C. R. Acad. Sci. Paris, Ser. I 356 (2018) 382–386

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Mathematical analysis

On Ozaki's condition for p-valency

Sur la condition d'Ozaki pour qu'une fonction soit p-valuée

Mamoru Nunokawa [a], Janusz Sokół [b], Derek K. Thomas [c]

[a] *University of Gunma, Hoshikuki-cho 798-8, Chuou-Ward, Chiba, 260-0808, Japan*
[b] *University of Rzeszów, Faculty of Mathematics and Natural Sciences, ul. Prof. Pigonia 1, 35-310 Rzeszów, Poland*
[c] *Department of Mathematics, Swansea University, Singleton Park, Swansea, SA2 8PP, UK*

ARTICLE INFO

Article history:
Received 3 January 2018
Accepted 19 February 2018
Available online 23 February 2018

Presented by the Editorial Board

ABSTRACT

Let f be an analytic function in a convex domain $D \subset \mathbb{C}$. A well-known theorem of Ozaki states that if f is analytic in D, and is given by $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ for $z \in D$, and

$$\mathfrak{Re}\{e^{i\alpha} f^{(p)}(z)\} > 0, \quad (z \in D),$$

for some real α, then f is at most p-valent in D. Ozaki's condition is a generalization of the well-known Noshiro–Warschawski univalence condition. The purpose of this paper is to provide some related sufficient conditions for functions analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ to be p-valent in $\mathbb{D}$, and to give an improvement to Ozaki's sufficient condition for p-valence when $z \in \mathbb{D}$.

RÉSUMÉ

Soit f une fonction analytique dans un domaine $D \subset \mathbb{C}$. Un théorème bien connu d'Ozaki affirme que, si f est analytique dans D, donnée par $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ pour $z \in D$ et

$$\mathfrak{Re}\{e^{i\alpha} f^{(p)}(z)\} > 0, \quad (z \in D),$$

pour un réel α, alors f est au plus p-valuée dans D. La condition d'Ozaki est une généralisation d'une condition de Noshiro–Warschawski pour qu'une fonction soit univaluée, également bien connue. Notre propos ici est de fournir des conditions suffisantes pour que des fonctions analytiques dans le disque unité $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ soient p-valuées dans $\mathbb{D}$ et d'améliorer la condition suffisante d'Ozaki correspondante quand $z \in \mathbb{D}$.

E-mail addresses: mamoru_nuno@doctor.nifty.jp (M. Nunokawa), jsokol@ur.edu.pl (J. Sokół), d.k.thomas@swansea.ac.uk (D.K. Thomas).

https://doi.org/10.1016/j.crma.2018.02.007
1631-073X/

1. Introduction

A function f analytic in a domain $D \in \mathbb{C}$ is called p-valent in D if, for every complex number w, the equation $f(z) = w$ has at most p roots in D, so that there exists a complex number w_0 such that the equation $f(z) = w_0$ has exactly p roots in D.

Let $\mathcal{A}$ denote the class of functions analytic in the unit $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and $f \in \mathcal{A}$ be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Denote by $\mathcal{A}(p)$ the class of functions that are analytic in $\mathbb{D}$, given by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (z \in \mathbb{D}),$$

where $p \in \mathbb{N} = \{1, 2, \ldots\}$.

The following theorem of Noshiro–Warschawski is well known.

Theorem 1.1. *[1,8, Noshiro–Warschawski Theorem] Suppose that f is analytic in a convex domain $D \subset \mathbb{C}$ and*

$$\mathfrak{Re}\left\{e^{i\alpha} f'(z)\right\} > 0, \quad (z \in D)$$

for some real α. Then f is univalent in D.

Ozaki [4], generalized the above theorem as follows.

Theorem 1.2. *[4, Ozaki's Theorem] Suppose that f is analytic in a convex domain $D \subset \mathbb{C}$ and*

$$\mathfrak{Re}\left\{e^{i\alpha} f^{(p)}(z)\right\} > 0, \quad (z \in D)$$

for some real α. Then f is at most p-valent in D.

We note that Ozaki's theorem shows that, if $f \in \mathcal{A}(p)$ and

$$\mathfrak{Re}\{f^{(p)}(z)\} > 0 \quad (z \in \mathbb{D}), \tag{1.1}$$

then f is at most p-valent in $\mathbb{D}$.

Also in [2, 454] it was shown that if $f \in \mathcal{A}(p)$, $p \geq 2$, and

$$|\arg\{f^{(p)}(z)\}| < \frac{3\pi}{4} \quad (z \in \mathbb{D}),$$

then f is at most p-valent in $\mathbb{D}$.

It is the purpose of this paper to improve Ozaki's theorem when $z \in \mathbb{D}$.

2. Main results

We shall need the following results.

Lemma 2.1. *[3, Theorem 1] Let p be analytic in $\mathbb{D}$ with $p(0) = 1$, and suppose that there exists a point $z_0 \in \mathbb{D}$ such that*

$$\mathfrak{Re}\{p(z)\} > 0 \quad for \ |z| < |z_0|$$

and

$$\mathfrak{Re}\{p(z_0)\} = 0.$$

Then $z_0 p'(z_0)$ is a negative real number such that

$$z_0 p'(z_0) \leq -\frac{1 + |p(z_0)|^2}{2} = -\frac{1 + |\mathfrak{Im}\, p(z_0)|^2}{2}.$$

Remark. In Lemma 2.1, we do not need the hypothesis

$$p(z_0) \neq 0, \quad (z_0 \in \mathbb{D}).$$

Lemma 2.2. *Let p be analytic in $\mathbb{D}$ with $p(0) = 1$, and suppose that*

$$\mathfrak{Re}\left\{p(z) + zp'(z)\right\} > -\frac{1 + |p(z)|^2}{2}, \quad (z \in \mathbb{D}).$$

Then

$$\mathfrak{Re}\{p(z)\} > 0, \quad (z \in \mathbb{D}).$$

Proof. Suppose there exists a point $z_0 \in \mathbb{D}$, such that

$$\mathfrak{Re}\{p(z)\} > 0 \ \text{ for } \ |z| < |z_0|$$

and

$$\mathfrak{Re}\{p(z_0)\} = 0,$$

then applying Lemma 2.1, we have

$$\mathfrak{Re}\left\{p(z_0) + z_0 p'(z_0)\right\} = \mathfrak{Re}\left\{z_0 p'(z_0)\right\} \le -\frac{1 + |p(z_0)|^2}{2}.$$

This contradicts hypothesis, and so proves the lemma. □

Using Lemmas 2.1 and 2.2, we now deduce the following.

Theorem 2.3. *Let $f \in \mathcal{A}(p)$ for $p \ge 2$, and suppose that*

$$\mathfrak{Re}\left\{f^{(p)}(z)\right\} > -\frac{p!}{2}\left(1 + \left|\frac{1}{p!}\frac{f^{(p-1)}(z)}{z}\right|^2\right), \quad (z \in \mathbb{D}).$$

Then

$$\mathfrak{Re}\left\{\frac{f^{(p-1)}(z)}{z}\right\} > 0, \quad (z \in \mathbb{D}).$$

Proof. Put

$$p(z) = \frac{1}{p!}\frac{f^{(p-1)}(z)}{z}, \quad p(0) = 1, \quad (z \in \mathbb{D}).$$

Then

$$f^{(p)}(z) = p!(p(z) + zp'(z)).$$

Suppose that there exists a point $z_0 \in \mathbb{D}$ such that

$$\mathfrak{Re}\{p(z)\} > 0 \ \text{ for } \ |z| < |z_0|$$

and

$$\mathfrak{Re}\{p(z_0)\} = 0.$$

Then applying Lemma 2.1, we have

$$\begin{aligned}\mathfrak{Re}\left\{f^{(p)}(z_0)\right\} &= p!\,\mathfrak{Re}\left\{p(z_0) + z_0 p'(z_0)\right\} \\ &= p!\,\mathfrak{Re}\left\{z_0 p'(z_0)\right\} \\ &\le -\frac{1 + |p(z_0)|^2}{2} \\ &= -\frac{p!}{2}\left(1 + \left|\frac{1}{p!}\frac{f^{(p-1)}(z_0)}{z_0}\right|^2\right).\end{aligned}$$

This contradicts the hypothesis, and so the theorem is proved. □

From Theorem 2.3, we deduce the following corollaries.

Corollary 2.4. *Let $f \in \mathcal{A}(p)$ for $p \geq 2$, and suppose that*

$$\mathfrak{Re}\left\{f^{(p)}(z)\right\} > -\frac{p!}{2}, \quad (z \in \mathbb{D}).$$

Then

$$\mathfrak{Re}\left\{\frac{f^{(p-1)}(z)}{z}\right\} > 0, \quad (z \in \mathbb{D}).$$

We now note that Pommerenke [5] and Sakaguchi [6] showed the following.

Lemma 2.5. *[5] If f and h are analytic in $\mathbb{D}$, and h is convex and univalent in $\mathbb{D}$, with*

$$\left|\arg\left\{\frac{f'(z)}{h'(z)}\right\}\right| \leq \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}),$$

for some real α, $0 \leq \alpha \leq 1$, then

$$\left|\arg\left\{\frac{f(z_2)-f(z_1)}{h(z_2)-h(z_1)}\right\}\right| \leq \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}),$$

for all $z_1, z_2 \in \mathbb{D}$.

Putting $z_1 = 0$, $z_2 = z$ in Lemma 2.5 gives

$$\left|\arg\left\{\frac{f'(z)}{h'(z)}\right\}\right| \leq \frac{\alpha\pi}{2}, \quad z \in \mathbb{D} \quad \Rightarrow \quad \left|\arg\left\{\frac{f(z)}{h(z)}\right\}\right| \leq \frac{\alpha\pi}{2}, \quad (z \in \mathbb{D}).$$

We also recall that Umezawa [7, p. 873, Corollary 2], showed the lemma below.

Lemma 2.6. *[7] If $f \in \mathcal{A}(p)$ for $p \geq 2$, and*

$$\mathfrak{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > 0, \quad (z \in \mathbb{D}).$$

Then f is at most p-valent in $\mathbb{D}$.

Applying Theorem 2.3, Lemma 2.5 and Lemma 2.6, we can now deduce the following.

Theorem 2.7. *If $f \in \mathcal{A}(p)$ for $p \geq 2$, and*

$$\mathfrak{Re}\left\{f^{(p)}(z)\right\} > -\frac{p!}{2}\left(1 + \left|\frac{1}{p!}\frac{f^{(p-1)}(z)}{z}\right|^2\right), \quad (z \in \mathbb{D}).$$

Then f is at most p-valent in $\mathbb{D}$.

Proof. From Theorem 2.3, we have

$$\mathfrak{Re}\left\{\frac{f^{(p-1)}(z)}{z}\right\} > 0, \quad (z \in \mathbb{D}).$$

Applying Lemma 2.5 repeatedly, we obtain

$$\mathfrak{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > 0, \quad (z \in \mathbb{D}).$$

Thus from Lemma 2.6, f is at most p-valent in $\mathbb{D}$, which completes the proof. □

Thus we finally deduce the following improvement to Ozaki's condition (1.1).

Corollary 2.8. *If $f \in \mathcal{A}(p)$ for $p \geq 2$ and*

$$\mathfrak{Re}\left\{f^{(p)}(z)\right\} > -\frac{p!}{2}, \quad (z \in \mathbb{D}),$$

then f is at most p-valent in $\mathbb{D}$.

References

[1] K. Noshiro, On the theory of schlicht functions, J. Fac. Sci. Hokkaido Univ. Jap. 2 (1) (1934–1935) 129–135.
[2] M. Nunokawa, A note on multivalent functions, Tsukuba J. Math. 13 (2) (1989) 453–455.
[3] M. Nunokawa, On properties of non-Carathéodory functions, Proc. Jpn. Acad., Ser. A 68 (6) (1992) 152–153.
[4] S. Ozaki, On the theory of multivalent functions, Sci. Rep. Tokyo Bunrika Daigaku, Sect. A. 2 (1935) 167–188.
[5] Ch. Pommerenke, On close to-convex functions, Trans. Amer. Math. Soc. 114 (1) (1965) 176–186.
[6] K. Sakaguchi, On certain univalent mapping, J. Math. Soc. Jpn. 11 (1959) 72–75.
[7] T. Umezawa, Multivalently close-to-convex functions, Proc. Amer. Math. Soc. 8 (5) (1957) 869–874.
[8] S. Warschawski, On the higher derivatives at the boundary in conformal mapping, Trans. Amer. Math. Soc. 38 (1935) 310–340.