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## A generalised comparison principle for the Monge–Ampère equation and the pressure in 2D fluid flows



*Un principe de comparaison généralisé pour l'équation de Monge–Ampère et la pression dans les écoulements fluides en dimension 2*

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## ABSTRACT

We extend the *generalised comparison principle* for the Monge–Ampère equation due to Rauch & Taylor (1977) [15] to nonconvex domains. From the generalised comparison principle, we deduce bounds (from above and below) on solutions to the Monge–Ampère equation with sign-changing right-hand side. As a consequence, if the right-hand side is nonpositive (and does not vanish almost everywhere), then the equation equipped with a constant boundary condition has no solutions. In particular, due to a connection between the two-dimensional Navier–Stokes equations and the Monge–Ampère equation, the pressure  $p$  in 2D Navier–Stokes equations on a bounded domain cannot satisfy  $\Delta p \leq 0$  in  $\Omega$  unless  $\Delta p \equiv 0$  (at any fixed time). As a result, at any time  $t > 0$  there exists  $z \in \Omega$  such that  $\Delta p(z, t) = 0$ .

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## R É S U M É

Nous étendons aux domaines non convexes le *principe de comparaison généralisé* pour l'équation de Monge–Ampère, dû à Rauch et Taylor. Nous en déduisons des bornes (supérieure et inférieure) pour les solutions de l'équation de Monge–Ampère avec second membre changeant de signe. En conséquence, si le second membre est négatif ou nul (et ne s'annule pas presque partout), alors l'équation avec condition au bord constante n'a pas de solution. En particulier, en raison d'une relation entre les équations de Navier–Stokes en dimension 2 et l'équation de Monge–Ampère, la pression  $p$  dans les équations de Navier–Stokes de dimension 2 sur un domaine borné  $\Omega$  satisfait  $\Delta p \leq 0$  dans  $\Omega$ , à moins que  $\Delta p \equiv 0$  (à tout temps donné). Il en résulte qu'à tout temps  $t > 0$ , il existe  $z \in \Omega$  tel que  $\Delta p(z, t) = 0$ .

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### 1. Introduction

The Monge–Ampère equation is

$$\det D^2\phi = f.$$

When the right-hand side  $f$  of this equation is positive, it constitutes an example of a nonlinear second order elliptic equation (see, for example, the Chapter 17 of Gilbarg & Trudinger [7]), and the study of the Dirichlet boundary value problem for this equation goes back to the works of Alexandrov [2], Bakelman [3], and Pogorelov [13], and it is related to the prescribed Gaussian curvature problem and to the Monge–Kantorovich mass transfer problem. Interesting results in this theory include Alexandrov’s maximum principle [1], the equivalence between the notion of generalised solution and the notion of viscosity solution [4], and, most notably, the interior regularity results (see [5]). See Gutiérrez [8] for a modern exposition of the theory of the Monge–Ampère equation.

Moreover, the Monge–Ampère equation with positive right-hand side  $f$  shares many striking similarities with the Laplace equation; take for instance the fact that both the Laplace operator  $\Delta\phi$  and the determinant of the Hessian  $\det D^2\phi$  are invariant under orthogonal transformations, the similarity between the comparison principle (see Corollary 4) and the maximum principle for subharmonic functions, or the occurrence of Perron’s method in finding solutions to the Dirichlet boundary value problem.

However, very little is known about the Monge–Ampère equation when the right-hand side  $f$  changes sign, since in this case it is a (nonlinear) mixed elliptic–hyperbolic problem. A step in this direction is a generalised comparison principle (see Theorem 7), which was first studied by Rauch & Taylor [15] in the case of a strictly convex domain  $\Omega$ , and which gives pointwise bounds, above and below, to the solution to the Monge–Ampère equation with a sign-changing right-hand side (see Corollary 8). This result gives the uniqueness of the solution  $\phi \equiv 0$  to the problem  $\det D^2\phi = 0$  in  $\Omega$  equipped with the boundary condition  $\phi|_{\partial\Omega} = 0$  (the standard existence and uniqueness theorem (see Theorem 5) gives uniqueness only among convex and concave solutions). This filled a gap in the uniqueness problem in the theory of the buckling thin elastic shell (which was also a partial motivation for Rauch & Taylor [15]; see the first section therein and Remark 2.2 in Rabinowitz [14]).

Here we further extend this comparison principle to cover the case of nonconvex domains  $\Omega$  and we point out an interesting application to the theory of two-dimensional Navier–Stokes equations.

In the next section, we recall some background theory of the Monge–Ampère equation. In Section 3, we prove the generalised comparison principle and discuss its consequences (bounds on the solution to the Monge–Ampère equation). In the last section (Section 4), we discuss the link between the two-dimensional Navier–Stokes equations and the Monge–Ampère equation, and we use the bounds on solution to the Monge–Ampère equation to show that the pressure  $p$  in 2D Navier–Stokes equations on a bounded domain cannot satisfy  $\Delta p \leq 0$ ,  $\Delta p \not\equiv 0$  at any  $t > 0$ .

### 2. Preliminary material

Let  $\Omega$  be a bounded, open subset of  $\mathbb{R}^n$ . We will use a number of properties of convex functions (and concave functions), the Monge–Ampère measure, and the Monge–Ampère equation. In this section we quickly recall the relevant definitions and results; the proofs can be found in the first chapter of Gutiérrez [8].

A function  $\phi: \Omega \rightarrow \mathbb{R}$  is convex if  $\phi(\lambda x + (1 - \lambda)y) \leq \lambda\phi(x) + (1 - \lambda)\phi(y)$  for every segment  $[x, y] \subset \Omega$  and  $\lambda \in [0, 1]$ . If  $\phi \in C^2(\overline{\Omega})$ , then  $\phi$  is convex in  $\Omega$  if and only if  $D^2\phi$  is positive semidefinite in  $\Omega$ . A set  $\Omega$  is convex if  $\lambda x + (1 - \lambda)y \in \overline{\Omega}$  for all  $x, y \in \overline{\Omega}$ ,  $\lambda \in [0, 1]$ ; it is strictly convex if  $\lambda x + (1 - \lambda)y \in \Omega$  for all  $x, y \in \overline{\Omega}$ ,  $\lambda \in (0, 1)$ . A supporting hyperplane to  $\phi$  at  $x_0 \in \Omega$  is an affine function  $\phi(x_0) + m \cdot (x - x_0)$  such that

$$\phi(x) \geq \phi(x_0) + m \cdot (x - x_0) \quad \text{for all } x \in \Omega.$$

**Definition 1.** The normal mapping of  $\phi$  (or subdifferential of  $\phi$ ) is the set-valued mapping  $\partial\phi: \Omega \rightarrow \mathcal{P}(\mathbb{R}^n)$ , which maps  $x_0 \in \Omega$  into the set of all those  $m$  for which  $\phi(x_0) + m \cdot (x - x_0)$  is a supporting hyperplane. Namely,

$$\partial\phi(x_0) := \{m \in \mathbb{R}^n : \phi(x) \geq \phi(x_0) + m \cdot (x - x_0) \quad \text{for all } x \in U_{x_0}\},$$

where  $U_{x_0}$  denotes some open neighbourhood of  $x_0$ . Given  $E \subset \Omega$ , we define

$$\partial\phi(E) = \cup_{x \in E} \partial\phi(x).$$

A convex function  $\phi$  has at least one supporting hyperplane at each point, that is  $\partial\phi(x_0) \neq \emptyset$  for all  $x_0 \in \Omega$ . If  $\phi \in C(\overline{\Omega})$  then the family of sets

$$S := \{E \subset \Omega : \partial\phi(E) \text{ is Lebesgue measurable}\}$$

is a Borel  $\sigma$ -algebra.

**Definition 2.** The set function  $M\phi : \mathcal{S} \rightarrow \overline{\mathbb{R}}$  defined by

$$M\phi(E) := |\partial\phi(E)|,$$

where  $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ , is the *Monge–Ampère measure* of  $\phi$ .

In a sense,  $M\phi(E)$  measures “how convex”  $\phi$  is on  $E$ . Moreover, this measure is finite on compact subsets of  $\Omega$  and it satisfies the following three properties.

- (i) If  $\phi \in C^2(\overline{\Omega})$ , then  $M\phi$  is absolutely continuous with respect to the Lebesgue measure and  $M\phi(E) = \int_E \det D^2\phi \, dx$  for all Borel sets  $E \subset \Omega$ . In particular, if  $\phi(x) := \delta|x - x_0|^2$  for some  $x_0 \in \Omega$ ,  $\delta > 0$  then  $D^2\phi = 2\delta I$ , where  $I$  denotes the unit matrix, and so

$$M\phi(E) = \int_E \det D^2\phi \, dx = (2\delta)^n |E|$$

for every Borel set  $E$ .

- (ii) If  $\phi, \psi$  are convex functions then  $M(\psi + \phi) \geq M\psi + M\phi$ . In particular, adding a constant function has no effect on the Monge–Ampère measure, and adding a quadratic polynomial  $\delta|x - x_0|^2$  strictly increases the Monge–Ampère measure, that is, if

$$\tilde{\psi} := \psi + \delta|\cdot - x_0|^2, \quad \text{then} \quad M\tilde{\psi}(E) \geq M\psi(E) + (2\delta)^n |E|$$

for every Borel set  $E$ .

- (iii) If  $\Omega \subset \mathbb{R}^n$  is a bounded open set and  $\phi, \psi \in C(\overline{\Omega})$  are such that  $\phi = \psi$  on  $\partial\Omega$  with  $\psi \leq \phi$ , then  $\partial\phi(\Omega) \subset \partial\psi(\Omega)$ , and hence also

$$M\phi(\Omega) \leq M\psi(\Omega).$$

The comparison principle ([Corollary 4](#)) is an important tool in studying the Monge–Ampère equation. Here we present a stronger version of the comparison principle. We focus on the case of convex functions; the case of concave functions follows analogously by replacing, respectively,  $M\phi, M\psi$  by  $M(-\phi), M(-\psi)$  and “minimum” by “maximum”.

**Theorem 3** (Strong comparison principle). *Let  $\Omega$  be open and bounded and  $\phi, \psi \in C(\overline{\Omega})$  be convex functions such that*

$$M\phi \leq M\psi \quad \text{on } \Omega. \tag{1}$$

*If  $\tilde{\psi} := \psi + Q$  for some quadratic polynomial  $Q(x) := \delta|x - x_0|^2$ , where  $\delta > 0$ , then  $\phi - \tilde{\psi}$  does not attain its minimum inside  $\Omega$ .*

This theorem will be important in obtaining our generalised comparison principle for nonconvex domains (see [Theorem 7](#)). We prove it by sharpening the proof of the standard comparison principle, see, e.g., [Gutiérrez \[8\]](#), p. 17.

**Proof.** Suppose that there exists  $z \in \Omega$  such that

$$\phi(z) - \tilde{\psi}(z) = \min_{\overline{\Omega}}(\phi - \tilde{\psi}) =: a$$

and let

$$\tilde{Q}(x) := \frac{\delta}{2}|x - (2x_0 - z)|^2 - \delta|x - x_0|^2.$$

This quadratic polynomial is tangent to  $Q(x)$  at  $z$  and supports it from below, that is,  $\tilde{Q}(z) = Q(z)$  and  $\tilde{Q}(x) < Q(x)$  for  $x \neq z$ . Indeed, direct calculation gives  $\tilde{Q}(z) = Q(z)$ ,  $\nabla\tilde{Q}(z) = \nabla Q(z)$ ,  $D^2(Q - \tilde{Q}) = \delta I$  and so Taylor’s expansion for  $x \neq z$  gives

$$(Q - \tilde{Q})(x) = (x - z) \cdot \frac{\delta I}{2}(x - z) = \frac{\delta}{2}|x - z|^2 > 0.$$

Hence, in particular,  $\tilde{Q}|_{\partial\Omega} < Q|_{\partial\Omega}$ , and we obtain

$$b := \min_{\partial\Omega}(\phi - \psi - \tilde{Q}) > \min_{\partial\Omega}(\phi - \psi - Q) \geq a,$$

see [Fig. 1](#). Now let

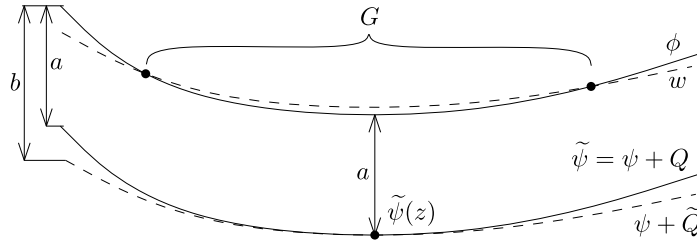


Fig. 1. The set  $G$  (note this is a 1D sketch of a multidimensional situation).

$$w(x) := \psi(x) + \tilde{Q}(x) + \frac{b+a}{2}$$

and

$$G := \{x \in \Omega : \phi(x) < w(x)\}.$$

We see that  $z \in G$  and so  $G$  is a nonempty open subset of  $\Omega$ . Hence,  $|G| > 0$  and property (ii) gives

$$M\psi(G) + \delta^n |G| \leq Mw(G). \tag{2}$$

Moreover,  $\partial G = \{x \in \Omega : w(x) = \phi(x)\}$  (see Fig. 1). Indeed, this is equivalent to  $\overline{G} \cap \partial\Omega = \emptyset$ , but for  $y \in \partial\Omega$ , we have

$$\phi(y) - \psi(y) - \tilde{Q}(y) \geq b > \frac{b+a}{2},$$

that is  $\phi(y) > w(y)$  and so  $y \notin \overline{G}$ . Therefore indeed  $\partial G = \{x \in \Omega : w(x) = \phi(x)\}$  and hence property (iii) gives

$$Mw(G) \leq M\phi(G).$$

This and (2) gives  $M\psi(G) < M\phi(G)$ , which contradicts the assumption (1).  $\square$

The standard comparison principle (Theorem 1.4.6 in Gutiérrez [8]) is a corollary of Theorem 3.

**Corollary 4** (Comparison principle). *Let  $\Omega$  be open and bounded and  $\phi, \psi \in C(\overline{\Omega})$  be convex functions such that  $M\phi \leq M\psi$  in  $\Omega$ . Then*

$$\min_{x \in \overline{\Omega}} (\phi - \psi)(x) = \min_{x \in \partial\Omega} (\phi - \psi)(x).$$

In particular, if  $\phi \geq \psi$  on  $\partial\Omega$  then  $\phi \geq \psi$  in  $\overline{\Omega}$ .

**Proof.** Suppose otherwise that there exists an  $x_0 \in \Omega$  such that

$$\phi(x_0) - \psi(x_0) = \min_{x \in \overline{\Omega}} (\phi - \psi)(x) < \min_{x \in \partial\Omega} (\phi - \psi)(x).$$

Because  $\Omega$  is bounded, for sufficiently small  $\delta > 0$  the function  $\phi - (\psi + \delta|x - x_0|^2)$  still attains its minimum inside  $\Omega$ , which contradicts the strong comparison principle (Theorem 3).  $\square$

If  $\mu$  is a Borel measure defined in  $\Omega$ , we say that a convex function  $v \in C(\Omega)$  is a *generalised solution* to the Monge–Ampère equation  $\det D^2 v = \mu$  if  $Mv = \mu$ . If  $v$  is concave, it is a generalised solution to  $\det D^2 v = \mu$  when  $M(-v) = \mu$ . We have the following existence and uniqueness result for the Dirichlet problem for the Monge–Ampère equation (Theorem 1.6.2 in Gutiérrez [8]).

**Theorem 5** (Existence theorem for the Monge–Ampère equation). *If  $\Omega \subset \mathbb{R}^n$  is open, bounded, and strictly convex,  $\mu$  is a Borel measure in  $\Omega$  with  $\mu(\Omega) < +\infty$  and  $g \in C(\partial\Omega)$ , then there exists a unique convex generalised solution  $\psi \in C(\overline{\Omega})$  to the problem*

$$\begin{cases} \det D^2 \psi = \mu & \text{in } \Omega, \\ \psi = g & \text{on } \partial\Omega. \end{cases}$$

Similarly, there exists a unique concave generalised solution to this problem.

Before turning to the generalised comparison principle, we recall the following weak convergence result for Monge–Ampère measures.

**Lemma 6.** *Let  $\Omega \subset \mathbb{R}^n$  be an open, bounded and strictly convex domain,  $\mu_j, \mu$  be Borel measures in  $\Omega$  with  $\mu_j(\Omega) \leq A$  for all  $j$  and some  $A > 0$  and  $\mu_j \rightarrow \mu$  as  $j \rightarrow \infty$ , that is,  $\int_{\Omega} f d\mu_j \rightarrow \int_{\Omega} f d\mu$  for all  $f \in C_0(\Omega)$ . Let  $g_j, g \in C(\partial\Omega)$  be such that  $\|g_j - g\|_{C(\partial\Omega)} \rightarrow 0$  as  $j \rightarrow \infty$ . If the family of convex functions  $\{\phi_j\} \subset C(\overline{\Omega})$  satisfies*

$$\begin{cases} M\phi_j = \mu_j & \text{in } \Omega, \\ \phi_j = g_j & \text{on } \partial\Omega, \end{cases}$$

then  $\{\phi_j\}$  contains a subsequence  $\{\phi_{j_k}\}$ , such that  $\phi_{j_k} \rightarrow \phi$  uniformly on compact subsets of  $\Omega$  as  $k \rightarrow \infty$ , where  $\phi \in C(\overline{\Omega})$  is convex and  $M\phi = \mu$  in  $\Omega$ ,  $\phi = g$  on  $\partial\Omega$ .

The above lemma is proved in Gutiérrez [8], pp. 21–22, in the case  $g_j \equiv g$ ,  $j = 1, 2, \dots$ . The case  $g_j \not\equiv g$  follows as a straightforward generalisation.

### 3. Generalised comparison principle

Let  $\phi \in H^2(\Omega)$ , that is,  $\phi: \Omega \rightarrow \mathbb{R}$  is such that  $\phi, \nabla\phi, D^2\phi \in L^2(\Omega)$  (in other words, these functions exist almost everywhere in  $\Omega$  and are square summable on  $\Omega$ ). Let

$$A_{\phi} := \{x \in \Omega : D^2\phi(x) \text{ is positive definite at } x\},$$

$$B_{\phi} := \{x \in \Omega : D^2\phi(x) \text{ is negative definite at } x\}.$$

We will denote by  $[M\phi]^+$  the measure that is absolutely continuous with respect to the Lebesgue measure with density  $(\det D^2\phi)^+$ . Observe that, since  $\phi \in H^2(\Omega)$  and  $\det D^2\phi = \phi_{xx}\phi_{yy} - \phi_{xy}^2$  consists only of products of two second-order derivatives, the Hölder inequality gives  $\det D^2\phi \in L^1(\Omega)$ , and consequently  $[M\phi]^+(\Omega) < \infty$ . Moreover, if additionally  $\phi \in C^2(\Omega)$ , then  $A_{\phi}$  is an open subset of  $\Omega$ ,  $\phi$  is convex on  $A_{\phi}$  and, using (i),  $[M\phi]^+$  is equal to the Monge–Ampère measure  $M\phi$  when restricted to  $A_{\phi}$ .

We also denote by  $[M\phi]^-$  the measure that is absolutely continuous with respect to the Lebesgue measure with density  $(\det D^2(-\phi))^+$ .

**Theorem 7 (Generalised comparison principle).** *Let  $\Omega$  be a bounded, open set in  $\mathbb{R}^n$ . Let  $\psi \in C(\overline{\Omega})$  be a convex function in  $\Omega$  with  $M\psi(\Omega) < \infty$  and let  $\phi \in H^2(\Omega)$  be such that*

$$[M\phi]^+ \leq M\psi \quad \text{in } \Omega. \tag{3}$$

Then

$$\min_{\overline{\Omega}}(\phi - \psi) = \min_{\partial\Omega}(\phi - \psi).$$

In particular, if  $\phi \geq \psi$  on  $\partial\Omega$ , then  $\phi \geq \psi$  in  $\overline{\Omega}$ . Similarly, if  $\psi$  is concave in  $\Omega$  and  $\phi \in H^2(\Omega)$  is such that  $[M\phi]^- \leq M(-\psi)$  in  $\Omega$ , then

$$\max_{\overline{\Omega}}(\phi - \psi) = \max_{\partial\Omega}(\phi - \psi).$$

We give a proof that does not use the solvability result of the Monge–Ampère equation on  $\Omega$ , and so does not require strict convexity of  $\Omega$  (see Theorem 5). Instead, we replace it with the solvability result on a neighbourhood  $B$  of a point in  $\Omega$  and an application of the strong comparison principle (Theorem 3). Since the resulting proof is therefore local in nature – it does not use any global properties of  $\Omega$  – it allows for  $\Omega$  to be nonconvex. (In fact, the original proof due to Rauch & Taylor [15] does not use the strict convexity of  $\Omega$  when  $\phi$  is assumed to be  $C^2(\Omega)$ ; but for  $\phi \in H^2(\Omega)$  their approximation argument requires the solvability result (Theorem 5), which is only valid for  $\Omega$  strictly convex.)

**Proof.** We focus on the case of  $\psi$  convex; the case of concave  $\psi$  follows by replacing  $\phi, \psi$  by  $-\phi, -\psi$  respectively. Assume first that  $\phi \in C^2(\Omega)$  (here we can follow Rauch & Taylor [15]). Suppose otherwise that there exists  $x_0 \in \Omega$  such that

$$(\phi - \psi)(x_0) = \min_{x \in \overline{\Omega}}(\phi - \psi)(x)$$

and consider the function

$$\tilde{\psi}(x) := \psi(x) + \varepsilon_0|x - x_0|^2 \tag{4}$$

for  $\varepsilon_0 > 0$ . Since  $\Omega$  is bounded, it is clear that, for  $\varepsilon_0$  sufficiently small, the function  $\phi - \tilde{\psi}$  still does not attain its minimum on  $\partial\Omega$ . This means that for such an  $\varepsilon_0$  fixed, there exists  $\tilde{x} \in \Omega$  such that

$$(\phi - \tilde{\psi})(\tilde{x}) = \min_{x \in \Omega} (\phi - \tilde{\psi})(x) < \min_{x \in \partial\Omega} (\phi - \tilde{\psi})(x). \tag{5}$$

Moreover, because on  $A_\phi$  both  $\tilde{\psi}$  and  $\phi$  are convex, and

$$M\tilde{\psi} \geq M\psi \geq M\phi,$$

the strong comparison principle (Theorem 3) gives  $\tilde{x} \notin A_\phi$ . In other words,  $D^2\phi(\tilde{x})$  is not positive definite. Now, because any symmetric matrix is positive definite if and only if all its eigenvalues are positive (see, e.g., Theorem 7.2.1 in Horn & Johnson [9]), we see that  $D^2\phi(\tilde{x})$  has at least one nonpositive eigenvalue. Let  $\lambda \leq 0$  be one such eigenvalue and let  $\alpha \in \mathbb{R}^n$ ,  $|\alpha| = 1$ , be the respective eigenvector. Then, by performing a Taylor expansion in the  $\alpha$  direction, we can write, for  $t \in \mathbb{R}$  with  $|t|$  small

$$\phi(\tilde{x} + t\alpha) - \phi(\tilde{x}) = a_1t + \lambda t^2 + o(t^2), \tag{6}$$

where  $a_1 \in \mathbb{R}$  and  $o(\cdot) : \mathbb{R} \rightarrow \mathbb{R}$  denotes any function such that  $o(y)/y \xrightarrow{y \rightarrow 0} 0$ . As  $\psi$  is convex, it has a supporting hyperplane at  $\tilde{x}$  (see (4)). Hence,

$$\begin{aligned} \tilde{\psi}(\tilde{x} + t\alpha) - \tilde{\psi}(\tilde{x}) &= \psi(\tilde{x} + t\alpha) - \psi(\tilde{x}) + \varepsilon_0(|\tilde{x} + t\alpha - x_0|^2 - |\tilde{x} - x_0|^2) \\ &\geq a_2t + \varepsilon_0(|\tilde{x} + t\alpha - x_0|^2 - |\tilde{x} - x_0|^2) = a_3t + \varepsilon_0t^2, \end{aligned}$$

where  $a_2, a_3 \in \mathbb{R}$ . Combining this with (6) and using (5), we obtain

$$(\phi - \tilde{\psi})(\tilde{x}) \leq (\phi - \tilde{\psi})(\tilde{x} + t\alpha) \leq (\phi - \tilde{\psi})(\tilde{x}) + (a_1 - a_3)t + (\lambda - \varepsilon_0)t^2 + o(t^2)$$

for small values of  $|t|$ . This means that the quadratic polynomial

$$(a_1 - a_3)t + (\lambda - \varepsilon_0)t^2$$

attains its minimum at  $t = 0$ . Hence  $a_1 = a_3$  and  $\lambda - \varepsilon_0 \geq 0$ , which contradicts  $\lambda \leq 0 < \varepsilon_0$ .

Now let  $\phi \in H^2(\Omega)$ , and similarly as before consider  $\tilde{\psi}$  and  $\tilde{x}$  given by (4) and (5). Let  $B$  be an open ball centered at  $\tilde{x}$  and such that  $\bar{B} \subset \Omega$ .

Let  $\{\phi_j\} \subset C_0^\infty(\mathbb{R}^2)$  be such that  $\|\phi_j - \phi\|_{H^2(B)} \rightarrow 0$  as  $j \rightarrow \infty$ . By the embedding  $H^2(B) \subset C^0(\bar{B})$ , we also have  $\|\phi_j - \phi\|_{C^0(\bar{B})} \rightarrow 0$  as  $j \rightarrow \infty$ . Let  $\mu_j, \mu$  be Borel measures on  $B$  defined by  $\mu_j := [M\phi_j]^+$ ,  $\mu := [M\phi]^+$  (note that  $\mu_j(\Omega), \mu(\Omega) < \infty$  due to the Hölder inequality). For each  $j$ , let  $\psi_j$  be the unique convex solution to the Dirichlet problem:

$$\begin{cases} M\psi_j = \mu_j & \text{in } B, \\ \psi_j = \phi_j & \text{on } \partial B. \end{cases}$$

The existence of such  $\psi_j$  is guaranteed by the existence theorem (Theorem 5). Because  $\phi_j \in C^2$ , the first part gives

$$\psi_j \leq \phi_j \quad \text{in } \bar{B}. \tag{7}$$

Furthermore, because  $\|(\det D^2\phi_j)^+ - (\det D^2\phi)^+\|_{L^1(B)} \rightarrow 0$  as  $j \rightarrow \infty$  gives  $\mu_j \rightarrow \mu$ , and because  $\|\phi_j - \phi\|_{C^0(\partial B)} \rightarrow 0$ , we can use the convergence lemma (Lemma 6) to obtain that  $\psi_j \rightarrow \Psi$  uniformly on compact subsets of  $B$  for some subsequence (which we relabel), where  $\Psi \in C^0(\bar{B})$  is convex and satisfies

$$\begin{cases} M\Psi = \mu & \text{in } B, \\ \Psi = \phi & \text{on } \partial B. \end{cases}$$

Taking the limit  $j \rightarrow \infty$  in (7) we get  $\Psi \leq \phi$  on  $\bar{B}$  and so in particular  $\Psi(\tilde{x}) \leq \phi(\tilde{x})$  and

$$(\Psi - \tilde{\psi})(\tilde{x}) \leq (\phi - \tilde{\psi})(\tilde{x}) = \min_{\Omega} (\phi - \tilde{\psi}) \leq \min_{\partial B} (\phi - \tilde{\psi}) = \min_{\partial B} (\Psi - \tilde{\psi}). \tag{8}$$

Because  $M\Psi = \mu = [M\phi]^+ \leq M\psi \leq M\tilde{\psi}$  on  $B$  and both  $\Psi$  and  $\tilde{\psi}$  are convex, we can use the comparison principle (Corollary 4) to write  $\min_{\partial B} (\Psi - \tilde{\psi}) = \min_{\bar{B}} (\Psi - \tilde{\psi})$ . Therefore, (8) becomes

$$(\Psi - \tilde{\psi})(\tilde{x}) \leq \min_{\bar{B}} (\Psi - \tilde{\psi}),$$

that is  $\Psi - \tilde{\psi}$  admits an internal minimum in  $B$ . This contradicts the strong comparison principle (Theorem 3).  $\square$

An immediate consequence of the generalised comparison principle is that a solution to the Monge–Ampère equation with sign-changing right-hand side can be bounded above and below by, respectively, the concave and the convex solutions to certain Monge–Ampère problems.

**Corollary 8.** *Let  $\Omega$  be a bounded, open subset of  $\mathbb{R}^n$ . If  $\phi \in H^2(\Omega)$ ,  $\Phi_{\text{conv}}$  is a convex generalised solution to  $\det D^2 \Phi_{\text{conv}} = (\det D^2 \phi)^+$  and  $\Phi_{\text{conc}}$  is a concave generalised solution to  $\det D^2(-\Phi_{\text{conc}}) = (\det D^2(-\phi))^+$  such that  $\Phi_{\text{conv}} = \Phi_{\text{conc}} = \phi$  on  $\partial\Omega$ , then*

$$\Phi_{\text{conv}} \leq \phi \leq \Phi_{\text{conc}} \quad \text{in } \overline{\Omega}.$$

**Proof.** This follows from the generalised comparison principle (Theorem 7), since  $M\phi \leq M\Phi_{\text{conv}} = M(-\Phi_{\text{conc}})$  and

$$M\Phi_{\text{conv}}(\Omega) = M(-\Phi_{\text{conc}})(\Omega) = \|(\det D^2(-\phi))^+\|_{L^1} \leq C\|\phi\|_{H^2} < \infty. \quad \square$$

Note that if  $\Omega$  is strictly convex then the functions  $\Phi_{\text{conv}}$ ,  $\Phi_{\text{conc}}$  are uniquely determined by the existence theorem (Theorem 5). What is more, if  $n$  is even, then  $\det D^2(-\phi) = \det D^2\phi$  and hence  $\Phi_{\text{conv}}$  and  $\Phi_{\text{conc}}$  are solutions to the same problem

$$\begin{cases} \det D^2 \Phi = (\det D^2 \phi)^+ & \text{in } \Omega, \\ \Phi = \phi & \text{on } \partial\Omega. \end{cases}$$

In other words, if  $n$  is even, then any  $\phi \in H^2(\Omega)$  can be bounded below and above using functions  $\Phi_{\text{conv}}$  and  $\Phi_{\text{conc}}$ , which depend only on the positive part of  $\det D^2\phi$  and on the boundary values of  $\phi$ . The power of Corollary 8 is demonstrated by the following nonexistence result.

**Corollary 9.** *Let  $n$  be even,  $\Omega$  a bounded, open subset of  $\mathbb{R}^n$ ,  $C \in \mathbb{R}$  and  $f$  a nonpositive function such that  $f \not\equiv 0$ . Then the problem*

$$\begin{cases} \det D^2 \phi = f & \text{in } \Omega, \\ \phi = C & \text{on } \partial\Omega \end{cases}$$

has no  $H^2(\Omega)$  solution.

**Proof.** Suppose that there exists  $\phi \in H^2(\Omega)$ , a solution to the above problem. The constant function  $\Phi \equiv C$  satisfies  $\det D^2 \Phi = 0 = f^+$  with  $\Phi|_{\partial\Omega} = C$ . Therefore, by Corollary 8,  $C \leq \phi \leq C$ , i.e.  $\Phi \equiv C$ . Hence  $0 \equiv \det D^2 \phi \equiv f \not\equiv 0$ , which is a contradiction.  $\square$

#### 4. An application to the 2D Navier–Stokes equations

Let us consider the two-dimensional Navier–Stokes equations

$$u_t + (u \cdot \nabla)u - \Delta u + \nabla p = 0$$

at any  $t > 0$  equipped with the incompressibility constraint  $\text{div } u = 0$ . Taking the divergence of the equations and using the incompressibility constraint, we obtain:

$$\nabla \cdot [(u \cdot \nabla)u] + \Delta p = 0.$$

Now, because any divergence-free 2D vector field can be represented as  $u = (\phi_y, -\phi_x)$  for some scalar function  $\phi$ , we can write

$$\begin{aligned} -\Delta p &= \partial_x(u_2 \partial_x u_1 + u_1 \partial_x u_1) + \partial_y(u_2 \partial_y u_2 + u_1 \partial_x u_2) \\ &= \partial_x(-\phi_x \phi_{yy} + \phi_y \phi_{xy}) + \partial_y(\phi_x \phi_{xy} - \phi_y \phi_{xx}) = -2\phi_{xx} \phi_{yy} + 2(\phi_{xy})^2, \end{aligned}$$

that is,

$$\phi_{xx} \phi_{yy} - (\phi_{xy})^2 = \frac{1}{2} \Delta p. \tag{9}$$

This is the Monge–Ampère equation

$$\det D^2 \phi = \frac{1}{2} \Delta p. \tag{10}$$

This connection between the pressure  $p$  and the velocity  $u$  in 2D Navier–Stokes equations was first studied by Larchevêque [10,11], who also observed that, in the regions of positive  $\Delta p$ , the velocity  $u$  has closed streamlines, which he related to the appearance of coherent structures (see also Roulstone et al. [18] for a connection between coherent structures and  $\Delta p$  in the case of three-dimensional Navier–Stokes equations). In contrast to this local analysis, here we use the results of the previous section to show that, if  $\Delta p \neq 0$ , then it is not possible that  $\Delta p \leq 0$  throughout  $\Omega$ . Indeed, because the global-in-time solution  $(u, p)$  to the two-dimensional Navier–Stokes equations is smooth, we have in particular that  $u \in H_0^1(\Omega)$ , that is  $\phi \in H^2(\Omega)$  and  $\nabla \phi = 0$  on  $\partial\Omega$ . Therefore, given  $C^1$  regularity of  $\partial\Omega$ , we obtain  $\phi|_{\partial\Omega} = C$  for some  $C \in \mathbb{R}$ . Hence the last corollary gives that  $\Delta p \leq 0$  (with  $\Delta p \neq 0$ ) cannot hold throughout  $\Omega$ .

We also note that  $\Delta p > 0$  cannot hold throughout  $\Omega$ , which can be shown using elementary methods. Indeed, because the solution  $(u, p)$  to the 2D Navier–Stokes equations is smooth (see, e.g., Lions & Prodi [12], Section 3.3 of Temam [19] or Section 9.6 of Robinson [16]) we have in particular that  $u \in C^1(\Omega)$ , that is,  $\phi \in C^2(\Omega)$ . Therefore, if  $\Delta p > 0$ , we can follow an idea from Section IV.6.3 of Courant & Hilbert [6] to write, using (9),

$$\phi_{xx} \phi_{yy} \geq \det D^2 \phi = \frac{1}{2} \Delta p > 0$$

and we see (by continuity) that either

$$\phi_{xx}, \phi_{yy} > 0 \text{ in } \Omega \quad \text{or} \quad \phi_{xx}, \phi_{yy} < 0 \text{ in } \Omega. \quad (11)$$

Supposing that  $\phi_{xx}, \phi_{yy} > 0$ , we can use the divergence theorem to obtain

$$0 < \int_{\Omega} \Delta \phi \, dx \, dy = \int_{\partial\Omega} \frac{\partial \phi}{\partial \nu} \, dS = 0,$$

which is a contradiction; we argue similarly if  $\phi_{xx}, \phi_{yy} < 0$ .

Therefore, if at any time  $t > 0$ , we have  $\Delta p \neq 0$ , then either  $\Delta p$  changes sign inside the domain or  $\Delta p \geq 0$  with  $\Delta p \not\equiv 0$ . In either case,  $\Delta p = 0$  at some interior point of the domain.

One of the questions related to the connection of the pressure  $p$  and velocity  $u$  in the 2D incompressible Navier–Stokes equations is whether the pressure determines the velocity uniquely (see the review article Robinson [17]). The answer to this question is negative, as the following example shows.

**Example.** Consider the shear flow  $u(x, y, t) = (U(y, t), 0)$  in a channel  $\Omega := \mathbb{T} \times [0, 1]$ , where  $U$  satisfies the 1D heat equation  $U_t - U_{yy} = 0$  in  $[0, 1] \times [0, \infty)$ , with boundary conditions  $U(0, t) = U(1, t) = 0$ . Note that  $U(y, t) := Ce^{-k^2 t} \sin(ky)$  is a solution to this problem for any  $C \neq 0$ ,  $k \in \mathbb{N}$ . Then the pair  $(u, p)$ , where  $p \equiv 0$ , satisfies the 2D incompressible Navier–Stokes equations as  $\operatorname{div} u$  vanishes, and  $u_t - \Delta u + (u \cdot \nabla)u + \nabla p = u_t - u_{yy} = 0$ .

This example also illustrates the relevance of boundary conditions in Corollary 8. Indeed, if the periodic boundary condition (in  $x$ ) was replaced by the homogeneous Dirichlet boundary condition, then Corollary 8 implies that the only velocity field  $u$  corresponding to  $p \equiv 0$  is  $u \equiv 0$ .

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