Number theory

On $AP_3$-covering sequences

Sur les suites d'entiers $AP_3$

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A R T I C L E   I N F O

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A B S T R A C T

Recently, motivated by Stanley’s sequences, Kiss, Sándor, and Yang introduced a new type sequence: a sequence $A$ of nonnegative integers is called an $AP_k$-covering sequence if there exists an integer $n_0$ such that, if $n > n_0$, then there exist $a_1, \ldots, a_{k-1} \in A$, $a_1 < a_2 < \cdots < a_{k-1} < n$ such that $a_1, \ldots, a_{k-1}, n$ form a $k$-term arithmetic progression. They prove that there exists an $AP_3$-covering sequence $A$ such that $\limsup_{n \to \infty} A(n)/\sqrt{n} \leq 34$. In this note, we prove that there exists an $AP_3$-covering sequence $A$ such that $\limsup_{n \to \infty} A(n)/\sqrt{n} = \sqrt{15}$. © 2018 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

R É S U M É

Motivés par la définition des suites de Stanley, Kiss, Sándor et Yang ont récemment introduit un nouveau type de suites: une suite d’entiers positifs ou nuls $A$ est dite $AP_k$ s’il existe un entier $n_0$ tel que, pour tout $n > n_0$, il existe $a_1, \ldots, a_{k-1} \in A$, $a_1 < a_2 < \cdots < a_{k-1} < n$ tels que $a_1, \ldots, a_{k-1}, n$ soient une progression arithmétique à $k$ termes. Ils démontrent qu’il existe une suite d’entiers $A$ qui est $AP_3$ et satisfait $\limsup_{n \to \infty} A(n)/\sqrt{n} \leq 34$. Nous montrons ici qu’il en existe une satisfaisant $\limsup_{n \to \infty} A(n)/\sqrt{n} = \sqrt{15}$. © 2018 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

Given an integer $k \geq 3$ and a set $A_0 = \{a_1, \ldots, a_t\} (a_1 < \cdots < a_t)$ of nonnegative integers such that $\{a_1, \ldots, a_t\}$ does not contain a $k$-term arithmetic progression. Define $a_{t+1}, \ldots$ by the greedy algorithm: for any $l \geq t$, $a_{l+1}$ is the smallest integer $a > a_l$ such that $\{a_1, \ldots, a_l, a\}$ does not contain a $k$-term arithmetic progression. The sequence $A = \{a_1, a_2, \ldots\}$ is called the Stanley sequence of order $k$ generated by $A_0$. It is known that if $A$ is a Stanley sequence of order 3, then

\[
\lim\inf_{n \to \infty} \frac{A(n)}{\sqrt{n}} \geq \sqrt{2}
\]

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(see [3] and [5]) and
\[ \limsup_{n \to \infty} \frac{A(n)}{\sqrt{n}} \geq 1.77 \]
(see [1]). For related results, one may refer to [2] and [6]. Recently, Kiss, Sándor and Yang [4] introduced the following notation: a sequence \( A \) of nonnegative integers is called an \( AP_k \)-covering sequence if there exists an integer \( n_0 \) such that, if \( n > n_0 \), then there exist \( a_1, \ldots, a_{k-1} \in A \), \( a_1 < a_2 < \cdots < a_{k-1} < n \) such that \( a_1, \ldots, a_{k-1}, n \) form a \( k \)-term arithmetic progression. They [4] observed that
\[ \liminf_{n \to \infty} \frac{A(n)}{\sqrt{n}} \geq \sqrt{2}, \quad \limsup_{n \to \infty} \frac{A(n)}{\sqrt{n}} \geq 1.77 \]
hold for any \( AP_3 \)-covering sequence \( A \) and proved that there exists an \( AP_3 \)-covering sequence \( A \) such that
\[ \limsup_{n \to \infty} \frac{A(n)}{\sqrt{n}} \leq 34. \]

In this note, the following result is proved.

**Theorem 1.1.** There exists an \( AP_3 \)-covering sequence \( A \) such that
\[ \limsup_{n \to \infty} \frac{A(n)}{\sqrt{n}} = \sqrt{15}. \tag{1.1} \]

If \( A \) is a Stanley sequence of order \( k \), then \( A \) does not contain a \( k \)-term arithmetic progression. If \( A \) is an \( AP_k \)-covering sequence of order \( k \), then \( A \) contains infinitely many \( k \)-term arithmetic progressions. So none of the sequences is both a Stanley sequence of order \( k \) and an \( AP_k \)-covering sequence. We pose a problem here.

**Problem 1.2.** Is there a Stanley sequence of order \( k+1 \) that is also an \( AP_k \)-covering sequence?

We introduce a new notation here that generalizes both Stanley sequences of order \( k \) and \( AP_k \)-covering sequences. A sequence \( A \) of nonnegative integers is called a weak \( AP_k \)-covering sequence if there exists an integer \( n_0 \) such that, if \( n > n_0 \) and \( n \not\in A \), then there exist \( a_1, \ldots, a_{k-1} \in A \), \( a_1 < a_2 < \cdots < a_{k-1} < n \) such that \( a_1, \ldots, a_{k-1}, n \) form a \( k \)-term arithmetic progression. Clearly, a Stanley sequence of order \( k \) is also a weak \( AP_k \)-covering sequence and an \( AP_k \)-covering sequence of order \( k \) is also a weak \( AP_k \)-covering sequence.

### 2. Proof of Theorem 1.1

Let
\[ T_l = \left\{ u4^l + \sum_{i=0}^{l-1} v_i4^i : u \in \{1, 2, 3, 4\}, v_i \in \{1, 2\} \right\}, \quad l = 0, 1, \ldots \]
and
\[ A = \bigcup_{l=0}^{\infty} T_l. \]

First, we prove that \( A \) is an \( AP_3 \)-covering sequence.

Let \( n \geq 32 \). We will prove that there exist \( a, b \in A \) with \( a < b < n \) such that \( a, b, n \) form a 3-term arithmetic progression. By \( n \geq 32 \), there exists an integer \( l \geq 2 \) such that \( 2 \cdot 4^l \leq n < 2 \cdot 4^{l+1} = 8 \cdot 4^l \). Let \( m \) be the integer with \( m4^l \leq n < (m+1)4^l \).

Then \( 2 \leq m \leq 7 \) and
\[ 0 \leq n - m4^l < 4^l. \]
Thus \( n - m4^l \) can be written as
\[ n - m4^l = \sum_{i=0}^{l-1} m_i4^i, \quad m_i \in \{0, 1, 2, 3\}. \]
If \( m_i = 0 \), then we take \( v_{1,i} = 1 \) and \( v_{2,i} = 2 \). If \( m_i \in \{1, 2\} \), then we take \( v_{1,i} = v_{2,i} = m_i \). If \( m_i = 3 \), then we take \( v_{1,i} = 2 \) and \( v_{2,i} = 1 \). If \( m = 2 \), then we take \( u_1 = 1 \) and \( u_2 = 0 \). If \( m = 3 \), then we take \( u_1 = 2 \) and \( u_2 = 1 \). If \( m = 4 \), then we take
\[ u_1 = 2 \text{ and } u_2 = 0. \text{ If } m = 5, \text{ then we take } u_1 = 3 \text{ and } u_2 = 1. \text{ If } m = 6, \text{ then we take } u_1 = 3 \text{ and } u_2 = 0. \text{ If } m = 7, \text{ then we take } u_1 = 4 \text{ and } u_2 = 1. \text{ Let } \]

\[ a = u_2 4^i + \sum_{i=0}^{l-1} v_{2,i} 4^i, \quad b = u_1 4^i + \sum_{i=0}^{l-1} v_{1,i} 4^i. \]

It is clear that \( 1 \leq a < b, a, b \in T_l \cup T_{l-1} \subseteq A \) and \( a, b, n \) form a 3-term arithmetic progression. Hence \( A \) is an \( AP_3 \)-covering sequence.

Now we prove that (1.1) holds. Let

\[ A = \{n_1, n_2, \ldots\}, \quad n_1 < n_2 < \cdots. \]

For \( n_j < m < n_{j+1} \), we have 

\[ \frac{A(m)}{\sqrt{m}} = \frac{A(n_j)}{\sqrt{m}} < \frac{A(n_j)}{\sqrt{n_j}}. \]

It follows that 

\[ \limsup_{n \to \infty} \frac{A(n)}{\sqrt{n}} = \limsup_{j \to \infty} \frac{A(n_j)}{\sqrt{n_j}}. \]

Let 

\[ n_j = u 4^i + \sum_{i=0}^{l-1} v_i 4^i, \quad u \in \{1, 2, 3, 4\}, v_i \in \{1, 2\} \quad (0 \leq i \leq l-1). \]

Then 

\[ A(n_j) = (u - 1)2^l + \sum_{i=1}^{l-1} (v_i - 1)2^i + v_0 + 4(2^{l-1} + \cdots + 2 + 1). \quad (2.1) \]

It is clear that 

\[ n_j \geq u 4^i + v_{l-1} 4^{l-1} + \frac{1}{3}(4^{l-1} - 1) = (4u + v_{l-1} + \frac{1}{3})4^{l-1} - \frac{1}{3}, \]

\[ A(n_j) \leq (u - 1)2^l + (v_{l-1} - 1)2^{l-1} + 2^{l-1} + 4(2^l - 1) = (2u + 6 + v_{l-1})2^{l-1} - 4. \]

Since 

\[ 2u + 6 + v_{l-1} < 4\sqrt{4u + v_{l-1} + \frac{1}{3}} \]

for \( u \in \{1, 2, 3, 4\} \) and \( v_{l-1} \in \{1, 2\} \), it follows that 

\[ A(n_j) \leq (2u + 6 + v_{l-1})2^{l-1} - 4 < 4\sqrt{n_j + \frac{1}{3}} - 4 < 4\sqrt{n_j}. \]

If \( v_i = 1 \) for some \( 0 \leq i \leq l-1 \), then \( n_j + 4^i \in A \) and by (2.1), we have 

\[ A(n_j + 4^i) = A(n_j) + 2^i. \]

Since \( n_j > 4^i \geq 4^{i+1} \), it follows that 

\[ \sqrt{n_j + 4^i} + \sqrt{n_j} > 4 \cdot 2^i. \]

That is, \( 2^i > 4\sqrt{n_j + 4^i} \). By \( A(n_j) < 4\sqrt{n_j} \), we have 

\[ A(n_j)(\sqrt{n_j + 4^i} - \sqrt{n_j}) < 4\sqrt{n_j}(\sqrt{n_j + 4^i} - \sqrt{n_j}) < 2^i \sqrt{n_j}. \]

So 

\[ (A(n_j) + 2^i)\sqrt{n_j} > A(n_j)\sqrt{n_j + 4^i}. \]

Hence 

\[ \frac{A(n_j + 4^i)}{\sqrt{n_j + 4^i}} = \frac{A(n_j) + 2^i}{\sqrt{n_j + 4^i}} > \frac{A(n_j)}{\sqrt{n_j}}. \]

So we need only consider those \( n_j \) with all \( v_i = 2 \). Let
\[ q_{u,l} = u4^l + \sum_{i=0}^{l-1} 2 \cdot 4^i = (u + \frac{2}{3})4^l - \frac{2}{3}. \]

By (2.1), \( A(q_{u,l}) = (u + 4)2^l - 4. \) It follows that
\[
\lim_{l \to \infty} \frac{A(q_{u,l})}{\sqrt{q_{u,l}}} = \frac{u + 4}{\sqrt{u + 2/3}}.
\]

Hence
\[
\limsup_{n \to \infty} \frac{A(n)}{\sqrt{n}} = \limsup_{j \to \infty} \frac{A(n_j)}{\sqrt{n_j}} = \max \left\{ \frac{u + 4}{\sqrt{u + 2/3}} : u = 1, 2, 3, 4 \right\} = \sqrt{15}.
\]

This completes the proof.

References