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A characterization of b_e-critical trees

Une caractérisation des arbres b_e-critiques

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ABSTRACT

The b-chromatic number of a graph G is the largest integer k such that G admits a proper coloring with k colors for which each color class contains a vertex that has at least one neighbor in all the other $k-1$ color classes. A graph G is called *b_e-critical* if the contraction of any edge e of G decreases the b-chromatic number of G. The purpose of this paper is the characterization of all *b_e-critical trees.*

RÉSUMÉ

Le nombre b-chromatique d'un graphe G est le plus grand entier k tel que G admette une coloration propre avec k couleurs, pour laquelle toute classe de couleur contient un sommet qui a au moins un voisin dans toutes les autres $k-1$ classes de couleur. Un graphe G est appelé *b_e-critique* si la contraction de toute arête e de G fait diminuer le nombre b-chromatique de G. Le but de cet article est la caractérisation de tous les arbres *b_e-critiques*.

1. Introduction

All graphs in this paper are finite and simple. For the terminology and the notations not defined here we refer to [2]. Let $G=(V(G), E(G))$ be a graph. For a non-empty set $A \subseteq V(G)$, we denote by $G[A]$ the subgraph of G induced by A, and by $G\setminus A$ the subgraph induced by $V(G)\setminus A$. If $A=\{v\}$ we may write $G\setminus v$ instead of $G\setminus\{v\}$. For a vertex v of G, the *open neighborhood* of v is $N_G(v)=\{u\in V(G): uv\in E(G)\}$ and the degree of v, denoted by $d_G(v)$, is $|N_G(v)|$. By $\Delta(G)$ and $d_G(u,v)$, we denote the *maximum degree* of the graph G and the *distance* between u and v in G, respectively. A *tree* is a connected graph without induced cycle. A *rooted tree* is a tree with a special vertex, called the *root* of the tree. A vertex of degree one is called a *leaf*, and its neighbor is called a *support* vertex. An edge incident with a leaf is called a *pendant edge*.

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A tree T is a *double star* $S_{p,q}$ $(p \geq q \geq 1)$ if it contains exactly two vertices x, y (called central vertices) that are not leaves such that $d_T(x) = p + 1$ and $d_T(y) = q + 1$. We let P_n and $K_{1,n-1}$ denote the *path* and *star* on n vertices, respectively.

A *proper coloring* of G is an assignment of colors (represented by natural numbers) to the vertices of G such that any two adjacent vertices have different colors. The minimum number $\chi(G)$ for which there exists a proper coloring (with $\chi(G)$ colors) is called the *chromatic number* of a graph G. A *b-coloring* of a graph by k colors is a proper coloring with the property that each color class contains a vertex that has at least one neighbor in all the other $k-1$ color classes. We call any such vertex a *b-vertex*. The *b-chromatic number* $b(G)$ of a graph G is the largest number k such that G has a b-coloring with k colors. This parameter has been defined by Irving and Manlove [7,10]. It is obvious that $\chi(G) \leq b(G) \leq \Delta(G)+1$. For arbitrary graphs, the problem of determining $b(G)$ is NP-complete [7,10], even when restricted to bipartite graphs [9]. For the special case of trees, Irving and Manlove [7,10] presented a linear time algorithm. A recent survey on the b-coloring in graphs can be found in [8].

It was observed in [7,10] that if a graph G admits a b-coloring with ℓ colors, G must have at least ℓ vertices with degree at least $\ell - 1$. The *m-degree* of a graph G, denoted $m(G)$, is the largest integer ℓ such that G has ℓ vertices of degree at least $\ell - 1$. Clearly, $m(G) \leq \Delta(G) + 1$. Irving and Manlove [7,10] show that this parameter bounds the b-chromatic number. So, every graph satisfies $b(G) \leq m(G)$. A vertex of G with degree at least $m(G) - 1$ is called a *dense vertex*. A *pivoted tree* is a tree T in which one vertex v of degree less than $m(G) - 1$ is distinguished and called the *pivot*.

Definition 1. [7,10] A tree T is pivoted if T has exactly $m(T)$ dense vertices and T contains a vertex v such that v is not dense and every dense vertex is adjacent either to v or to a neighbor of v of degree $m(T) - 1$.

The following observation is straightforward.

Observation 2. *Every non-dense vertex of a pivoted tree T, except the pivot, may be adjacent to at most one dense vertex of T.*

D.F. Manlove and R.W. Iring [7,10] have proved that, for trees, the b-chromatic number can be computed as follows.

Theorem 3. *[7] If T is a pivoted tree, then $b(T) = m(T) - 1$; else, $b(T) = m(T)$.*

The concept of critical graphs with respect to the b-chromatic number has received more attention in recent years. The graphs for which the b-chromatic number decreases on the deletion of any edge were first studied in [4,6]. Further, a characterization of all such graphs is given in [1]. On the other hand, the authors of [3] characterized the trees whose b-chromatic number decreases when any vertex is removed. The graphs for which the b-chromatic number increases upon the removal of any edge (or vertex) were explored in [5].

In this paper, we study those graphs where the b-chromatic number decreases on the contraction of any edge. Before stating our results, we need some definitions and notation. For a given graph G, the *contraction* of an edge $e = uv$ means removing u and v from the vertex-set $V(G)$ and replacing it by a new vertex z and attaching z to all vertices that are adjacent to u or v in G. We denote by G_e the graph obtained from G by contracting the edge e.

Definition 4. A graph is called b_e-critical if the b-chromatic number decreases upon the contraction of any edge.

More precisely, we say that a graph G is b_e-critical if $b(G_e) < b(G)$ holds for every edge e in G. The aim of the paper is to characterize all b_e-critical trees.

2. Preliminary results

This section presents some results that will be useful in the characterization of b_e-critical trees.

Observation 5. *Let e be an edge of a b_e-critical tree T and let T_e be the tree obtained from T by contracting e. Then,*

(*i*) *$m(T_e) \leq m(T)$, with equality if e is a non-pendant edge such that one of its endpoints is a non-dense vertex.*
(*ii*) *If T_e is not a pivoted tree, then $m(T_e) \leq m(T) - 1$.*

Proof. (*i*) If the first part is not true, then Theorem 3 yields $b(T_e) \geq m(T_e) - 1 \geq m(T) \geq b(T)$, which is a contradiction. The second part follows immediately because contracting such edge does not decrease the m-degree of T.

(*ii*) Using again Theorem 3, we get $m(T_e) = b(T_e) \leq b(T) - 1 = m(T) - 1$. □

For the remainder of this paper, we denote by D and L, respectively, the set of dense vertices and the set of leaves in T. Denote also by D_e and L_e, respectively, the set of dense vertices and the set of leaves in T_e.

Theorem 6. *Let $T = (V, E)$ be a b_e-critical tree, B be the set of all b-vertices of a b-coloring c of T with $b(T)$ colors, S be the set of all support vertices of T and D be the set of dense vertices in T. Then:*

(*i*) *T is not a pivoted tree;*
(*ii*) *$S \subseteq B$. Moreover, there are no two neighbors of a support vertex s with the same color such that one of them is a leaf;*
(*iii*) *there are no two b-vertices of the same color. So, $|B| = b(T)$;*
(*iv*) *$b(T) = \Delta(T) + 1$;*
(*v*) *$D = B$.*

Proof. Set $b(T) = k$. When $k = 2$, it is easy to see that $T = P_2$ and the theorem holds. So, we can assume that $k \geq 3$. Let e be any edge of T and let T_e denote the tree obtained from T by contracting the edge e in a new vertex v_e

(*i*) Suppose on the contrary that T is a pivoted tree with pivot v. So, Theorem 3 gives $b(T) = m(T) - 1$. Let u be a dense vertex adjacent to v. Pick $e = uv$. Observation 5 (*i*) yields $m(T_e) = m(T)$. Therefore, Theorem 3 implies that $b(T_e) \geq m(T_e) - 1 \geq m(T) - 1 = b(T)$, which is a contradiction.

(*ii*) If any part of (*ii*) does not hold, then contracting of some pendant edge incident to a support vertex s does not decrease the b-chromatic number.

(*iii*) Let x and y be two b-vertices of c such that $c(x) = c(y)$. Neither x nor y is a support vertex, because if not, contracting of some pendant edge incident to x or y does not decrease the b-chromatic number. Let us root T at vertex x. Let $u_1, u_2, ..., u_h$ be the neighbors of x. Since x is a b-vertex $h \geq k - 1$. For each $i \in \{1, 2, ..., h\}$, let T_i be the component of $T \setminus \{x\}$ that contains u_i. As x is not a support vertex, T_i has a support z_i for each $i \in \{1, 2, ..., h\}$. The first part of (*ii*) implies that z_i is a b-vertex of c. Thereby, z_i is the only b-vertex of c of color $c(z_i)$ in T (in particular, $c(z_i) \neq c(x)$), otherwise contracting of some pendant edge incident to z_i does not decrease the b-chromatic number. Therefore T contains exactly $k - 1$ support b-vertices $z_1, z_2, ..., z_h$ of distinct colors. So, $h = k - 1$ and for each $i \in \{1, 2, ..., k - 1\}$, T_i contains exactly one support vertex z_i. Let u be any vertex in $V(T) \setminus \{x, z_1, z_2, ..., z_{k-1}\}$. Assume that $u \in T_i$ for some integer i in $\{1, 2, ..., k - 1\}$. The degree of u is at most 2, because if not, T_i has two support vertices of T, which is a contradiction. So, $d_T(u) \leq 2$; in particular, we have also $d_T(y) \leq 2$. Consequently, $k = 3$. This means that $d_T(u) = d_T(y) = d_T(x) = 2$. Also, by the second part of (*ii*), we have $d_T(z_1) = d_T(z_2) = 2$. Hence, T is a path. Since T contains at least 4 b-vertices such that two of them (x and y) are non-support vertices of the same color, it follows that T is a path of at least 7 vertices. But in this case, T is not b_e-critical, which is a contradiction.

(*iv*) The upper bound trivially holds for any graph, so let us prove the lower bound. To do this, we claim that each vertex $x \in V(T)$ with $d_T(x) \geq 3$ satisfies the following,

$$\text{if } d_T(x) \geq 3, \text{ then every two neighbors of } x \text{ have distinct colors.} \tag{1}$$

Suppose, on the contrary, that x has two neighbors of the same color. Let $u_1, u_2, ..., u_p$ ($p \geq 3$) be the neighbors of x. Assume, without loss of generality, that u_1 and u_2 have the same color t, and that u_3 has color ℓ. Let us root T at x. Let T_i be the component of $T \setminus x$ that contains u_i for each $i \in \{1, ..., p\}$. So, there are two cases to consider; in such cases, our goal is to modify the b-coloring c and extend it to a b-coloring of T_e with k colors. To do this, we first interchange two colors of c in some of components of T. This might make c improper coloring. In this case, if two adjacent vertices x and y have the same color, the edge xy is called a conflicting edge.

Case 1: x is not a b-vertex.

We distinguish between two subcases.

Case 1.1: $\ell = t$.

Then one of T_1, T_2, T_3, say T_1 has no b-vertex of colors $c(x)$ and t. In this case, we interchange colors t and $c(x)$ in the component T_1, all other vertices of T keep their color. We obtain an improper coloring c' of T with k colors such that $e = u_1x$ is the unique conflicting edge (since u_1 and x are colored the same). Let c_e be the coloring c' restricted to $T \setminus \{u_1, x\}$. Since $T \setminus \{u_1, x\}$ is an induced subgraph of T_e, c_e can be extended to a proper coloring of T_e by assigning color $c(x)$ to v_e. It is easy to check that c_e yields a b-coloring of T_e with k colors, a contradiction.

Case 1.2: $\ell \neq t$.

If one of T_1, T_2 has no b-vertex of colors $c(x)$ and t, then we have a contradiction as in Case 1.1. So, assume that T_1 contains the b-vertex of color $c(x)$. Thereby, the b-vertex of color t belongs to T_2. If T_3 has no b-vertex of color ℓ, then we interchange colors ℓ and $c(x)$ in T_3; otherwise we interchange colors ℓ and $c(x)$ in $T_1 \cup T_3$. In both cases, we obtain an improper coloring of T with k colors such u_3 and x have the same color $c(x)$. Proceeding as in Case 1.1 above, taking $e = u_3x$, we get again a contradiction.

Case 2: x is a b-vertex.

Note that, by the second part of (*ii*), u_1 and u_2 are not leaves in T. Therefore, by (*iii*), one of T_1 or T_2, say T_1, has no b-vertex of colors t and $c(x)$. In this case, we can interchange colors t and $c(x)$ in T_1 and all other vertices of T keep their color. We obtain an improper coloring of T with k colors such that u_1 and x are colored the same. By taking $e = u_1x$ and proceeding as in Case 1.1, we obtain a contradiction.

In each case, we have a contradiction; thus Claim (1) is proved. As a consequence, each vertex x in T has at most $k - 1$ neighbors, because if not, x has two neighbors of the same color with $d_T(x) \geq k \geq 3$, which contradicts Claim (1). Hence, $d_T(x) \leq k - 1$ for every vertex x in T, and in particular, we have $\Delta(G) \leq k - 1$.

(v) It is obvious that $B \subset D$. Let $x \in D$. As, by (i) and (vi), $m(T) = \Delta(T) + 1$, it follows that $d_T(x) = \Delta(T) = k - 1$. If $k = 3$, then $T = P_5$ and then (v) holds. So, assume that $d_T(x) = k - 1 \geq 3$. Therefore, by Claim (1), all neighbors of x have distinct colors. This means that $x \in B$. Thus $D = B$. This concludes the proof. □

In the rest of this section, we denote by $\overline{D} = V(T) \setminus D$, the set of non-dense vertices in T; and by $\overline{D}_e = V(T_e) \setminus D_e$ the set of non-dense vertices in T_e.

We next proceed to characterize all b_e-critical trees. For this purpose, we prove the following lemmas.

Lemma 7. *Let T be a b_e-critical tree and $v \in \overline{D}$. Then v is not a support vertex and has at least one neighbor in D.*

Proof. Consider a b-coloring c of T with $b(T)$ colors. Observe first that if v is a leaf, then v is adjacent to a support vertex which belongs to D by Theorem 6 (ii) and (v). Hence, we can suppose that v is not a leaf. Also, Theorem 6 (v) implies that v is not a b-vertex of c. So, v is not a support vertex in T by Theorem 6 (ii). Suppose that v has no neighbor in D. We root T at vertex v. Let v_1 and v_2 two neighbors of v, and let T_i ($i = 1, 2$) be the component of $T \setminus v$ that contains v_i. Since v is not a support vertex, $d_T(v_i) \geq 2$. This means that each component T_i has at least one support vertex z_i. Then, by Theorem 6 (ii) and (v), z_i is a dense vertex in T. Let T_e be the tree obtained from T by contracting the edge $e = vv_1$ in a new vertex v_e. Clearly, z_i remains a support vertex in T_e. Thus, by Observation 5 (i) and (ii), we have $m(T_e) = m(T)$, and T_e is a pivoted tree. Let w be the unique pivot of T_e. Then $w = v_e$ or v_2, for otherwise w would be a pivot of T, which contradicts Theorem 6 (i). In this case, one of z_1, z_2 is not adjacent to w or to a dense vertex adjacent to w, this contradicts Definition 1. Thus T_e is not a pivoted tree, a contradiction again. □

Lemma 8. *Let T be a b_e-critical tree with $b(T) = k \geq 3$. Then $\overline{D} \setminus L$ is either an empty-set, or has exactly two non-support vertices, each of degree 2, and at distance at most 2, and $k \geq 4$.*

Proof. Consider a b-coloring c of T with k colors. Suppose first that $\overline{D} \setminus L$ is an empty set. Then each vertex of $V(T)$ is either a dense vertex or a leaf. It is clear that the contraction of any edge of T decreases the m-degree of T, and so its b-chromatic number. Hence such tree exists. Assume now that $\overline{D} \setminus L$ is a non-empty set. If $k = 3$, Theorem 6 (iv) implies that each vertex in D has degree 2. Therefore, each vertex in $D \cup (\overline{D} \setminus L)$ is a dense vertex, which is a contradiction. Thus $k \geq 4$. Let $v_1 \in \overline{D} \setminus L$. In view of Lemma 7, v_1 is not a support vertex, and has a neighbor $u \in D$. Let e be any edge of T and let T_e be the tree obtained from T by contracting e in a new vertex v_e. Pick $e = uv_1$. Obviously, $v_e \in D_e$ and $L_e = L$. Since e is not a pendant edge in T, Observation 5 (i) and (ii) imply that

$$m(T_e) = m(T) \text{ and } T_e \text{ is a pivoted tree.} \tag{2}$$

As $v_e \in D_e$ and each dense vertex in T different from u remains a dense vertex in T_e, it follows that $D_e = (D \cup \{v_e\}) \setminus u$ and $\overline{D}_e = \overline{D} \setminus v_1$. If v_1 is the unique vertex of $\overline{D} \setminus L$, then $\overline{D}_e \setminus L_e = \emptyset$ (each non-dense vertex in T_e is a leaf). This implies, by Definition 1, that T_e is not a pivoted tree, which contradicts (2). Hence,

$$\left|\overline{D} \setminus L\right| \geq 2. \tag{3}$$

Let $v_2 \neq v_1$ be a vertex of $\overline{D} \setminus L$. Then $v_2 \in \overline{D}_e \setminus L_e$. Assume, without loss of generality, that v_2 is the pivot of T_e. Then v_e is adjacent to v_2 or to a dense vertex adjacent to v_2. Denote by D_1 the set of dense vertices in T_e that are adjacent to v_2, and by D_2 the remaining dense vertices in T_e. So $D_1 \cup D_2 = D_e$. As T_e is a pivoted tree, Definition 1 implies that D_i ($i = 1, 2$) is a stable set. Also, each vertex in D_2 is adjacent to exactly one vertex in D_1 and not to v_2. Suppose that $\left|\overline{D} \setminus L\right| \geq 3$. Let $v_3 \neq v_1, v_2$ be a vertex of $\overline{D} \setminus L$. So $v_3 \in \overline{D}_e \setminus L_e$. By Lemma 7 and Observation 2, v_3 has exactly one neighbor, say x_1 in D, and so in D_e. As, v_3 is not a leaf in T (and in T_e), it has a neighbor v_4 in $\overline{D}_e \setminus L_e$. Then again, Lemma 7 and Observation 2 imply that v_4 has exactly one neighbor, say x_2 in D and so in D_e. Clearly, $x_2 \neq x_1$. Vertices x_1, x_2 cannot be both in D_1, for otherwise v_3, v_4, x_2, v_2, x_1 induce a cycle of length 5 in T_e. Likewise, x_1, x_2 cannot both be in D_2. Indeed, if x_1 and x_2 have a common neighbor y in D_1, then v_3, v_4, x_2, y, x_1 induce a cycle of length 5 in T_e; otherwise, x_1, v_3, v_4, x_2 together with v_2 and the two neighbors of x_1, x_2 in D_1 induce a cycle of length 7 in T_e. If x_i ($i = 1, 2$) belongs to D_i, then x_1, v_3, v_4, v_2 together with x_2 and its neighbor in D_1 induce a cycle of length 6 in T_e, a contradiction. Thus $\left|\overline{D} \setminus L\right| \leq 2$. This means, by (3), that $\overline{D} \setminus L$ contains exactly two non-support vertices v_1, v_2.

Now, we shall show that both v_1 and v_2 have degree 2. Since v_1 is not a leaf in T (and in T_e), it has a neighbor $u' \neq u$. Then u' must be adjacent to v_e in T_e. Suppose that $u' \in D_e \setminus v_e$. In this case, v_e and u' cannot both be in D_i ($i = 1, 2$), because it is a stable set. This implies that either $u' = v_2$ or exactly one of v_e, u', say u' belongs to D_1. In each case, v_1 can not has an other neighbor, which is in D, because $\left|\overline{D} \setminus L\right| = 2$, for otherwise, let $u'' \in D$ be the third neighbor of v_1. In the first one, we have $v_e \in D_1$ and $u'' \in D_2$, hence v_e is adjacent to the pivot and to the dense vertex u'', but $d_{T_e}(v_e) \geq m(T_e)$, which contradicts Definition 1. In the last one, $v_e \in D_2$, as D_i is a stable set (for $i = 1, 2$) $u'' \in D_1$, so v_e has two neighbors in D_1, a contradiction with the fact that each vertex in D_2 is adjacent to exactly one vertex in D_1. Therefore, v_1 has degree 2 in T. Similarly, v_2 has degree equal to 2 in T. Proceeding similarly as above, we conclude that $d_T(v_2) = 2$.

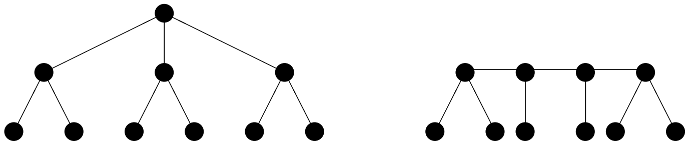

Fig. 1. Examples of trees that belong to $\mathcal{T}_1^4$.

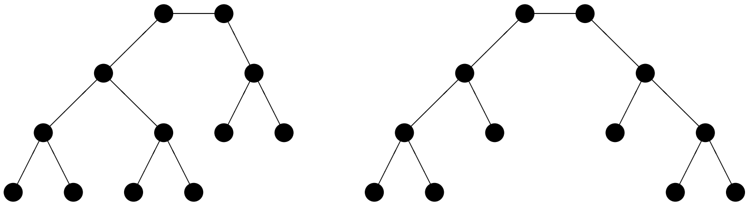

Fig. 2. Examples of trees which belong to $\mathcal{T}_2^4$.

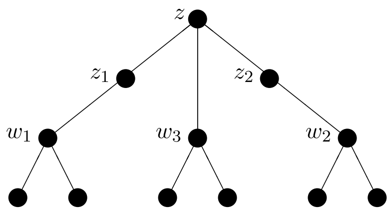

Fig. 3. The tree T_0^4.

Finally, it remains to prove that $d_T(v_1, v_2) \leq 2$. If this is not true, then v_e (dense vertex in T_e) would not be adjacent to the pivot v_2 of T_e, or to a dense vertex adjacent to v_2, which contradicts (2). Hence, $d_T(v_1, v_2) \leq 2$. □

Now we are ready to characterize all b_e-critical trees.

3. Characterization of b_e-critical trees

The main result of this section is a characterization of trees for which contracting any edge decreases its b-chromatic number. For this purpose, we define two families of trees $\mathcal{T}_1^k$, $\mathcal{T}_2^k$ and a special tree T_0^k as follows. Let k be a positive integer. A tree T is in the family $\mathcal{T}_1^k$ (with $k \geq 3$) if it has k vertices each of degree $k-1$, and the other vertices are leaves. A tree T is in the family $\mathcal{T}_2^k$ (with $k \geq 4$) if it can be obtained from a double star $S_{k-2,k-2}$ with central vertices w_1 and w_2 and subdividing the edge w_1w_2 twice by inserting two new vertices z_1 and z_2 (i.e. adding two new vertices z_1, z_2 and edges w_1z_1, z_1z_2 and z_2w_2 in $G - w_1w_2$) and attaching $k-2$ new vertices to each of the $k-2$ leaves of $S_{k-2,k-2}$. We define T_0^k (with $k \geq 4$) to be the tree obtained from $k-1$ disjoint stars with central vertices $w_1, w_2, \ldots, w_{k-1}$, each of order $k-1$, by adding a new vertex z attached to w_i for each $i \in \{1, \ldots, k-1\}$ and for $i = 1, 2$ subdividing the edge zw_i once by inserting one vertex z_i.

Note that each tree T in $\{\mathcal{T}_1^k, k \geq 3\} \cup \{\mathcal{T}_2^k \cup \{T_0^k\}, k \geq 4\}$ has exactly k vertices, each of degree $k-1$, but no vertex of T is a pivot. This means that $|D| = k$ and T is not pivoted. So, by Theorem 3, we have $b(T) = k$.

Put $\mathcal{T} = \{\mathcal{T}_1^k, k \geq 3\} \cup \{\mathcal{T}_2^k \cup \{T_0^k\}, k \geq 4\}$ with $k = b(T)$. Notice that the only tree which belongs to $\mathcal{T}$ with $k = 3$ is P_5.

In Figs. 1–3, we give examples of trees belonging to $\mathcal{T}_1^4$, $\mathcal{T}_2^4$, or $\{T_0^4\}$.

Theorem 9. *A graph T is b_e-critical if and only if T is a P_2 or $T \in \mathcal{T}$.*

Proof. To establish the theorem, we will first prove the sufficiency condition. Let T be a member of $\mathcal{T}$, with the same notation as above. It is obvious that P_2 is b_e-critical, so we can assume that $T \neq P_2$. By the remark before the Theorem, $b(T) = m(T) = k \geq 3$. Let T_e be the tree obtained from T by contracting the edge e of T. If $T \in \mathcal{T}_1^k$, then contracting e decreases the m-degree of T by one. Therefore $m(T_e) = k-1$, implying that $b(T_e) \leq k-1 < k = b(T)$. Assume now that $k \geq 4$ and $T \in \mathcal{T}_2^k \cup \{T_0^k\}$. If one of the endpoints of e is z_1 or z_2, then the contraction of e does not decrease the m-degree of T, and T_e is a pivoted tree. Then $m(T_e) = k$, which means by Theorem 3 that $b(T_e) = k-1 < k = b(T)$. If the endpoints of e are dense vertices or one of them is a leaf, then the contraction of e decreases the m-degree of T by one. Therefore $m(T_e) = k-1$ and $b(T_e) \leq k-1 < k = b(T)$. Hence T is a b_e-critical tree.

To prove the necessity, let T be a b_e-critical tree with $k = b(G)$. Consider a b-coloring c of T with k colors. Let B, D denote, respectively, the set of all b-vertices of c and the set of all dense vertices of T. According to clauses (iii) and (iv) of Theorem 6, we have $|B| = k = \Delta(T) + 1$, and each vertex of B has degree $k - 1$. If $k \in \{2, 3\}$, then $\Delta(T) \in \{1, 2\}$, and it is easy to verify that $T = P_2$ or $T = P_5 \in \mathcal{T}_1^3$. So, assume that $k \geq 4$. Clause (v) of Theorem 6 yields $D = B$. Let $D = \{x_1, x_2, ..., x_k\}$ and $\overline{D} = V \setminus D$. In view of Lemma 8, either $\overline{D} \setminus L$ is empty, which means that each vertex of T is either a leaf or has degree equal to $k - 1$, thus $T \in \mathcal{T}_1^k$; or $\overline{D} \setminus L$ has two non-support vertices v_1 and v_2, both of degree 2. In this case, $V = D \cup L \cup \{v_1, v_2\}$, and by Lemma 8, we have two cases to consider.

Case 1: $d_T(v_1, v_2) = 2$.

As v_1 and v_2 are non-support vertices, their neighbors are in D. Therefore, since $d_T(v_1, v_2) = 2$, we can assume, without loss of generality, that $N_T(v_1) \cap N_T(v_2) = \{x_3\}$ and for $i = 1, 2$, $v_i \in N_T(x_i)$. We claim that, for each $t \in \{4, ..., k\}$, x_t has no neighbor in $\{x_1, x_2\}$. Suppose, on the contrary, that x_t is adjacent to x_1. Pick $e = v_1 x_3$. Let T_e be the tree obtained from T by contracting the edge e. Observe that each vertex in T_e, except v_2, is either a leaf or a dense vertex. Since e is not a pendant edge in T, clauses (i) and (ii) of Observation 5 imply that $m(T_e) = m(T)$ and T_e is a pivoted tree with pivot v_2 (since v_2 is the unique non-dense vertex that is not a leaf in T_e). But, in this case, x_t is not adjacent to v_2 or to a dense vertex adjacent to v_2, leading to contradicting the fact that T_e is a pivoted tree. Consequently, x_1 is not adjacent to x_t in T_e, and thus in T. Likewise, x_2 is not adjacent to x_t in T. In this case, Definition 1 implies that x_t must be adjacent to x_3 in T_e, and thus in T. Hence T is isomorphic to T_0^k.

Case 2: $d_T(v_1, v_2) = 1$.

For $i = 1, 2$, let x_i be the unique dense vertex adjacent to v_i in T. Pick $e = v_1 v_2$, and let T_e be the tree obtained from T by contracting edge e in a new vertex v_e. Using a similar argument as in Case 1, we conclude that T_e is a pivoted tree with pivot v_e (since each other vertex in T_e is either a dense vertex or a leaf). Therefore, each dense vertex in T_e (and so in T) different from x_1 and x_2 is either adjacent to x_1 or to x_2. So $T \in \mathcal{T}_2^k$. □

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