Mathematical analysis

Non-uniformly hyperbolic horseshoes in the standard family

Fers à cheval hyperboliques non uniformes dans la famille standard

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\textbf{A R T I C L E I N F O}

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\textbf{A B S T R A C T}

We show that the non-uniformly hyperbolic horseshoes of Palis and Yoccoz occur in the standard family of area-preserving diffeomorphisms of the two-torus.

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\textbf{R É S U M É}

Nous montrons que les fers à cheval hyperboliques non uniformes de Palis et Yoccoz apparaissent dans la famille standard des difféomorphismes du tore de dimension 2 préservant l’aire.

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1. Introduction

In their tour-de-force work about the dynamics of surface diffeomorphisms, Palis and Yoccoz [2] proved that the so-called non-uniformly hyperbolic horseshoes are very frequent in the generic unfolding of a first heteroclinic tangency associated with periodic orbits in a horseshoe with Hausdorff dimension slightly bigger than one.

In the same article, Palis and Yoccoz gave an ad hoc example of a 1-parameter family of diffeomorphisms of the two-sphere fitting the setting of their main results, and thus exhibiting non-uniformly hyperbolic horseshoes: see page 3 (and, in particular, Figure 1) of [2].

In this note, we show that the standard family $f_k: \mathbb{T}^2 \to \mathbb{T}^2$, $k \in \mathbb{R}$,

$$f_k(x, y) := (-y + 2x + k \sin(2\pi x), x)$$

of area-preserving diffeomorphisms of the two-torus $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ displays non-uniformly hyperbolic horseshoes.

More precisely, our main theorem is:

\textbf{Theorem 1.1.} There exists $k_0 > 0$ such that, for all $|k| > k_0$, the subset of parameters $r \in \mathbb{R}$ such that $|r - k| < 4k^{1/3}$ and $f_r$ exhibits a non-uniformly hyperbolic horseshoe (in the sense of Palis–Yoccoz [2]) has positive Lebesgue measure.

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The remainder of this text is divided into three sections: in Section 2, we briefly recall the context of Palis–Yoccoz work [2]; in Section 3, we revisit some elements of Duarte's construction [1] of tangencies associated with certain (uniformly hyperbolic) horseshoes of \( f_k \); finally, we establish Theorem 1.1 in Section 4 by modifying Duarte's constructions (from Section 3) in order to apply the Palis–Yoccoz results (from Section 2).

2. Non-uniformly hyperbolic horseshoes

Suppose that \( F \) is a smooth diffeomorphism of a compact surface \( M \) displaying a first heteroclinic tangency associated with periodic points of a horseshoe \( K \), that is:

- \( p_s, p_u \in K \) belong to distinct periodic orbits of \( F \);
- \( W^s(p_s) \) and \( W^u(p_u) \) have a quadratic tangency at a point \( q \in M \setminus K \);
- for some neighborhoods \( U \) of \( K \) and \( V \) of the orbit \( \mathcal{O}(q) \), the maximal invariant set of \( U \cup V \) is precisely \( K \cup \mathcal{O}(q) \).

Assume that \( K \) is slightly thick in the sense that its stable and unstable dimensions \( d_s \) and \( d_u \) satisfy \( d_s + d_u > 1 \) and

\[
(d_s + d_u)^2 + \max(d_s, d_u)^2 < d_s + d_u + \max(d_s, d_u)
\]

Remark 2.1. Since the stable and unstable dimensions of a horseshoe of an area-preserving diffeomorphism \( F \) always coincide, a slightly thick horseshoe \( K \) of an area-preserving diffeomorphism \( F \) has stable and unstable dimensions:

\[
0.5 < d_s = d_u < 0.6
\]

In this setting, the results proved by Palis and Yoccoz [2] imply the following statement.

Theorem 2.2 (Palis–Yoccoz). Given a 1-parameter family \( (F_t)_{t \in [t_0]} \) with \( F_0 = F \) and generically unfolding the heteroclinic tangency at \( q \), the subset of parameters \( t \in (−t_0, t_0) \) such that \( F_t \) has a non-uniformly hyperbolic horseshoe\(^1\) has positive Lebesgue measure.

3. Horseshoes and tangencies in the standard family

The standard family \( f_k \) generically unfolds tangencies associated with very thick horseshoes \( \Lambda_k \); this phenomenon was studied in details by Duarte [1] during his proof of the almost denseness of elliptic islands of \( f_k \) for large generic parameters \( k \).

In the sequel, we review some facts from Duarte’s article about \( \Lambda_k \) and its tangencies (for later use in the proof of our Theorem 1.1).

For technical reasons, it is convenient to work with the standard family \( f_k \) and their singular perturbations

\[
g_k(x, y) = (−y + 2x + k \sin(2\pi x) + \rho_k(x), x),
\]

where \( \rho_k \) is defined in Section 4 of [1]. Here, it is worth to recall that the key features of \( \rho_k \) are:

- \( \rho_k \) has poles at the critical points \( v_\pm = \pm 1/4 + O(1/k) \) of the function \( 2x + k \sin(2\pi x) \);
- \( \rho_k \) vanishes outside \(|x \pm \frac{1}{4}| \leq \frac{2}{k^{1/4}} \).

In Section 2 of [1], Duarte constructs the stable and unstable foliations \( \mathcal{F}^s \) and \( \mathcal{F}^u \) for \( g_k \). As it turns out, \( \mathcal{F}^s \), resp. \( \mathcal{F}^u \), is an almost vertical, resp. horizontal, foliation in the sense that it is generated by a vector field \( (\alpha^s(x, y), 1) \), resp. \( (1, \alpha^u(x, y)) \), satisfying all properties described in Section 2 of Duarte’s paper [1]. In particular, \( \mathcal{F}^s \), resp. \( \mathcal{F}^u \), describe the local stable, resp. unstable, manifolds for the standard map \( f_k \) at points whose future, resp. past, orbits stay in the region \( \{f_k = g_k\} \), resp. \( \{f_k^{-1} = g_k^{-1}\} \).

In Section 3 of [1], Duarte analyzes the projections \( \pi^s \) and \( \pi^u \) obtained by thinking the foliations \( \mathcal{F}^s \) and \( \mathcal{F}^u \) as fibrations over the singular circles \( C_s = \{(x, v_+) \in \mathbb{T}^2\} \) and \( C_u = \{(v_+, y) \in \mathbb{T}^2\} \). Among many things, Duarte shows that the circle map \( \Psi_k : \mathbb{S}^1 \to \mathbb{S}^1 \) defined by

\[
(\Psi_k(x), v_+) := \pi^s(g_k(x, v_+)) \quad \text{or, equivalently,} \quad (v_+, \Psi_k(y)) = \pi^u(g_k^{-1}(v_+, y))
\]

is singular expansive with small distortion.

\(^1\) We are not going to recall the definition of non-uniformly hyperbolic horseshoes here: instead, we refer the reader to the original article [2] for the details.
In Section 4 of [1], Duarte considers a Cantor set

$$K_k = \bigcap_{n \in \mathbb{N}} \Psi_k^{-1}(J_0 \cup J_1)$$

of the circle map $\Psi_k$ associated with a Markov partition $J_0 \cup J_1 \subset [-1/4, 3/4]$ with the following properties:

- the extremities of the intervals $J_0 = [a, b]$ and $J_1 = [b', a' + 1]$ satisfy $a + \frac{1}{4}, b - \frac{1}{4} - a', b' - \frac{1}{4} \in \{\frac{1}{4k}, \frac{4}{k}\}$, so that $J_0$ and $J_1$ are contained in the region $\{\rho_k = 0\}$;
- $\Psi_k(a) = a = \Psi_k(a')$, $\Psi_k(b) = a' = \Psi_k(b')$.

In particular, Duarte uses these features of $K_k$ to prove that

$$\Lambda_k = (\pi^s)^{-1}(K_k) \cap (\pi^u)^{-1}(K_k)$$

is a horseshoe of both $\Phi_k$ and $f_k$.

In Section 5 of [1], Duarte studies the tangencies associated with the invariant foliations of $\Lambda_k$. More concretely, denote by $\mathcal{G}^s = (f_k)_* (\mathcal{F}^s)$ the foliation obtained by pushing the almost horizontal foliation $\mathcal{F}^s$ by the standard map $f_k$. The vector fields $(\beta^s(x, y), 1)$ defining $\mathcal{G}^s$ and $(\alpha^s(x, y), 1)$ defining $\mathcal{F}^s$ coincide along two (almost horizontal) circles of tangencies $\{(x, \sigma_\pm(x) : x \in S^1)\} \cup \{(x, \sigma_\mp(x) : x \in S^1)\}$ (with $|\sigma_\pm(x) - v_\pm| \leq \frac{1}{270k^{3/4}}$ and $|\sigma_\mp(x) - v_\mp| \leq \frac{1}{12k^{3/4}}$ for all $x \in S^1$). The projections of $\Lambda_k$ along $\mathcal{F}^s$ and $\mathcal{G}^s$ on the circle of tangencies $\{(x, \sigma_+(x)) : x \in S^1\}$ define two Cantor sets

$$K^1_n = \{(x, \sigma_+(x)) : x \in S^1\} \cap (\pi^s)^{-1}(K_k)$$

and

$$K^u_n = \{(x, \sigma_+(x)) : x \in S^1\} \cap f_k((\pi^u)^{-1}(K_k))$$

whose intersection points $x \in K^1_n \cap K^u_n$ are points of tangencies between the invariant manifolds of $\Lambda_k$. Furthermore, it is shown in Propositions 18 and 20 of [1] that these tangencies are quadratic and unfold generically.

4. Proof of Theorem 1.1

After these preliminaries on the works of Palis–Yoccoz and Duarte, we are ready to prove the main result of this note. The standard map $f_k$ has fixed points at $p_0 = (0, 0) \in \Lambda_k$ and $p_u = (-1/2 + O(1/k), -1/2 + O(1/k)) \in \Lambda_k$.

The local stable leaf $\mathcal{F}^s(p_0)$ is tangent to some leaf of $\mathcal{G}^s$ at a point $q$. Since $K_k$ is $\frac{1}{2k^{3/4}}$-dense in $S^1$ (cf. page 394 of [1]), and $f_k$ sends the vertical circle $S^1$-dense in $S^1$ into the horizontal circle $\{(x, \sigma_+(x)) : x \in S^1\}$ as a $C^1$-perturbation of size $\frac{1}{8k^{3/4}}$ of a rigid rotation (cf. page 397 of [1]), we can find a point of $K^u_k$ in the $\frac{7}{2k^{3/4}}$-neighborhood of the tangency point $q = \{(x, \sigma_+(x)) : x \in S^1\}$.

Therefore, the fact that the tangency at $q$ unfolds generically (cf. footnote 3) permits to take a parameter $|\epsilon - k| < \frac{4}{k^{3/4}}$ such that the local stable leaf $\mathcal{F}^s(p_0)$ is tangent to the unstable manifold of some point of $\Lambda_k$.

Because the unstable manifold of the fixed point $p_u$ is dense in $\Lambda_k$ (and the tangencies unfold generically), we can replace $k$ by a parameter $|\epsilon - k| < \frac{4}{k^{3/4}}$ such that the local stable manifold $\mathcal{F}^s(p_0)$ has a quadratic tangency with the unstable manifold of $p_u$ at $q$, which is unfolded generically.

Next, we observe that the right part of a small neighborhood of $q$ in the circle of tangencies is transversal to leaves of $\mathcal{F}^s$ to the right of $p_u$, and the left part of a small neighborhood of $q$ in the circle of tangencies is transversal to a certain (fixed) iterate of the leaves of $\mathcal{G}^s$ which are either all above or all below $p_u$. In the former, resp. latter case, we consider a Markov partition $I_\pm \cup I_0 \cup I_1$ for the singular expansive map $\Psi_t : S^1 \to S^1$ where:

- $I_0$ has extremities $\pi^s(p_u)$ and $a \in \frac{15}{32} - \frac{1}{k^{3/4}}$;
- $I_1$ has extremities $b \in \frac{15}{32} - \frac{1}{k^{3/4}}$ and $c \in \frac{19}{32} + \frac{1}{k^{3/4}}$;
- $I_- \cup I_1$ has extremities $\pi^u(p_u)$ and $d \in \frac{1}{38} - \frac{1}{38} + \frac{1}{k^{3/4}}$, resp. $d \in \frac{1}{38} - \frac{1}{38} - \frac{1}{k^{3/4}}$;
- $\Psi_t(c) = \pi^u(p_u)$, $\Psi_t(b) = c = \Psi_t(d)$ and $\Psi_t(a) = d$, resp. $\Psi_t(a) = \pi^u(p_u)$, $\Psi_t(d) = \pi^u(p_u)$, $\Psi_t(c) = d$ and $\Psi_t(b) = c$.

This defines a Cantor set

$$L_\tau := \bigcap_{n \in \mathbb{N}} \Psi_t^{-n}(I_- \cup I_0 \cup I_1)$$

2 The difference in curvatures at tangency points is $\geq 4\pi^2 k - \frac{3}{k^{3/4}}$.

3 The leaves of $\mathcal{F}^s$ move with speed $\leq \frac{1}{k^{3/4}}$ and the leaves of $\mathcal{G}^s$ move with speed $\geq 1 - \frac{1}{k^{3/4}}$. 
and a horseshoe

$$\Theta_r := (\pi^r)^{-1}(L_r) \cap (\pi^u)^{-1}(L_r)$$

containing $p_s$ and $p_u$.

By definition, we can select neighborhoods $U$ of $\Theta_r$ and $V$ of the orbit $O(q)$ of $q$ such that the $f_r$-maximal invariant set of $U \cup V$ is exactly $\Theta_r \cup O(q)$; this happens because our choices were made so that the local stable leafs of $\Theta_r$ approach $q$ only from the right, while certain (fixed) iterates of the local unstable manifolds of $\Theta_r$ approach $q$ only from the left.

Therefore, we can conclude Theorem 1.1 from the Palis–Yoccoz work (cf. Theorem 2.2) once we verify that $\Theta_r$ is slightly thick.

In view of Remark 2.1, our task is reduced to check that the stable and unstable Hausdorff dimensions of $\Theta_r$ are comprised between 0.5 and 0.6. In this direction, note that these Hausdorff dimensions coincide with the Hausdorff dimension $d(r)$ of $L_r$. Moreover, the distortion constant $C_1(r)$ of $\Psi_r$ is small (namely, $0 \leq C_1(k) \leq \frac{9}{877}$, cf. page 388 of [1]). Hence, $d(r)$ is close to the solution $\kappa(r)$ to the “Bowen’s equation”

$$(\text{length } I_{-})^{\kappa(r)} + (\text{length } I_{0})^{\kappa(r)} + (\text{length } I_{1})^{\kappa(r)} = (\text{length } I)^{\kappa(r)}$$

where $I$ is the convex hull of $I_{-} \cup I_{0} \cup I_{1}$. Since length $I_{-} = \frac{1}{16} + O(\frac{1}{k^{1/3}})$, length $I_{0} = \text{length } I_{1} = \frac{1}{8} + O(\frac{1}{k^{1/3}})$,

$$\text{length } I = \frac{19}{32} + \frac{7}{48} + O(\frac{1}{k^{1/3}})$$

and

$$\frac{1}{16}^{0.5809\ldots} + \frac{1}{8}^{0.5809\ldots} + \frac{1}{8}^{0.5809\ldots} = 0.5809\ldots$$

we derive that $0.554 < d(r) < 0.581$. This completes the argument.

References