Mathematical analysis

On properties and applications of \((p,q)\)-extended \(\tau\)-hypergeometric functions

\[\text{Sur les propriétés et applications des fonctions } \tau\text{-hypergéo-}\]
\[\text{métriques } (p,q)\text{-étendues}\]

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**A B S T R A C T**

We introduce the \((p,q)\)-extended \(\tau\)-hypergeometric and confluent hypergeometric functions along with their integral representations. We also present closed integral expressions for the Mathieu-type \(a\)-series and for the associated alternating versions whose terms contain the \((p,q)\)-extended \(\tau\)-hypergeometric functions with related contiguous functional relations.

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**R É S U M É**

Nous introduisons les fonctions \(\tau\)-hypergéométriques et hypergéo-métriques confluentes \((p,q)\)-étendues, avec leurs représentations intégrales. Nous présentons également des formules intégrales closes pour les \(a\)-séries de type Mathieu et les versions alternées associées, dont les termes contiennent les fonctions \(\tau\)-hypergéométriques \((p,q)\)-étendues, avec les relations fonctionnelles de contiguïté.

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1. Introduction and motivation

In the recent years, a series of papers have been published by many authors, including Pogány either alone and/or with his co-workers Srivastava and Tomovski [12–17], in which special general Mathieu-type series and their alternating variants have been considered, whose terms contain various special functions, for example, the Gauss hypergeometric function \(\text{_{2}}F_{1}\), the generalized hypergeometric function \(\text{_{p}F_{q}}\), Meijer \(G\)-functions, and so on. The derived results concern, among others,
closed integral form expressions for the considered series and bilateral bounding inequalities. Recently, extensions, generalizations and unifications of various special functions of \((p, q)\)-variant, and in turn, when \(p = q\) the \(p\)-variant, have been studied widely together with the set of related higher transcendental hypergeometric-type special functions by several authors; consult, for instance, \([2–6,9,10]\). In particular, Choi et al. \([5]\) introduced and studied the \((p, q)\)-extended Beta, the \((p, q)\)-extended hypergeometric, and the \((p, q)\)-extended confluent hypergeometric functions in the following manner:

\[
B(x, y; p, q) = \int_{0}^{1} t^{x-1}(1 - t)^{y-1} e^{-\frac{z}{\Phi_1} + \frac{\tau}{\Phi_1}} dt, \tag{1.1}
\]

when \(\min[\Re(x), \Re(y)] > 0\); \(\min[\Re(p), \Re(q)] \geq 0\), and by means of \((1.1)\),

\[
F_{p,q}(a, b; c; z) = \sum_{n \geq 0} (a)_{n} \frac{B(b + n, c - b; p, q) z^n}{B(b, c - b)} \frac{\tau^n}{n!} \quad |z| < 1; \quad \Re(c) > \Re(b) > 0, \tag{1.2}
\]

and

\[
\Phi_{p,q}(b; c; z) = \sum_{n \geq 0} \frac{B(b + n, c - b; p, q) z^n}{B(b, c - b)} \frac{\tau^n}{n!} \quad \Re(c) > \Re(b) > 0. \tag{1.3}
\]

Here we remark that the definition \((1.1)\) is a special case of the definition in \([18, \text{Eq. (6.1)}]\). Related properties, various integral representations, differentiation formulæ, Mellin transform, recurrence relations, summations are also given in \([5]\). On the other hand, \(\tau\)-extension of hypergeometric and confluent hypergeometric functions have been introduced by Virchenko \([19,20]\) \(\text{[also see} \ [7]\) and studied recently by Parmar \([11]\).

Inspired by certain recent extensions of the various special functions of \((p, q)\)-variants, we introduce \((p, q)\)-extended \(\tau\)-hypergeometric and confluent hypergeometric functions along with their integral representations. We also present contiguous functional relations by closed-form integral expressions for the Mathieu-type \(a\)-series and for the associated alternating versions whose terms contain the \((p, q)\)-extended \(\tau\)-hypergeometric function.

2. \((p, q)\)-extended \(\tau\)-hypergeometric functions

In this section, we introduce and investigate the \((p, q)\)-extended \(\tau\)-hypergeometric function and \((p, q)\)-extended \(\tau\)-confluent hypergeometric function by means of the \((p, q)\)-extended beta function as follows:

\[
R_{p,q}^{\tau}(a, b; c; z) = \sum_{n \geq 0} (a)_{n} \frac{B(b + n, c - b; p, q) z^n}{B(b, c - b)} \frac{\tau^n}{n!}, \tag{2.1}
\]

where \(\min[\Re(p), \Re(q)] > 0\), \(\tau \geq 0\); \(0 < |z| < 1\) while \(\Re(c) > \Re(b) > 0\) when \(p = 0 = q\), and

\[
\Phi_{p,q}^{\tau}(b; c; z) = \sum_{n \geq 0} \frac{B(b + n, c - b; p, q) z^n}{B(b, c - b)} \frac{\tau^n}{n!}. \tag{2.2}
\]

with the parameter range and domain being \(\min[\Re(p), \Re(q)] > 0\), \(\tau \geq 0\), and if \(p = 0 = q\) it is \(\Re(c) > \Re(b) > 0\), respectively. The case \(p = 0 = q\) reduces for series to Virchenko’s \(\tau\)-hypergeometric function \([20]\) and \(\tau\)-confluent hypergeometric function \([19]\):

\[
2R_{p,q}^{\tau}(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n \geq 0} (a)_{n} \frac{\Gamma(b + n + \tau)}{\Gamma(c + \tau n)} \frac{z^n}{n!}.
\]

Here \(\tau > 0, \Re(a) > 0\), \(\Re(c) > \Re(b) > 0\); \(0 < |z| < 1\) and

\[
1\Phi_{p,q}^{\tau}(b; c; z) = \frac{\Gamma(c)}{\Gamma(b)} \sum_{n \geq 0} \frac{\Gamma(b + n + \tau)}{\Gamma(c + \tau n)} \frac{z^n}{n!}, \quad \tau > 0, \Re(c) > \Re(b) > 0,
\]

respectively. Further, definitions \((2.1)\) and \((2.2)\) reduce to \((1.2)\) and \((1.3)\), when specifying \(\tau = 1\).

We begin the exposition of our main results by presenting a set of Laplace integral representations for \((p, q)\)-extended \(\tau\)-hypergeometric function.

**Theorem 1.** For all \(\min[\Re(p), \Re(q)] > 0\), \(\tau \geq 0\); \(\Re(z) < 1\) or \(\Re(a) > 0\) when \(p = 0 = q\) the following Laplace-type integral representation holds true:

\[
R_{p,q}^{\tau}(a, b; c; z) = \frac{1}{\Gamma(a)} \int_{0}^{\infty} e^{-zt} t^{a-1} \Phi_{p,q}^{\tau}(b; c; zt) dt.
\]
Proof. Using the definition of the Pochhammer symbol \((a)_n\) in (2.1), by considering the Gamma function integral

\[
\Gamma(\eta) \xi^{-\eta} = \int_{0}^{\infty} e^{-\xi t} t^{\eta-1} dt, \quad \min\{\Re(\xi), \Re(\eta)\} > 0
\]

(2.3)

and (2.2), we are led to the desired result. □

**Theorem 2.** For all \(\min\{\Re(p), \Re(q)\} > 0, \tau \geq 0; |\arg(1 - z)| < \pi; \) and \(\Re(c) > \Re(b) > 0\) when \(p = 0 = q\), we have the following Euler-type integral representation:

\[
R^\tau_{p,q}(a, b; c; z) = \frac{1}{B(b, c - b)} \int_{0}^{1} t^{b-1} (1 - t)^{c-b-1} (1 - zt^\tau)^{-a} e^{-\frac{t}{t^\tau} - \frac{a}{t^\tau}} dt.
\]

Proof. By the definition (1.1) of the \((p, q)\)-extended Beta applied in (2.1) and interchanging summation and integration, we conclude

\[
R^\tau_{p,q}(a, b; c; z) = \frac{1}{B(b, c - b)} \int_{0}^{1} t^{b-1} (1 - t)^{c-b-1} e^{-\frac{t}{t^\tau} - \frac{a}{t^\tau}} \sum_{n \geq 0} (a)_n \frac{(zt^\tau)^n}{n!} dt.
\]

Employing the generalized binomial expansion

\[
(1 - zt^\tau)^{-a} = \sum_{n \geq 0} (a)_n \frac{(zt^\tau)^n}{n!},
\]

which obviously converges for all \(|z| < 1, \tau \geq 0\), we finish the proof. □

A similar consideration gives the integral form of \(\Phi^\tau_{p,q}\), viz.

**Theorem 3.** For all \(p, q \in \mathbb{C} \setminus \{0\}, \min\{\Re(p), \Re(q)\} > 0, \tau \geq 0,\) and all \(b, c \in \mathbb{C}, \Re(c) > \Re(b) > 0\) when \(p = 0 = q\) we have the integral expression

\[
\Phi^\tau_{p,q}(b; c; z) = \frac{1}{B(b, c - b)} \int_{0}^{1} t^{b-1} (1 - t)^{c-b-1} e^{zt^\tau - \frac{a}{t^\tau}} dt.
\]

3. On Mathieu-type series built by \(R^\tau_{p,q}\)

Extending the Mathieu-type series studied in [12] by imposing the \(R^\tau_{p,q}(a, b; c; z)\) input–kernel in the summands, we define the Mathieu-type \(a\)-series \(R_{\lambda, \eta}(R^\tau_{p,q}; a; r)\) and its alternating variant \(\tilde{R}_{\lambda, \eta}(R^\tau_{p,q}; a; r)\) in the form of series

\[
R_{\lambda, \eta}(R^\tau_{p,q}; a; r) := \sum_{n \geq 1} R^\tau_{p,q}(\lambda, b; c; -\frac{r^2}{a^n}) \frac{a^n}{a^n(a + r^2)^\eta},
\]

and

\[
\tilde{R}_{\lambda, \eta}(R^\tau_{p,q}; a; r) := \sum_{n \geq 1} (-1)^{n-1} R^\tau_{p,q}(\lambda, b; c; -\frac{r^2}{a^n}) \frac{a^n}{a^n(a + r^2)^\eta},
\]

being in both series the parameters’ range \(r \geq 0; \lambda, \eta, r > 0\). Now we establish a contiguous integral form expressions for the series \(R_{\lambda, \eta}(R^\tau_{p,q}; a; r)\) and \(\tilde{R}_{\lambda, \eta}(R^\tau_{p,q}; a; r)\) with respect to parameters \(\lambda, \eta\). We note that the function \(z \mapsto R^\tau_{p,q}\) (the Laplace transform treated in Theorem 1) is homogeneous of degree \(-a\), that is,

\[
R^\tau_{p,q}(a, b; c; \omega z) = \omega^{-a} R^\tau_{p,q}(a, b; c; z), \quad \omega \in \mathbb{R}.
\]

(3.1)
Theorem 4. Let \( \lambda > 0, \eta > 0, r > 0 \) and suppose the real sequence \( a = (a_n)_{n \geq 1} \) is monotone increasing and tends to \( \infty \). Then for \( \tau \geq 0 \), and \( \min\{\Re(p), \Re(q)\} \geq 0 \), we have

\[
\Re_{\lambda, \eta}(R_{p,q}^\tau a; r) = \lambda \cdot \mathcal{F}_{p,q}^\tau(\lambda + 1, \eta) + \eta \cdot \mathcal{F}_{p,q}^\tau(\lambda, \eta + 1) \\
\tilde{\Re}_{\lambda, \eta}(R_{p,q}^\tau a; r) = \lambda \cdot \tilde{\mathcal{F}}_{p,q}^\tau(\lambda + 1, \eta) + \eta \cdot \tilde{\mathcal{F}}_{p,q}^\tau(\lambda, \eta + 1),
\]

where

\[
\mathcal{F}_{p,q}^\tau(\lambda, \eta) = \int_{a_1}^\infty \frac{R_{p,q}^\tau(\lambda, b; c; \ -\frac{r^2}{x})[a^{-1}(x)]}{x^\lambda(x + r^2)^\eta} \, dx
\]

\[
\tilde{\mathcal{F}}_{p,q}^\tau(\lambda, \eta) = \int_{a_1}^\infty \frac{R_{p,q}^\tau(\lambda, b; c; \ -\frac{r^2}{x}) \sin^2 \left( \frac{\pi}{2}[a^{-1}(x)] \right)}{x^\lambda(x + r^2)^\eta} \, dx
\]

and \( a : \mathbb{R}^+ \mapsto \mathbb{R}^+ \) is an increasing function such that \( a(x)_{x \in \mathbb{N}} = a, a^{-1}(x) \) denotes the inverse of \( a(x) \) and \([a^{-1}(x)]\) stands for the integer part of the quantity \( a^{-1}(x)\).

Proof. Taking \( \xi = a_0 + r^2 \) in the familiar gamma formula (2.3), after rearrangement by specifying \( \omega = -r^2, z = a_0 \), in (3.1), the function \( \Re_{\lambda, \eta}(R_{p,q}^\tau a; r) \) becomes

\[
\Re_{\lambda, \eta}(R_{p,q}^\tau a; r) = \int_0^\infty e^{-r_t^2 \pi^{-1} s^{-1}} \Gamma^2(\lambda) \Gamma(\eta) \sum_{n \geq 1} e^{-a_0(t + s)} \Phi_{p,q}^\tau(b; c; \ -r_t^2 s) \, dt \, ds.
\]

Using the Cahen formula for Dirichlet series [1, p. 97], [8, p. 11, Theorem 11],

\[
\sum_{n \geq 1} \mu_n e^{-a_n x} = x \int_0^\infty e^{-xt} \sum_{n \colon a_n \leq t} \mu_n \, dt, \quad \Re(x) > 0,
\]

according to the technique developed in [16], we obtain

\[
D_a(u) = \sum_{n \geq 1} e^{-a_n u} = u \int_{a_1}^\infty e^{-ux}[a^{-1}(x)] \, dx,
\]

which results, for \( u = t + s \), in

\[
\Re_{\lambda, \eta}(R_{p,q}^\tau a; r) = \frac{1}{\Gamma(\lambda)\Gamma(\eta)} \int_0^\infty \int_0^\infty e^{-\theta(x + r)^2 t} s^{-1} \Phi_{p,q}^\tau(b; c; \ -r_t^2 s) \, dt \, ds
\]

\[
= \lambda \int_{a_1}^\infty \frac{[a^{-1}(x)]}{x^{\lambda + 1} + (x + r^2)^\eta} R_{p,q}^\tau(\lambda + 1, b; c; \ -\frac{r^2}{x}) \, dx
\]

\[
= \lambda \int_{a_1}^\infty \frac{[a^{-1}(x)]}{x^{\lambda + 1} + (x + r^2)^\eta} R_{p,q}^\tau(\lambda + 1, b; c; \ -\frac{r^2}{x}) \, dx
\]

and in similar way follows \( D_a = \eta \cdot \mathcal{F}_{p,q}^\tau(\lambda, \eta + 1) \). These give (3.2).

The Cahen integral form of the alternating Dirichlet series \( D_a(x) \) reads [16]:

\[
\tilde{D}_a(v) = \sum_{n \geq 1} (-1)^{n-1} e^{-a_n v} = v \int_{a_1}^\infty e^{-vx} \sin^2 \left( \frac{\pi}{2}[a^{-1}(x)] \right) \, dx.
\]

The rest is obvious by following the lines of establishing (3.2).
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