4, Theorem 23.1] with respect to the moment map

$$\widetilde{\mu}_I : U_I \times \mathbb{C}^I \longrightarrow \mathfrak{t}^*_{I;\bullet}, \qquad \widetilde{\mu}_I(x,z) = \mu_I(x) - \mu_{\mathbb{C}^I;\bullet}(z),$$

where $\mu_{\mathbb{C}^I;\bullet}$ is the moment map for the restriction of the standard $(S^1)^I$-action on $\mathbb{C}^I$ to $(S^1)^I_\bullet$ such that $0 \in \mathbb{C}^I$ is its critical point. The restriction of π to $\mathcal{Z}_I^\circ$ is a positive multiple of the map

$$\mathcal{Z}_I^\circ \longrightarrow \mathbb{C}, \qquad \left[x, (z_i)_{i\in I}\right] \longrightarrow \prod_{i\in I} z_i \,.$$

The intersection of $\mathcal{Z}_I^\circ$ with X_i corresponds to the region $z_i = 0$ inside $\mathcal{Z}_I^\circ$.

The $S = \{1,2\}$ case of Definition 13 corresponds to the symplectic cut construction of [18] by identifying $(S^1)^S_\bullet$ with S^1 via the projection to the first component of $(S^1)^S$ and taking

$$W = U_S, \quad X_- = U_1^{\le}, \quad X_+ = U_2^{\le}, \quad \widetilde{V} = U_{12}^{\le}\,.$$

Figs. 2 and 1 show a 3-fold cutting configuration and the associated 3-fold symplectic cut, respectively.

The SC case of the symplectic sum/smoothing construction of Theorem 9 and the symplectic cut/degeneration construction of [11] are intuitively mutual inverses. However, we are unaware of any work where even the 2-fold case of this statement (relating the constructions of [13] and [18]) is made precise. The purpose of [6] is to establish this statement as formulated below.

Fix $n, N \in \mathbb{Z}^+$. Let $\mathrm{SCV}(n,N)$ be the space of tuples $(X_\emptyset, \omega_\emptyset, \hbar)$ consisting of a compact N-fold SC symplectic variety $(X_\emptyset, \omega_\emptyset)$ of real dimension $2n$ and a homotopy class $\hbar$ of trivializations of an associated line bundle $\mathcal{O}_{X_\partial}(X_\emptyset)$. Let $\mathrm{SCC}(n,N)$ be the space of tuples $(X, \omega, \mathscr{C})$ consisting of a compact symplectic manifold (X,ω) of real dimension $2n$ and an N-fold cutting configuration $\mathscr{C}$ for (X,ω). By [6], a generic fiber (X,ω) of a nearly regular symplectic fibration arising from the proof of the SC case of Theorem 9 admits a natural N-fold cutting configuration $\mathscr{C}$. By [8, Proposition 5.1], the semistable degeneration $(\mathcal{Z}', \omega_{\mathcal{Z}}|_{\mathcal{Z}'}, \pi|_{\mathcal{Z}'})$ of (X,ω) arising from a cutting configuration $\mathscr{C}$ determines a homotopy class $\hbar$ of trivialization of a line bundle $\mathcal{O}_{X_\partial}(X_\emptyset)$ associated with the central fiber. Thus, there are natural maps

$$\mathcal{S}_{n,N} : \mathrm{SCV}(n,N) \longrightarrow \mathrm{SCC}(n,N), \qquad \mathcal{C}_{n,N} : \mathrm{SCC}(n,N) \longrightarrow \mathrm{SCV}(n,N), \tag{16}$$

which we call smoothing and cutting maps respectively.

The aim of [6] is to show that the two maps in (16) are weak homotopy inverses. It is fairly straightforward to show that $\mathcal{C}_{n,N} \circ \mathcal{S}_{n,N}$ is homotopy equivalent to the identity. Along with Conjecture 10, this implies that $\mathcal{S}_{n,N} \circ \mathcal{C}_{n,N}$ is weakly homotopy equivalent to the identity, but Conjecture 10 is more than what is needed. The multifold symplectic cut construction of [11], the maps (16), and their being weak homotopy inverses should generalize to the arbitrary NC case as well.

6. Directions for further research

Log smooth degenerations to log smooth algebraic varieties play important roles in such areas of modern algebraic geometry as Hodge theory and log Gromov–Witten theory [16,1]. The almost Kähler analogue of the log smooth category provided by the exploded manifold category of [29] underpins a similar study of GW-invariants in [30]. The works [16,1,30] extend GW-invariants to algebraic varieties with so-called fine saturated log structures and show that these invariants do not change under deformations that are smooth in the category of such varieties. An effective decomposition formula splitting GW-invariants of a fine saturated log algebraic variety into GW-invariants of the irreducible components of the underlying variety would generalize the renowned formula of [19] and is key to the Gross–Siebert program [15], but has turned out to be difficult to work out. As GW-invariants of smooth algebraic varieties are fundamentally symplectic topology invariants, it is natural to expect the same of fine saturated log algebraic varieties. This should in turn provide a more robust setting for effectively generalizing the decomposition formula of [19]. We believe that the deformation equivalence philosophy stated on page 425 and the methods used to implement it in the case of NC singularities can be used to define a category of fine saturated log symplectic varieties, which in turn should provide a suitable setting for extending GW-invariants from symplectic manifolds and studying their properties under semistable degeneration. NC singularities are the most basic type of fine saturated log structures. Our work [7–11,6] introduces symplectic topology analogues of these structures, including the associated log tangent bundles, and lays the foundation for defining the long-awaited GW-invariants relative to NC symplectic divisors.

The (2-fold) symplectic sum construction of [13] has been used to build vast classes of non-Kähler symplectic manifolds. For example, it is shown in [13] that every finitely presented group can be realized as the fundamental group of a compact symplectic manifold of real dimension 4. There are still many open questions about the geography of symplectic manifolds and the classification of symplectic manifolds with certain topological properties. The smoothing and degeneration constructions of [8,10,11] may shed light on some of these questions. For example, it is well known that a rationally connected compact Kähler manifold (i.e. a Kähler manifold with a rational curve through every pair of points) is simply connected; see [3, Theorem 3.5]. As noted by J. Starr, the fundamental group of a compact almost Kähler manifold (X, ω, J) with a rational J-holomorphic curve of a fixed homology class through every pair of points is finite.

Question 14. *Is every compact almost Kähler manifold with a rational J-holomorphic curve of a fixed homology class through every pair of points simply connected?*

The multifold sum/smoothing construction of [8] and the multifold cut/degeneration construction of [11] may be useful in answering this question negatively and positively, respectively. The existence of curves as Question 14 is implied by the existence of a nonzero GW-invariant of (X, ω) with two point insertions, but the converse is not known to be true, even in the projective category. The constructions of [8] and [11] may be also useful in studying whether the simple connectedness holds under the stronger condition of the existence of a nonzero GW-invariant with two point insertions.

References

[1] D. Abramovich, Q. Chen, Stable logarithmic maps to Deligne–Faltings pairs II, Asian J. Math. 18 (3) (2014) 465–488.
[2] D. Abramovich, Q. Chen, M. Gross, B. Siebert, Decomposition of degenerate Gromov–Witten invariants, arXiv:1709.09864 [math].
[3] F. Campana, On twistor spaces of the class $\mathscr{C}$, J. Differ. Geom. 33 (2) (1991) 541–549.
[4] A. Canas da Silva, Lectures on Symplectic Geometry, Lect. Notes Math., vol. 1764, Springer-Verlag, 2001 (revised 2006).
[5] S. Donaldson, Symplectic submanifolds and almost-complex geometry, J. Differ. Geom. 44 (4) (1996) 666–705.
[6] M. Farajzadeh Tehrani, A. Zinger, On the multifold symplectic sum and cut constructions, work in progress.
[7] M. Farajzadeh-Tehrani, M. McLean, A. Zinger, Normal crossings singularities for symplectic topology, arXiv:1410.0609 [math].
[8] M. Farajzadeh-Tehrani, M. McLean, A. Zinger, The smoothability of normal crossings symplectic varieties, arXiv:1410.2573 [math].
[9] M. Farajzadeh-Tehrani, M. McLean, A. Zinger, Normal crossings singularities for symplectic topology, II, in preparation.
[10] M. Farajzadeh-Tehrani, M. McLean, A. Zinger, The smoothability of normal crossings symplectic varieties, II, in preparation.
[11] M. Farajzadeh-Tehrani, A. Zinger, Normal crossings degenerations of symplectic manifolds, arXiv:1603.0766 [math].
[12] R. Friedman, Global smoothings of varieties with normal crossings, Ann. of Math. (2) 118 (1) (1983) 75–114.
[13] R. Gompf, A new construction of symplectic manifolds, Ann. of Math. 142 (3) (1995) 527–595.
[14] M. Gromov, Partial Differential Relations, Springer-Verlag, 1986.
[15] M. Gross, B. Siebert, Affine manifolds, log structures, and mirror symmetry, Turk. J. Math. 27 (1) (2003) 33–60.
[16] M. Gross, B. Siebert, Logarithmic Gromov–Witten invariants, J. Amer. Math. Soc. 26 (2) (2013) 451–510.
[17] E. Ionel, T. Parker, Relative Gromov–Witten invariants, Ann. of Math. (2) 157 (1) (2003) 45–96.
[18] E. Lerman, Symplectic cuts, Math. Res. Lett. 2 (3) (1995) 247–258.
[19] J. Li, A degeneration formula for GW-invariants, J. Differ. Geom. 60 (1) (2002) 199–293.
[20] A.-M. Li, Y. Ruan, Symplectic surgery and Gromov–Witten invariants of Calabi–Yau 3-folds, Invent. Math. 145 (1) (2001) 151–218.
[21] J. Li, G. Tian, Virtual moduli cycles and Gromov–Witten invariants of general symplectic manifolds, in: Topics in Symplectic 4-Manifolds, First Int. Press Lect. Ser., I, International Press, 1998, pp. 47–83.
[22] J. McCarthy, J. Wolfson, Symplectic normal connect sum, Topology 33 (4) (1994) 729–764.
[23] D. McDuff, D. Salamon, Introduction to Symplectic Topology, 2nd ed., Oxford University Press, 1998.
[24] D. McDuff, D. Salamon, J-Holomorphic Curves and Symplectic Topology, Colloq. Publ., vol. 52, AMS, 2012.
[25] M. McLean, The growth rate of symplectic homology and affine varieties, Geom. Funct. Anal. 22 (2) (2012) 369–442.
[26] M. McLean, Reeb orbits and the minimal discrepancy of an isolated singularity, Invent. Math. 204 (2) (2016) 505–594.
[27] D. Morrison, Compactifications of moduli spaces inspired by mirror symmetry, arXiv:math/9304007.
[28] A. Newlander, L. Nirenberg, Complex analytic coordinates in almost complex manifolds, Ann. of Math. (2) 65 (3) (1957) 391–404.
[29] B. Parker, Exploded fibrations, in: Proceedings of Gokova Geometry–Topology Conference 2006, 2007, pp. 52–90.
[30] B. Parker, Gromov–Witten invariants of exploded manifolds, arXiv:1102.0158 [math].
[31] B. Parker, Gluing formula for Gromov–Witten invariants in a triple product, arXiv:1511.00779 [math].
[32] U. Persson, H. Pinkham, Some examples of non-smoothable varieties with normal crossings, Duke Math. J. 50 (2) (1983) 477–486.
[33] N. Sheridan, Homological mirror symmetry for Calabi–Yau hypersurfaces in projective space, Invent. Math. 199 (1) (2015) 1–186.
[34] M. Symington, New Constructions of Symplectic Four-Manifolds, Ph.D. Thesis, Stanford University, Stanford, CA, USA, 1996.
[35] M. Symington, A new symplectic surgery: the 3-fold sum, Topol. Appl. 88 (1) (1998) 27–53.
[36] G. Tian, The Quantum Cohomology and Its Associativity, Curr. Dev. Math., International Press, 1995, pp. 361–401.
[37] L.-S. Tseng, S.-T. Yau, Non-Kähler Calabi–Yau manifolds, in: String-Math 2011, Proc. Symp. Pure Math. 85 (2011) 241–254.