C. R. Acad. Sci. Paris, Ser. I 356 (2018) 243–252

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Number theory/Mathematical analysis

Computation and theory of Euler sums of generalized hyperharmonic numbers

Théorie et calcul des sommes d'Euler des nombres hyper-harmoniques généralisés

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ARTICLE INFO

Article history:
Received 4 September 2017
Accepted after revision 10 January 2018
Available online 1 February 2018

Presented by the Editorial Board

ABSTRACT

Recently, Dil and Boyadzhiev [10] proved an explicit formula for the sum of multiple harmonic numbers whose indices are the sequence $(\{0\}_r, 1)$. In this paper, we show that the sums of multiple harmonic numbers whose indices are the sequence $\left(\{0\}_r, 1; \{1\}_{k-1}\right)$ can be expressed in terms of (multiple) zeta values, (multiple) harmonic numbers, and Stirling numbers of the first kind, and give an explicit formula.

RÉSUMÉ

Récemment, Dil et Boyadzhiev [10] ont établi une formule explicite pour les sommes de nombres hyper-harmoniques multiples, dont les indices sont les suites $(\{0\}_r, 1)$. Nous montrons ici que les sommes de nombres harmoniques multiples dont les indices sont $(\{0\}_r, 1; \{1\}_{k-1})$ peuvent être exprimées en termes de valeurs zêta (multiples), de nombres harmoniques (multiples) et de nombres de Stirling de première espèce. Nous donnons une formule explicite.

1. Introduction

Let $s_1, \ldots, s_m$ and n be positive integers. The classical multiple harmonic numbers (MHNs) and multiple harmonic star numbers (MHSNs) are defined by the partial sums (see [15,22]):

$$\zeta_n(s_1, s_2, \cdots, s_m) := \sum_{n \geq n_1 > \cdots > n_m \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_m^{s_m}}, \tag{1.1}$$

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https://doi.org/10.1016/j.crma.2018.01.004

$$\zeta_n^\star(s_1, s_2, \cdots, s_m) := \sum_{n\geq n_1\geq\cdots\geq n_m\geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_m^{s_m}}, \tag{1.2}$$

when $n < m$, then $\zeta_n(s_1, s_2, \cdots, s_m) = 0$, and $\zeta_n(\emptyset) = \zeta_n^\star(\emptyset) = 1$. The limit cases of MHNs and MHSNs give rise to multiple zeta values (MZVs) and multiple zeta star values (MZSVs) (see [15,16,19,22,27]):

$$\zeta(s_1, s_2, \cdots, s_m) = \lim_{n\to\infty} \zeta_n(s_1, s_2, \cdots, s_m),$$
$$\zeta^\star(s_1, s_2, \cdots, s_m) = \lim_{n\to\infty} \zeta_n^\star(s_1, s_2, \cdots, s_m)$$

defined for $s_2, \ldots, s_m \geq 1$ and $s_1 \geq 2$ to ensure convergence of the series. For non-negative integers $s_1, \ldots, s_{m+k}$, we define the following generalized multiple harmonic numbers

$$\zeta_n\left(s_1, \cdots, s_m; s_{m+1}, \cdots, s_{m+k}\right) := \sum_{\substack{0<n_{m+k}<\cdots<n_{m+1}\\ <n_m\leq n_{m-1}\cdots\leq n_1\leq n}} \frac{1}{n_1^{s_1} \cdots n_{m+k}^{s_{m+k}}}. \tag{1.3}$$

Obviously, if $m = 0$ or $k = 0$ in (1.3) and $s_j \in \mathbb{N} := \{1, 2, \cdots\}$, then

$$\zeta_n(\emptyset; s_1, \cdots, s_k) = \zeta_n(s_1, \cdots, s_k),$$
$$\zeta_n(s_1, \cdots, s_m; \emptyset) = \zeta_n^\star(s_1, \cdots, s_m).$$

There are a lot of recent contributions on MZVs and MZSVs (for example, see [15,16,19,22,27]). The earliest results on MZVs or MZSVs are due to Euler, who elaborated a method to reduce double sums $\zeta(s_1, s_2)$ (also called linear Euler sums [13,25]) of small weight to certain rational linear combinations of products of zeta values. In [13], Flajolet and Salvy introduced the following generalized series:

$$S_{\mathbf{S},q} := \sum_{n=1}^\infty \frac{H_n^{(s_1)} H_n^{(s_2)} \cdots H_n^{(s_r)}}{n^q}, \tag{1.4}$$

which is called the generalized (nonlinear) Euler sums. Here $\mathbf{S} := (s_1, s_2, \ldots, s_r)$ $(r, s_i \in \mathbb{N}, i = 1, 2, \ldots, r)$ with $s_1 \leq s_2 \leq \ldots \leq s_r$ and $q \geq 2$. The quantity $w := s_1 + \cdots + s_r + q$ is called the weight and the quantity r is called the degree. The notation $H_n^{(p)}$ denotes the ordinary harmonic numbers defined by

$$H_n^{(p)} \equiv \zeta_n(p) := \sum_{j=1}^n \frac{1}{j^p} \quad (p, n \in \mathbb{N}).$$

It has been discovered in the course of the years that many nonlinear Euler sums admit expressions involving finitely "zeta values" (that is say values of the Riemann zeta function

$$\zeta(s) := \sum_{n=1}^\infty \frac{1}{n^s} \quad (\Re(s) > 1)$$

with positive integer arguments) and linear Euler sums. The relationship between the values of the Riemann zeta values and Euler sums has been studied by many authors. For details and historical introductions, please see [1,4–7,11,13,17,20,21,24–26] and references therein.

From [2,9,10,12,14,18], we know that the classical hyperharmonic numbers are defined by

$$h_n^{(m)} := \sum_{1\leq n_m\leq\cdots\leq n_1\leq n} \frac{1}{n_m} = \zeta_n^\star\left(\{0\}_{m-1}, 1\right). \tag{1.5}$$

In [23], we define the generalized hyperharmonic numbers $h_n^{(m)}(k)$ by

$$h_n^{(m)}(k) := \sum_{\substack{1\leq n_{m+k-1}<\cdots<n_{m+1}\\ <n_m\leq n_{m-1}\leq\cdots\leq n_1\leq n}} \frac{1}{n_m n_{m+1} \cdots n_{m+k-1}} = \zeta_n\left(\{0\}_{m-1}, 1; \{1\}_{k-1}\right), \tag{1.6}$$

where $m, k \in \mathbb{N}$. (The notation $\{\}_p$ means that the sequence in the bracket is repeated p times.) In this paper, we prove the result: for positive integers m and k, the Euler-type sums with hyperharmonic numbers

$$S(k, m; p) := \sum_{n=1}^\infty \frac{h_n^{(m)}(k)}{n^p} \quad (p \geq m+1)$$

are related to the multiple zeta values, multiple harmonic numbers and Stirling numbers of the first kind. For $k = 1, 2, 3$, the above results have been proved in Dil et al. [10] and our paper [23]. The purpose of the present paper is to prove the following two theorems.

Theorem 1.1. *For integers $k \in \mathbb{N}$ and $r \in \mathbb{N}_0$ with $r + 2 \le p \in \mathbb{N}$, then the following identity holds:*

$$S(k, r+1; p) = \frac{1}{r!}\sum_{l=1}^{r+1}\begin{bmatrix} r+1 \\ l \end{bmatrix}\sum_{\substack{i+j=k \\ i,j\ge 0}} (-1)^i \zeta_r^\star(\{1\}_i)\, U_{j,r}(p+1-l), \tag{1.7}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}$ denotes the (unsigned) Stirling number of the first kind, which is defined by [8,9]

$$n!x(1+x)\left(1+\frac{x}{2}\right)\cdots\left(1+\frac{x}{n}\right) = \sum_{k=0}^{n}\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} x^{k+1}, \tag{1.8}$$

with $\begin{bmatrix} n \\ k \end{bmatrix} := 0$, if $n < k$ and $\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ k \end{bmatrix} := 0$, $\begin{bmatrix} 0 \\ 0 \end{bmatrix} := 1$, or equivalently, by the generating function:

$$\log^k(1-x) = (-1)^k k! \sum_{n=1}^{\infty}\begin{bmatrix} n \\ k \end{bmatrix}\frac{x^n}{n!},\ x \in [-1, 1). \tag{1.9}$$

Here $U_{j,r}(p)$ denotes the infinite sum

$$U_{j,r}(p) := \sum_{n=1}^{\infty}\frac{\zeta_{n+r}\left(\{1\}_j\right)}{n^p}. \tag{1.10}$$

For a closed-form representation of $U_{j,r}(p)$, see Theorem 1.2.

According to the definition (1.8), it is obvious that the (unsigned) Stirling number of the first kind can be expressed in terms of a rational linear combination of products of harmonic numbers. Some illustrative examples are as follows.

$$\begin{aligned}
\begin{bmatrix} n \\ 1 \end{bmatrix} &= (n-1)!, \\
\begin{bmatrix} n \\ 2 \end{bmatrix} &= (n-1)! H_{n-1}, \\
\begin{bmatrix} n \\ 3 \end{bmatrix} &= \frac{(n-1)!}{2}\left[H_{n-1}^2 - H_{n-1}^{(2)}\right], \\
\begin{bmatrix} n \\ 4 \end{bmatrix} &= \frac{(n-1)!}{6}\left[H_{n-1}^3 - 3H_{n-1}H_{n-1}^{(2)} + 2H_{n-1}^{(3)}\right], \\
\begin{bmatrix} n \\ 5 \end{bmatrix} &= \frac{(n-1)!}{24}\left[H_{n-1}^4 - 6H_{n-1}^{(4)} - 6H_{n-1}^2 H_{n-1}^{(2)} + 3(H_{n-1}^{(2)})^2 + 8H_{n-1}H_{n-1}^{(3)}\right].
\end{aligned}$$

These formulas can also be found in Comtet's book [8].

Theorem 1.2. *For integers $m > 0$, $p > 1$ and $r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we have*

$$\begin{aligned}
&U_{m,r}(p) = \zeta(p, \{1\}_m) + \zeta\left(p+1, \{1\}_{m-1}\right) \\
&+ (-1)^{p-1}\sum_{1\le i_m<\cdots<i_1\le r}\sum_{l=1}^{m}\frac{H_{i_l}}{i_l^p \prod\limits_{\substack{a=1 \\ a\ne l}}^{m}(i_a - i_l)} \\
&+ \sum_{1\le i_m<\cdots<i_1\le r}\sum_{b=1}^{p-1}\sum_{l=1}^{m}\frac{\zeta(p+1-b)}{i_l^b \prod\limits_{\substack{a=1 \\ a\ne l}}^{m}(i_a - i_l)}(-1)^{b-1}
\end{aligned}$$

$$+(-1)^{p-1}\sum_{j=1}^{m-1}\sum_{1\le i_j<\cdots<i_1\le r}\sum_{l=1}^{j}\frac{\zeta(m-j+1)+\zeta^\star_{i_l}\left(\{1\}_{m-j+1}\right)-\frac{\zeta^\star_{i_l}(\{1\}_{m-j})}{i_l}}{i_l^p\prod\limits_{a=1,\,a\ne l}^{j}(i_a-i_l)}$$

$$+\sum_{j=1}^{m-1}\sum_{1\le i_j<\cdots<i_1\le r}\sum_{b=1}^{p-1}\sum_{l=1}^{j}\frac{\zeta\left(p+1-b,\{1\}_{m-j}\right)+\zeta\left(p+2-b,\{1\}_{m-j-1}\right)}{i_l^b\prod\limits_{a=1,\,a\ne l}^{j}(i_a-i_l)}(-1)^{b-1},\tag{1.11}$$

where H_{i_l} is harmonic number H_n with $n=i_l$.

Since the mzvs $\zeta\left(m+1,\{1\}_{n-1}\right)$ $(m,n\in\mathbb{N})$ can be expressed as a rational linear combination of products of zeta values and the mhns $\zeta_n^\star\left(\{1\}_m\right)$ can be expressed in terms of harmonic numbers (see [22]). Hence, the Theorem 1.2 implies that the sums $U_{m,r}(p)$ can be expressed in terms of series of Riemann zeta function and (multiple) harmonic numbers. Further, from Theorem 1.1, we know that the Euler type sums $S(k,m;p)$ can be evaluated in closed from.

2. Some lemmas and theorems

To prove the Theorem 1.1 and Theorem 1.2, we need the following lemmas.

Lemma 2.1. *([23]) For positive integers n and k, then the following identity holds:*

$$\begin{bmatrix} n \\ k \end{bmatrix}=(n-1)!\zeta_{n-1}\left(\{1\}_{k-1}\right).\tag{2.1}$$

Lemma 2.2. *([23]) For positive integers m, n and k, we have the recurrence relation*

$$h_n^{(m)}(k)=\frac{(-1)^{k-1}}{k}\sum_{i=0}^{k-1}(-1)^i h_n^{(m)}(i)\left\{H_{m+n-1}^{(k-i)}-H_{m-1}^{(k-i)}\right\},\tag{2.2}$$

where

$$h_n^{(m)}(0):=\binom{m+n-1}{m-1}.$$

Lemma 2.3. *([22]) For positive integers m and n, then the recurrence relation holds:*

$$\bar{B}_m(n)=\frac{(-1)^{m-1}}{m}\sum_{i=0}^{m-1}(-1)^i\bar{B}_i(n)X_n(m-i),\tag{2.3}$$

where

$$X_n(m):=\sum_{i=1}^{n}x_i^m\quad(x_i\in\mathbb{C}),$$

$$\bar{B}_m(n):=\sum_{k_1=1}^{n}x_{k_1}\sum_{k_2=1}^{k_1-1}x_{k_2}\cdots\sum_{k_m=1}^{k_{m-1}-1}x_{k_m}\quad(\bar{B}_0(n):=1).$$

Lemma 2.4. *([23,26]) For integers $k\in\mathbb{N}$ and $p\in\mathbb{N}\setminus\{1\}:=\{2,3,\ldots\}$, then the following identity holds:*

$$(p-1)!\sum_{n=1}^{\infty}\frac{\begin{bmatrix} n+1 \\ p \end{bmatrix}}{n!n(n+k)}=\frac{1}{k}\left\{(p-1)!\zeta(p)+\frac{Y_p(k)}{p}-\frac{Y_{p-1}(k)}{k}\right\},\tag{2.4}$$

where $Y_k(n):=Y_k\left(H_n,1!H_n^{(2)},2!H_n^{(3)},\cdots,(r-1)!H_n^{(r)},\cdots\right)$ and $Y_k(x_1,x_2,\cdots)$ stands for the complete exponential Bell polynomial defined by (see [8])

$$\exp\left(\sum_{m\ge1}x_m\frac{t^m}{m!}\right)=1+\sum_{k\ge1}Y_k(x_1,x_2,\cdots)\frac{t^k}{k!}.\tag{2.5}$$

Noting that, in [22,25], we find the relation

$$\zeta_n^\star\left(\{1\}_m\right)=\frac{1}{m!}Y_m\left(n\right)\quad (n,m\in\mathbb{N}_0) \tag{2.6}$$

and give

$$Y_1\left(n\right)=H_n, Y_2\left(n\right)=H_n^2+H_n^{(2)}, Y_3\left(n\right)=H_n^3+3H_nH_n^{(2)}+2H_n^{(3)},$$
$$Y_4\left(n\right)=H_n^4+8H_nH_n^{(3)}+6H_n^2H_n^{(2)}+3(H_n^{(2)})^2+6H_n^{(4)}.$$

Lemma 2.5. *([3]) For positive integers m and n, then*

$$\sum_{\substack{i+j=m\\ i,j\geq 0}}(-1)^i\bar{A}_i\left(n\right)A_j\left(n\right)=0, \tag{2.7}$$

where

$$\bar{A}_m\left(n\right):=\sum_{1\leq k_m<\cdots<k_1\leq n}a_{k_1}\cdots a_{k_m}\quad (a_k\in\mathbb{C}),$$
$$A_m\left(n\right):=\sum_{1\leq k_m\leq\cdots\leq k_1\leq n}a_{k_1}\cdots a_{k_m}\quad (a_k\in\mathbb{C}).$$

For convenience, we set $\bar{A}_0\left(n\right)=A_0\left(n\right)=1$. If $n<m$, we let $\bar{A}_m\left(n\right)=0$.

Proof. By a direct calculation, the following identities are easily derived:

$$\prod_{i=1}^n(1-a_it)^{-1}=\sum_{m=0}^\infty A_m\left(n\right)t^m,$$
$$\prod_{i=1}^n(1+a_it)=\sum_{m=0}^\infty \bar{A}_m\left(n\right)t^m.$$

Hence, by using Cauchy product of power series, we have

$$\begin{aligned}1&=\prod_{i=1}^n(1-a_it)^{-1}\prod_{i=1}^n(1-a_it)\\ &=\left(\sum_{m=0}^\infty A_m\left(n\right)t^m\right)\left(\sum_{m=0}^\infty(-1)^m\bar{A}_m\left(n\right)t^m\right)\\ &=\sum_{m=0}^\infty\left\{\sum_{\substack{i+j=m\\ i,j\geq 0}}(-1)^i\bar{A}_i\left(n\right)A_j\left(n\right)\right\}t^m.\end{aligned}$$

Thus, comparing the coefficients of t^m in the above equation, we obtain the formula (2.7). The proof of Lemma 2.5 is finished. □

The above lemmas will be useful in the development of the main theorems. Next, we give some important theorems and theirs proofs by using these lemmas.

Theorem 2.6. *For integers $r\geq 0$ and $m,n>1$, then*

$$\sum_{1\leq k_m<\cdots<k_1\leq n}\frac{1}{(k_1+r)\cdots(k_m+r)}=\sum_{\substack{i+j=m\\ i,j\geq 0}}(-1)^i\zeta_r^\star\left(\{1\}_i\right)\zeta_{n+r}\left(\{1\}_j\right), \tag{2.8}$$

where

$$\zeta_0^\star\left(\emptyset\right)=\zeta_0\left(\emptyset\right):=1\quad and\quad \zeta_0^\star\left(\{1\}_i\right)=\zeta_0\left(\{1\}_j\right):=0\quad (i,j\geq 1).$$

Proof. The proof is by induction on m. For $m=1$ we have $\sum\limits_{1\le k_1\le n}\frac{1}{k_1+r}=\zeta_{n+r}(1)-\zeta_r^\star(1)$, and the formula is true. For $m>1$ we proceed as follow. Let

$$\zeta_n(s_1,s_2,\cdots,s_m\,|r+1):=\sum_{1\le k_m<\cdots<k_1\le n}\frac{1}{(k_1+r)^{s_1}\cdots(k_m+r)^{s_m}},\tag{2.9}$$

$$\zeta_n(\emptyset\,|r+1)=1.$$

Then by the definition (2.9) and the induction hypothesis, we have that

$$\begin{aligned}\zeta_n\left(\{1\}_{m+1}\,|r+1\right)&=\sum_{k=1}^n\frac{\zeta_{k-1}(\{1\}_m\,|r+1)}{k+r}\\&=\sum_{k=1}^n\frac{1}{k+r}\sum_{\substack{i+j=m\\i,j\ge0}}(-1)^i\zeta_r^\star(\{1\}_i)\,\zeta_{k+r-1}\left(\{1\}_j\right)\\&=\sum_{\substack{i+j=m\\i,j\ge0}}(-1)^i\zeta_r^\star(\{1\}_i)\sum_{k=1}^n\frac{\zeta_{k+r-1}\left(\{1\}_j\right)}{k+r}\\&=\sum_{\substack{i+j=m\\i,j\ge0}}(-1)^i\zeta_r^\star(\{1\}_i)\left\{\zeta_{n+r}\left(\{1\}_{j+1}\right)-\zeta_{n+r}\left(\{1\}_{j+1}\right)\right\}\\&=\sum_{\substack{i+j=m+1\\i,j\ge0}}(-1)^i\zeta_r^\star(\{1\}_i)\,\zeta_{n+r}\left(\{1\}_j\right)-\sum_{\substack{i+j=m+1\\i,j\ge0}}(-1)^i\zeta_r^\star(\{1\}_i)\,\zeta_r\left(\{1\}_j\right).\end{aligned}\tag{2.10}$$

On the other hand, from Lemma 2.5, setting $a_k=\frac{1}{k}$ and $n=r$, we get:

$$\sum_{\substack{i+j=m\\i,j\ge0}}(-1)^i\zeta_r^\star(\{1\}_i)\,\zeta_r\left(\{1\}_j\right)=0\quad(m\ge1).\tag{2.11}$$

Hence, combining (2.10) and (2.11), we prove that formula (2.8) holds. □

Similarly, by a similar argument as in the proof of Theorem 2.6 and with the help of formula (5.2) in reference [22], we obtain the more general theorem.

Theorem 2.7. *For integers $r\ge0$, $m,n>1$ and real $p>0$, then*

$$\sum_{1\le k_m<\cdots<k_1\le n}\frac{1}{(k_1+r)^p\cdots(k_m+r)^p}=\sum_{\substack{i+j=m\\i,j\ge0}}(-1)^i\zeta_r^\star\left(\{p\}_i\right)\zeta_{n+r}\left(\{p\}_j\right),\tag{2.12}$$

$$\sum_{1\le k_m\le\cdots\le k_1\le n}\frac{1}{(k_1+r)^p\cdots(k_m+r)^p}=\sum_{\substack{i+j=m\\i,j\ge0}}(-1)^i\zeta_r\left(\{p\}_i\right)\zeta_{n+r}^\star\left(\{p\}_j\right).\tag{2.13}$$

Remark 2.1. In fact, in the same way as above, the results of Theorems 2.6 and 2.7 can be extended to the following generalized conclusion.

$$A_m(n;r):=\sum_{1\le k_m<\cdots<k_1\le n}a_{k_1+r}\cdots a_{k_m+r}=\sum_{\substack{i+j=m\\i,j\ge0}}(-1)^iA_i(r)\,\bar A_j(n+r),\tag{2.14}$$

$$\bar A_m(n;r):=\sum_{1\le k_m\le\cdots\le k_1\le n}a_{k_1+r}\cdots a_{k_m+r}=\sum_{\substack{i+j=m\\i,j\ge0}}(-1)^i\bar A_i(r)\,A_j(n+r),\tag{2.15}$$

where $r\in\mathbb{N}$, $A_m(n)$ and $\bar A_m(n)$ are defined in Lemma 2.5.

Proof. Next, we only prove the formula (2.14), since the proof of (2.15) is similar as the proof of (2.14). To prove the first identity, we proceed by induction on m. For $m = 0$, it is valid. Assume that the result is valid up to p ($p \in \mathbb{N}_0$). From the definition of $A_m(n; r)$, we have

$$A_{p+1}(n;r) = \sum_{k=1}^{n} a_{k+r} A_p(k;r).$$

Then, by the induction hypothesis, an elementary calculation gives

$$\begin{aligned} A_{p+1}(n;r) &= \sum_{\substack{i+j=p,\\ i,j\geq 0}} (-1)^i \bar{A}_i(r) \sum_{k=1}^{n} a_{k+r} A_j(k+r) \\ &= \sum_{\substack{i+j=p,\\ i,j\geq 0}} (-1)^i \bar{A}_i(r)(A_{j+1}(n+r) - A_{j+1}(r)) \\ &= \sum_{\substack{i+j=p+1,\\ i,j\geq 0}} (-1)^i \bar{A}_i(r) A_j(n+r) - \sum_{\substack{i+j=p+1,\\ i,j\geq 0}} (-1)^i \bar{A}_i(r) A_j(r). \end{aligned} \tag{2.16}$$

Hence, with the help of Lemma 2.5, we may deduce the desired result (2.14). Proceeding in the same fashion as in the proof of (2.14), the formula (2.15) can also be obtained. □

It is clear that Theorems 2.6 and 2.7 are immediate corollaries of Remark 2.1.

Theorem 2.8. *For integers $r \geq 0$ and $m, n \geq 1$, then the following identity holds:*

$$h_n^{(r+1)}(m) = \binom{n+r}{r} \sum_{\substack{i+j=m\\ i,j\geq 0}} (-1)^i \zeta_r^{\star}(\{1\}_i)\, \zeta_{n+r}\left(\{1\}_j\right). \tag{2.14}$$

Proof. Take $x_j = \frac{1}{j+r}$ in Lemma 2.3; then we have:

$$X_n(m) = \sum_{j=1}^{n} \left(\frac{1}{j+r}\right)^m = H_{n+r}^{(m)} - H_r^{(m)}, \tag{2.17}$$

$$\zeta_n(\{1\}_m | r+1) = \frac{(-1)^{m-1}}{m} \sum_{i=0}^{m-1} (-1)^i \zeta_n(\{1\}_i | r+1) \left(H_{n+r}^{(m-i)} - H_r^{(m-i)}\right). \tag{2.18}$$

From Lemma 2.1 and formula (2.18), we deduce that

$$h_n^{(r+1)}(m) = \binom{n+r}{r} \zeta_n(\{1\}_m | r+1). \tag{2.19}$$

Substituting (2.8) into (2.19), we may easily obtain the desired result. This completes the proof of Theorem 2.7. □

3. Proof of Theorem 1.1

By replacing x by n and n by r in (1.9), we deduce that

$$\binom{n+r}{r} = \frac{1}{r!} \sum_{l=1}^{r+1} \begin{bmatrix} r+1 \\ l \end{bmatrix} n^{l-1}. \tag{3.1}$$

Therefore, from (2.12) and (3.1), we obtain

$$h_n^{(r+1)}(k) = \frac{1}{r!} \sum_{l=1}^{r+1} \begin{bmatrix} r+1 \\ l \end{bmatrix} n^{l-1} \sum_{\substack{i+j=k\\ i,j\geq 0}} (-1)^i \zeta_r^{\star}(\{1\}_i)\, \zeta_{n+r}\left(\{1\}_j\right). \tag{3.2}$$

Thus, by the definition of $S(k, m; p)$ and (3.2), we can prove (1.7). □

4. Proof of Theorem 1.2

By the definition of multiple harmonic number (1.1), we can find that

$$\begin{aligned}U_{m,r}(p) &= \sum_{n=1}^{\infty}\frac{\zeta_{n+r}(\{1\}_m)}{n^p} = \sum_{n=1}^{\infty}\frac{1}{n^p}\sum_{k=1}^{n+r}\frac{\zeta_{k-1}\left(\{1\}_{m-1}\right)}{k}\\
&= \sum_{n=1}^{\infty}\frac{\zeta_n(\{1\}_m)}{n^p} + \sum_{k=1}^{r}\sum_{n=1}^{\infty}\frac{\zeta_{n+k-1}\left(\{1\}_{m-1}\right)}{n^p(n+k)}\\
&= \zeta(p,\{1\}_m) + \zeta\left(p+1,\{1\}_{m-1}\right) + \sum_{i_1=1}^{r}\sum_{n=1}^{\infty}\frac{\zeta_n\left(\{1\}_{m-1}\right)}{n^p(n+i_1)} + \sum_{i_1=1}^{r}\sum_{i_2=1}^{i_1-1}\sum_{n=1}^{\infty}\frac{\zeta_{n+i_2-1}\left(\{1\}_{m-2}\right)}{n^p(n+i_1)(n+i_2)}\\
&= \cdots\\
&= \zeta(p,\{1\}_m) + \zeta\left(p+1,\{1\}_{m-1}\right) + \sum_{j=1}^{m}\sum_{1\le i_j<\cdots<i_1\le r}\sum_{n=1}^{\infty}\frac{\zeta_n\left(\{1\}_{m-j}\right)}{n^p(n+i_1)\cdots\left(n+i_j\right)}.\end{aligned} \tag{4.1}$$

On the other hand, consider the expansion

$$\frac{1}{\prod\limits_{i=1}^{k}(n+a_i)} = \sum_{j=1}^{k}\frac{A_j}{n+a_j}\quad (k\in\mathbb{N}_0,\ a_i\in\mathbb{C}\setminus\mathbb{Z}^-) \tag{4.2}$$

where

$$A_j = \lim_{n\to -a_j}\frac{n+a_j}{\prod\limits_{i=1}^{k}(n+a_i)} = \prod_{i=1,i\ne j}^{k}\left(a_i-a_j\right)^{-1}. \tag{4.3}$$

Then, the equation (4.1) can be written as

$$\begin{aligned}U_{m,r}(p) =&\zeta(p,\{1\}_m) + \zeta\left(p+1,\{1\}_{m-1}\right)\\
&+\sum_{j=1}^{m}\left(\sum_{l=1}^{j}\prod_{a=1,a\ne l}^{j}(i_a-i_l)^{-1}\right)\sum_{1\le i_j<\cdots<i_1\le r}\sum_{n=1}^{\infty}\frac{\zeta_n\left(\{1\}_{m-j}\right)}{n^p(n+i_l)}.\end{aligned} \tag{4.4}$$

For $r>0$, we have the partial fraction decomposition

$$\frac{1}{n^p(n+r)} = \sum_{b=1}^{p-1}\frac{(-1)^{b-1}}{r^b}\cdot\frac{1}{n^{p+1-b}} + \frac{(-1)^{p-1}}{r^{p-1}}\cdot\frac{1}{n(n+r)}. \tag{4.5}$$

Moreover, from identities (2.1), (2.4) and (2.6), we deduce the following result

$$\sum_{n=1}^{\infty}\frac{\zeta_n\left(\{1\}_{p-1}\right)}{n(n+k)} = \frac{1}{k}\left\{\zeta(p) + \zeta_k^\star\left(\{1\}_p\right) - \frac{\zeta_k^\star\left(\{1\}_{p-1}\right)}{k}\right\}\quad (k\in\mathbb{N},\ p\in\mathbb{N}\setminus\{1\}). \tag{4.6}$$

Hence, combining (4.4), (4.5) and (4.6), by a simple calculation, we obtain the desired result. This completes the proof of Theorem 1.2. □

Similarly, applying the same arguments as in the proof of formula (4.1), we also deduce a similar result

$$\begin{aligned}V_{m,r}(p) :&= \sum_{n=1}^{\infty}\frac{\zeta_{n+r}^\star(\{1\}_m)}{n^p}\\
&= \zeta^\star(p,\{1\}_m) + \sum_{j=1}^{m}\sum_{1\le i_j\le\cdots\le i_1\le r}\sum_{n=1}^{\infty}\frac{\zeta_n^\star\left(\{1\}_{m-j}\right)}{n^p(n+i_1)\cdots\left(n+i_j\right)}.\end{aligned} \tag{4.7}$$

In fact, we hope to obtain a similar result of Theorem 1.2, but, so far, we have been unable to achieve any progress with these sums.

5. Conclusion

From [16,19], we know that the Aomoto–Drinfel'd–Zagier formula reads

$$\sum_{n,m=1}^{\infty} \zeta\left(m+1,\{1\}_{n-1}\right) x^m y^n = 1 - \exp\left(\sum_{n=2}^{\infty} \zeta(n) \frac{x^n + y^n - (x+y)^n}{n}\right), \tag{5.1}$$

which implies that, for any $m,\ n \in \mathbb{N}$, the multiple zeta value $\zeta\left(m+1,\{1\}_{n-1}\right)$ can be represented as a polynomial of zeta values with rational coefficients, and we have the duality formula

$$\zeta\left(n+1,\{1\}_{m-1}\right) = \zeta\left(m+1,\{1\}_{n-1}\right).$$

In particular, one can find explicit formulas for small weights:

$$\zeta(2,\{1\}_m) = \zeta(m+2),$$

$$\zeta(3,\{1\}_m) = \frac{m+2}{2}\zeta(m+3) - \frac{1}{2}\sum_{k=1}^{m} \zeta(k+1)\,\zeta(m+2-k).$$

Hence, from formulas (1.11) and (5.1), we see that the sums $U_{m,r}(p)$ can be expressed in terms of series of Riemann zeta function and harmonic numbers. Thus, we show that the Euler-type sums with hyperharmonic numbers $S(k,r+1;p)$ can be expressed in terms of zeta values and Stirling numbers of the first kind, for integers $k \in \mathbb{N}$ and $r \in \mathbb{N}_0$ with $r+2 \le p \in \mathbb{N}$.

To conclude, note that it would be useful to be able to extend the approach described above to include other similar and related sums. In particular, it would be very interesting to consider sums of the form

$$S^\star(k,m;p) := \sum_{n=1}^{\infty} \frac{H_n^{(m)}(k)}{n^p} \quad (p \ge m+1),$$

where $H_n^{(m)}(k)$ is called the generalized hyperharmonic star number, defined by

$$H_n^{(m)}(k) := \sum_{\substack{1 \le n_{m+k-1} \le \cdots \le n_m \\ \le n_{m-1} \le \cdots \le n_1 \le n}} \frac{1}{n_m n_{m+1} \cdots n_{m+k-1}}.$$

A straightforward calculation gives the generating function of hyperharmonic star number $H_n^{(m)}(k)$:

$$\sum_{n=1}^{\infty} H_n^{(m)}(k) z^n = \frac{(-1)^{k-1}}{(k-1)!} \frac{1}{(1-z)^m} \int_0^1 \frac{\log^{k-1}(1-t)}{1-zt} z\,dt \quad (z \in (-1,1)).$$

Obviously, by the definitions of $H_n^{(m)}(k)$ and $h_n^{(m)}(k)$, we have

$$H_n^{(m)}(1) = h_n^{(m)}(1) = \zeta_n^\star\left(\{0\}_{m-1},1\right).$$

Hence $S(1,m;p) = S^\star(1,m;p)$. However, we have been unable, so far, to make any progress with these sums. Unfortunately, it appears that, even in the case $k=2$, the method used in this work gives rise to several complex and intractable summations.

Acknowledgements

The authors would like to thank the anonymous referee for his/her helpful comments, which improved the presentation of the paper.

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