



Analytic geometry/Differential geometry

Parametric CR-umbilical locus of ellipsoids in \mathbb{C}^2 *Détermination paramétrique du lieu CR-ombilic d'ellipsoïdes dans \mathbb{C}^2*

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ABSTRACT

For every real numbers $a \geq 1$, $b \geq 1$ with $(a, b) \neq (1, 1)$, the curve parametrized by $\theta \in \mathbb{R}$ valued in $\mathbb{C}^2 \cong \mathbb{R}^4$

$$\gamma: \theta \mapsto (x(\theta) + \sqrt{-1}y(\theta), u(\theta) + \sqrt{-1}v(\theta))$$

with components:

$$\begin{aligned} x(\theta) &:= \sqrt{\frac{a-1}{a(ab-1)}} \cos \theta, & y(\theta) &:= \sqrt{\frac{b(a-1)}{ab-1}} \sin \theta, \\ u(\theta) &:= \sqrt{\frac{b-1}{b(ab-1)}} \sin \theta, & v(\theta) &:= -\sqrt{\frac{a(b-1)}{ab-1}} \cos \theta, \end{aligned}$$

has image contained in the CR-umbilical locus:

$$\gamma(\mathbb{R}) \subset \text{UmbCR}(E_{a,b}) \subset E_{a,b}$$

of the ellipsoid $E_{a,b} \subset \mathbb{C}^2$ of equation $ax^2 + y^2 + bu^2 + v^2 = 1$, where the CR-umbilical locus of a Levi nondegenerate hypersurface $M^3 \subset \mathbb{C}^2$ is the set of points at which the Cartan curvature of M vanishes.

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R É S U M É

Pour tous nombres réels $a \geq 1$, $b \geq 1$ avec $(a, b) \neq (1, 1)$, la courbe paramétrée par $\theta \in \mathbb{R}$ à valeurs dans $\mathbb{C}^2 \cong \mathbb{R}^4$

$$\gamma: \theta \mapsto (x(\theta) + \sqrt{-1}y(\theta), u(\theta) + \sqrt{-1}v(\theta))$$

ayant pour composantes :

$$\begin{aligned} x(\theta) &:= \sqrt{\frac{a-1}{a(ab-1)}} \cos \theta, & y(\theta) &:= \sqrt{\frac{b(a-1)}{ab-1}} \sin \theta, \\ u(\theta) &:= \sqrt{\frac{b-1}{b(ab-1)}} \sin \theta, & v(\theta) &:= -\sqrt{\frac{a(b-1)}{ab-1}} \cos \theta, \end{aligned}$$

est d'image contenue dans le lieu CR-ombilic :

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$$\gamma(\mathbb{R}) \subset \text{UmbCR}(E_{a,b}) \subset E_{a,b}$$

de l'ellipsoïde $E_{a,b} \subset \mathbb{C}^2$ d'équation $ax^2 + y^2 + bu^2 + v^2 = 1$, où le lieu CR-ombilic d'une hypersurface Levi non dégénérée $M^3 \subset \mathbb{C}^2$ est l'ensemble des points en lesquels la courbure de Cartan de M s'annule.

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1. Introduction

In 1932, Élie Cartan [1–3] showed that a local real-analytic (\mathcal{C}^ω) hypersurface $M^3 \subset \mathbb{C}^2$ is determined up to local biholomorphic equivalence by a single invariant function:

$$\mathcal{J}_{\text{Cartan}}^M : M \longrightarrow \mathbb{C},$$

together with its (covariant) derivatives with respect to a certain coframe of differential 1-forms on an 8-dimensional principal bundle $P^8 \rightarrow M$. In coordinates $(z, w) = (x + \sqrt{-1}y, u + \sqrt{-1}v)$ on \mathbb{C}^2 , whenever M is:

- either a *complex graph*:

$$\{(z, w) \in \mathbb{C}^2 : w = \Theta(z, \bar{z}, \bar{w})\},$$

- or a *real graph*:

$$\{(z, w) \in \mathbb{C}^2 : v = \varphi(x, y, u)\},$$

- or represented in *implicit form*:

$$\{(z, w) \in \mathbb{C}^2 : \rho(z, w, \bar{z}, \bar{w}) = 0\},$$

it is known that $\mathcal{J}_{\text{Cartan}}^M$ depends on the respective 6-jets:

$$J_{z, \bar{z}, \bar{w}}^6 \Theta, \quad J_{x, y, u}^6 \varphi, \quad J_{z, w, \bar{z}, \bar{w}}^6 \rho.$$

The *invariancy* of $\mathcal{J}_{\text{Cartan}}^M$ means that, for any local biholomorphism $h : \mathbb{C}^2 \rightarrow \mathbb{C}^2$, setting $M' := h(M)$, it holds at every point $p \in M$ that:

$$\mathcal{J}_{\text{Cartan}}^{M'}(h(p)) = \nu(p) \mathcal{J}_{\text{Cartan}}^M(p), \tag{\forall p \in M}$$

for some nowhere vanishing (local) function $\nu : M \rightarrow \mathbb{C} \setminus \{0\}$. This guarantees that the locus of *CR-umbilical* points:

$$\text{Umb}_{\text{CR}}(M) := \{p \in M : \mathcal{J}_{\text{Cartan}}^M(p) = 0\}$$

is intrinsic. Furthermore, when M is connected, it is well known that $\text{Umb}_{\text{CR}}(M)$ contains an open set $\emptyset \neq V \subset M$ if and only if M is *spherical*, in the sense of being locally biholomorphic to the unit sphere $S^3 \subset \mathbb{C}^2$.

In 1974, Chern and Moser [4] raised the problem whether $\emptyset \neq \text{Umb}_{\text{CR}}(M)$ for compact Levi nondegenerate \mathcal{C}^ω hypersurfaces $M^{2N-1} \subset \mathbb{C}^N$ when $N \geq 2$. This paper studies the more specific aspects below.

Question 1. Can $\text{Umb}_{\text{CR}}(M)$ be described explicitly?

But because $\mathcal{J}_{\text{Cartan}}^M$ is ‘*complicated*’, as confirmed in [8,9], the question is nontrivial even in simplest nonspherical examples, like, e.g., the real ellipsoids introduced and studied by Webster in [11–13].

In $\mathbb{C}^{N \geq 2} \cong \mathbb{R}^{2N \geq 4}$ equipped with coordinates $z_i = x_i + \sqrt{-1}y_i$, an *ellipsoid* is the image of the unit sphere:

$$S^{2N-1} := \{z \in \mathbb{C}^N : |z_1|^2 + \dots + |z_N|^2 = 1\},$$

through a real affine transformation of \mathbb{R}^{2N} ; hence its equation takes the form:

$$\sum_{1 \leq i \leq N} (\alpha_i x_i^2 + \beta_i y_i^2) = 1, \tag{E_{\alpha, \beta}}$$

with real constants $\alpha_i \geq \beta_i > 0$ – replace $z_i \mapsto \sqrt{-1}z_i$ if necessary.

The complex geometry of ellipsoids (Segre varieties, dynamics) began in Webster’s seminal article [11], in which it was verified that two ellipsoids $E_{\alpha, \beta} \cong E_{\alpha', \beta'}$ are biholomorphically equivalent if and only if up to permutation:

$$\frac{\alpha_i - \beta_i}{\alpha_i + \beta_i} = \frac{\alpha'_i - \beta'_i}{\alpha'_i + \beta'_i} \tag{1 \leq i \leq N}$$

Replacing $z_i \mapsto \frac{1}{\sqrt{\beta_i}} z_i$ and setting $a_i := \frac{\alpha_i}{\beta_i}$, whence $1 \leq a_i$, leads to a convenient representation:

$$\sum_{1 \leq i \leq N} (a_i x_i^2 + y_i^2) = 1. \tag{E_{a_1, \dots, a_N}}$$

Yet an alternative view, due to Webster in [12], is:

$$\sum_{1 \leq i \leq N} (z_i \bar{z}_i + A_i (z_i^2 + \bar{z}_i^2)) = 1, \tag{E_{A_1, \dots, A_N}}$$

obtained by setting $A_i := \frac{a_i - 1}{2a_i + 2}$, whence $0 \leq A_i < \frac{1}{2}$, so that $a_i = \frac{1 + 2A_i}{1 - 2A_i}$, then by changing coordinates $z_i =: \sqrt{1 - 2A_i} z'_i$, and then by dropping primes.

In \mathbb{C}^N when $N \geq 3$, what corresponds to the invariant $\mathfrak{I}_{\text{Cartan}}^M$ is the Hachtroudi–Chern tensor $S_{\rho\sigma}^{\alpha\beta}$ with indices $1 \leq \alpha, \beta, \rho, \sigma \leq N$, and the concerned CR-umbilical locus:

$$\text{Umb}_{\text{CR}}(M) := \{p \in M : S_{\alpha\rho}^{\beta\sigma}(p) = 0, \forall \alpha, \rho, \beta, \sigma\},$$

is known, through local biholomorphisms $h : \mathbb{C}^N \rightarrow \mathbb{C}^N$ as above, to enjoy

$$h(\text{Umb}_{\text{CR}}(M)) = \text{Umb}_{\text{CR}}(h(M)).$$

Theorem 2. ([12]) *In $\mathbb{C}^{N \geq 3}$, if $0 < A_1 < \dots < A_N < \frac{1}{2}$, then:*

$$\emptyset = \text{Umb}_{\text{CR}}(E_{A_1, \dots, A_N}).$$

This motivated Huang–Ji in [5] to study the question for compact \mathcal{C}^ω hypersurfaces $M \subset \mathbb{C}^2$. If $M = \{\rho = 0\}$, the expected dimension of:

$$\text{Umb}_{\text{CR}}(M) = \{0 = \rho = \mathbb{R}e \mathfrak{I}_{\text{Cartan}} = \Im \mathfrak{I}_{\text{Cartan}}\}$$

should be $4 - 3 = 1$, although this is not rigorous, for \mathbb{R} is not algebraically closed.

Theorem 3. (Implicitly proved in [5].) *Every real ellipsoid $E_{a,b} \subset \mathbb{C}^2$ of equation:*

$$ax^2 + y^2 + bu^2 + v^2 = 1 \tag{a \geq 1, b \geq 1, (a, b) \neq (1, 1)}$$

enjoys:

$$\dim_{\mathbb{R}} \text{Umb}_{\text{CR}}(M) \geq 1.$$

In other words, it contains at least some (real algebraic) curve. This article finds an explicit curve.

Theorem 4. *For every real numbers $a \geq 1, b \geq 1$ with $(a, b) \neq (1, 1)$, the curve parametrized by $\theta \in \mathbb{R}$ valued in $\mathbb{C}^2 \cong \mathbb{R}^4$:*

$$\gamma : \theta \mapsto (x(\theta) + \sqrt{-1}y(\theta), u(\theta) + \sqrt{-1}v(\theta))$$

with components:

$$\begin{aligned} x(\theta) &:= \sqrt{\frac{a-1}{a(ab-1)}} \cos \theta, & y(\theta) &:= \sqrt{\frac{b(a-1)}{ab-1}} \sin \theta, \\ u(\theta) &:= \sqrt{\frac{b-1}{b(ab-1)}} \sin \theta, & v(\theta) &:= -\sqrt{\frac{a(b-1)}{ab-1}} \cos \theta, \end{aligned}$$

has image contained in the CR-umbilical locus:

$$\gamma(\mathbb{R}) \subset \text{Umb}_{\text{CR}}(E_{a,b}) \subset E_{a,b}$$

of the ellipsoid $E_{a,b} \subset \mathbb{C}^2$ of equation $ax^2 + y^2 + bu^2 + v^2 = 1$.

In other words:

$$\mathfrak{J}_{\text{Cartan}}^{E_{a,b}}(\gamma(\theta)) = 0. \tag{(\forall \theta \in \mathbb{R})}$$

As is known for ellipsoids, Cartan’s invariant $\mathfrak{J}_{\text{Cartan}}^{E_{a,b}}$ exhibits a high complexity, e.g., $\sim 40\,000$ terms in [8]. So this theorem might be interpreted as a somewhat unexpectedly nice and simple description of $\text{Umb}_{\text{CR}}(E_{a,b})$.

All computations of this paper were done by hand on a computer, which shows $\dim \text{Umb}_{\text{CR}}(E_{a,b}) = 1$.

2. Explicit expression of Cartan’s CR-Invariant \mathfrak{J}

In \mathbb{C}^2 equipped with coordinates $(z, w) = (x + \sqrt{-1}y, u + \sqrt{-1}v)$, consider a connected real-analytic (\mathcal{C}^ω) 3-dimensional hypersurface:

$$M^3 := \{(z, w) \in \mathbb{C}^2 : \rho(z, w, \bar{z}, \bar{w}) = 0\},$$

with $\bar{\rho} = \rho$, and with $d\rho|_M$ never zero. Local or global M , compact or open, bounded or unbounded, can be equally treated. The two vector fields:

$$L := -\rho_w \frac{\partial}{\partial z} + \rho_z \frac{\partial}{\partial w} \quad \text{and} \quad \bar{L} := -\rho_{\bar{w}} \frac{\partial}{\partial \bar{z}} + \rho_{\bar{z}} \frac{\partial}{\partial \bar{w}}$$

generate $T^{1,0}M$ and $T^{0,1}M$.

If $h: \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is a local biholomorphism:

$$(z, w) \mapsto (f(z, w), g(z, w)) =: (z', w'),$$

if $M = \{\rho = 0\}$ and $M' = \{\rho' = 0\}$ are two \mathcal{C}^ω hypersurfaces; if $h(M) \subset M'$, there is a nowhere vanishing function $\mu: M \rightarrow \mathbb{C} \setminus \{0\}$ such that:

$$\mu(z, w, \bar{z}, \bar{w}) \rho(z, w, \bar{z}, \bar{w}) \equiv \rho'(f(z, w), g(z, w), \bar{f}(\bar{z}, \bar{w}), \bar{g}(\bar{z}, \bar{w})),$$

whence in $\mathbb{C}\{z, w, \bar{z}, \bar{w}\}$:

$$\mu \left(-\rho_w \frac{\partial}{\partial z} + \rho_z \frac{\partial}{\partial w} \right) = (f_z g_w - f_w g_z) \left(-\rho'_{w'} \frac{\partial}{\partial z'} + \rho'_{z'} \frac{\partial}{\partial w'} \right).$$

Furthermore, the *Levi determinant*:

$$\begin{aligned} L(\rho) &:= - \begin{vmatrix} 0 & \rho_z & \rho_w \\ \rho_{\bar{z}} & \rho_{z\bar{z}} & \rho_{w\bar{z}} \\ \rho_{\bar{w}} & \rho_{z\bar{w}} & \rho_{w\bar{w}} \end{vmatrix} \\ &= \rho_{\bar{z}}\rho_z\rho_{w\bar{w}} - \rho_{\bar{z}}\rho_w\rho_{z\bar{w}} - \rho_{\bar{w}}\rho_z\rho_{z\bar{w}} + \rho_{\bar{w}}\rho_w\rho_{z\bar{z}}, \end{aligned}$$

enjoys:

$$\mu^3 L(\rho) = (f_z g_w - f_w g_z) (\bar{f}_{\bar{z}} \bar{g}_{\bar{w}} - \bar{f}_{\bar{w}} \bar{g}_{\bar{z}}) L(\rho'). \tag{(on M)}$$

Definition 5. A smooth hypersurface $M^3 \subset \mathbb{C}^2$ is called *Levi nondegenerate* at a point $p \in M$ if:

$$0 \neq L(p).$$

From now on, all M will be assumed smooth and Levi nondegenerate at every point, without further mention.

When $0 \neq \rho_w(p) = \rho_{\bar{w}}(p)$ at a point $p = (z_p, w_p) \in M$, the implicit function theorem represents M as a complex graph:

$$w = \Theta(z, \bar{z}, \bar{w}) \quad \text{or equivalently:} \quad \bar{w} = \bar{\Theta}(\bar{z}, z, w),$$

in terms of a \mathcal{C}^ω defining function Θ . A similar graphed representation exists at points $q = (z_q, w_q) \in M$ at which $0 \neq \rho_z(q) = \rho_{\bar{z}}(q)$.

Differentiating the identity:

$$0 \equiv \rho(z, \Theta(z, \bar{z}, \bar{w}), \bar{z}, \bar{w}), \tag{(in } \mathbb{C}\{z, \bar{z}, \bar{w}\})$$

once with respect to z, \bar{z}, \bar{w} yields:

$$\begin{aligned} 0 &\equiv \rho_z + \Theta_z \rho_w, \\ 0 &\equiv \rho_{\bar{z}} + \Theta_{\bar{z}} \rho_w, \\ 0 &\equiv \rho_{\bar{w}} + \Theta_{\bar{w}} \rho_w, \end{aligned}$$

and next twice with respect to $zz, z\bar{z}, z\bar{w}, \bar{z}\bar{z}, \bar{z}\bar{w}, \bar{w}\bar{w}$ gives:

$$\begin{aligned} 0 &\equiv \rho_{zz} + 2\Theta_z \rho_{z\bar{w}} + \Theta_z \Theta_z \rho_{ww} + \Theta_{zz} \rho_w, \\ 0 &\equiv \rho_{z\bar{z}} + \Theta_z \rho_{z\bar{w}} + \Theta_{\bar{z}} \rho_{z\bar{w}} + \Theta_z \Theta_{\bar{z}} \rho_{ww} + \Theta_{z\bar{z}} \rho_w, \\ 0 &\equiv \rho_{z\bar{w}} + \Theta_z \rho_{w\bar{w}} + \Theta_{\bar{w}} \rho_{z\bar{w}} + \Theta_z \Theta_{\bar{w}} \rho_{ww} + \Theta_{z\bar{w}} \rho_w, \\ 0 &\equiv \rho_{\bar{z}\bar{z}} + 2\Theta_{\bar{z}} \rho_{\bar{z}\bar{w}} + \Theta_{\bar{z}} \Theta_{\bar{z}} \rho_{ww} + \Theta_{\bar{z}\bar{z}} \rho_w, \\ 0 &\equiv \rho_{\bar{z}\bar{w}} + \Theta_{\bar{z}} \rho_{w\bar{w}} + \Theta_{\bar{w}} \rho_{\bar{z}\bar{w}} + \Theta_{\bar{z}} \Theta_{\bar{w}} \rho_{ww} + \Theta_{\bar{z}\bar{w}} \rho_w, \\ 0 &\equiv \rho_{\bar{w}\bar{w}} + 2\Theta_{\bar{w}} \rho_{w\bar{w}} + \Theta_{\bar{w}} \Theta_{\bar{w}} \rho_{ww} + \Theta_{\bar{w}\bar{w}} \rho_w. \end{aligned} \tag{6}$$

It holds that:

$$\{\rho_w \neq 0\} = \{\Theta_{\bar{w}} \neq 0\}. \tag{in M}$$

Definition 7. Call M spherical if it is locally biholomorphic to:

$$S^3 := \{(z, w) \in \mathbb{C}^2 : z\bar{z} + w\bar{w} = 1\}.$$

When M is connected, the principle of analytic continuation guarantees the propagation of this property. Next, set:

$$\Delta := -\Theta_{\bar{w}} \Theta_{z\bar{z}} + \Theta_{\bar{z}} \Theta_{z\bar{w}}.$$

Lemma 8. At a point $p \in \{\Theta_{\bar{w}} \neq 0\}$:

$$M \text{ is Levi nondegenerate at } p \iff \Delta(p) \neq 0.$$

Levi’s nondegeneracy being a biholomorphically invariant feature, spherical M are so since S^3 is.

A characterization of sphericity in terms of some defining function for a hypersurface $M^3 \subset \mathbb{C}^2$ is as follows (cf. also [6, 10,5]). To recall it, set:

$$\square := \frac{\Delta}{-\Theta_{\bar{w}}},$$

and use instead:

$$\begin{aligned} \bar{\mathcal{L}} &:= -\frac{1}{\rho_{\bar{w}}} \bar{L} \\ &= \frac{\partial}{\partial \bar{z}} - \frac{\Theta_{\bar{z}}}{\Theta_{\bar{w}}} \frac{\partial}{\partial \bar{w}}. \end{aligned}$$

Theorem 9. ([7]) At a point $p \in \{\Theta_{\bar{w}} \neq 0\}$, the hypersurface M is spherical if and only if, near p :

$$0 \equiv \frac{1}{\square} \bar{\mathcal{L}} \left(\frac{1}{\square} \bar{\mathcal{L}} \left(\frac{1}{\square} \bar{\mathcal{L}} \left(\frac{1}{\square} \bar{\mathcal{L}} (\Theta_{zz}) \right) \right) \right).$$

Exchanging $z \longleftrightarrow w$ yields a similar formula at points $q \in \{\rho_z \neq 0\}$.

Corollary 10. In $\{\rho_w \neq 0\} = \{\Theta_{\bar{w}} \neq 0\}$, a partly expanded characterization of sphericity is:

$$0 \equiv \frac{\overline{\mathcal{L}}^4(\Theta_{zz})}{\square^4} - 6 \frac{\overline{\mathcal{L}}(\square) \overline{\mathcal{L}}^3(\Theta_{zz})}{\square^5} - 4 \frac{\overline{\mathcal{L}}^2(\square) \overline{\mathcal{L}}^2(\Theta_{zz})}{\square^5} - \frac{\overline{\mathcal{L}}^3(\square) \overline{\mathcal{L}}(\Theta_{zz})}{\square^5} + 15 \frac{[\overline{\mathcal{L}}(\square)]^2 \overline{\mathcal{L}}^2(\Theta_{zz})}{\square^6} + 10 \frac{\overline{\mathcal{L}}(\square) \overline{\mathcal{L}}^2(\square) \overline{\mathcal{L}}(\Theta_{zz})}{\square^6} - 15 \frac{[\overline{\mathcal{L}}(\square)]^3 \overline{\mathcal{L}}(\Theta_{zz})}{\square^7}.$$

Without presenting details, it is known that Cartan’s treatment of the concerned biholomorphic equivalence problem brings a single invariant function:

$$\mathfrak{J}_{\text{Cartan}}^M : M \longrightarrow \mathbb{C},$$

other invariants being (covariant) derivations of it, and that:

$$M \text{ is spherical} \iff 0 \equiv \mathfrak{J}_{\text{Cartan}}^M.$$

Notation 11. For two functions $I_1 : M \longrightarrow \mathbb{C}$ and $I_2 : M \longrightarrow \mathbb{C}$, write:

$$I_2 \doteq I_1,$$

when there is a nowhere vanishing function $\mu : M \longrightarrow \mathbb{C} \setminus \{0\}$ such that:

$$I_2 = \mu I_1.$$

For instance:

$$\mathfrak{J}_{\text{Cartan}}^M \doteq \left(\frac{1}{\square} \overline{\mathcal{L}} \right)^4 (\Theta_{zz}).$$

Now, translate the formula of [Corollary 10](#) to the case where M is given in implicit representation:

$$0 = \rho(z, w, \bar{z}, \bar{w}).$$

Set:

$$H(\rho) := \rho_z \rho_z \rho_{w\bar{w}} - 2 \rho_z \rho_w \rho_{z\bar{w}} + \rho_w \rho_w \rho_{z\bar{z}},$$

with on $\{\rho_w \neq 0\}$:

$$\Theta_{zz} = - \frac{H(\rho)}{\rho_w \rho_w \rho_w}.$$

Remind the *Levi determinant*:

$$L(\rho) := \rho_{\bar{z}} \rho_z \rho_{w\bar{w}} - \rho_{\bar{z}} \rho_w \rho_{z\bar{w}} - \rho_{\bar{w}} \rho_z \rho_{\bar{z}w} + \rho_{\bar{w}} \rho_w \rho_{z\bar{z}},$$

which satisfies on $\{\rho_w \neq 0\}$:

$$L(\rho) \doteq \Delta,$$

i.e. more precisely thanks to [\(6\)](#):

$$L(\rho) = - \rho_w \rho_w \rho_w \Delta.$$

Corollary 12. On $\{\rho_w \neq 0\}$, up to a nowhere vanishing function:

$$\mathfrak{J}_{\text{Cartan}}^M \doteq I_{[w]},$$

where:

$$\begin{aligned}
 I_{[w]} := & 12(\rho_w)^9 \left\{ \left[\frac{L(\rho)}{\rho_w^2} \right]^3 \bar{L}^4 \left(\frac{H(\rho)}{\rho_w^3} \right) - \right. \\
 & - 6 \left[\frac{L(\rho)}{\rho_w^2} \right]^2 \bar{L} \left(\frac{L(\rho)}{\rho_w^2} \right) \bar{L}^3 \left(\frac{H(\rho)}{\rho_w^3} \right) - 4 \left[\frac{L(\rho)}{\rho_w^2} \right]^2 \bar{L}^2 \left(\frac{L(\rho)}{\rho_w^2} \right) \bar{L}^2 \left(\frac{H(\rho)}{\rho_w^3} \right) - \left. \left[\frac{L(\rho)}{\rho_w^2} \right]^2 \bar{L}^3 \left(\frac{L(\rho)}{\rho_w^2} \right) \bar{L} \left(\frac{H(\rho)}{\rho_w^3} \right) + \right. \\
 & + 15 \frac{L(\rho)}{\rho_w^2} \left[\bar{L} \left(\frac{L(\rho)}{\rho_w^2} \right) \right]^2 \bar{L}^2 \left(\frac{H(\rho)}{\rho_w^3} \right) + 10 \frac{L(\rho)}{\rho_w^2} \bar{L} \left(\frac{L(\rho)}{\rho_w^2} \right) \bar{L}^2 \left(\frac{L(\rho)}{\rho_w^2} \right) \bar{L} \left(\frac{H(\rho)}{\rho_w^3} \right) - \\
 & \left. - 15 \left[\bar{L} \left(\frac{L(\rho)}{\rho_w^2} \right) \right]^3 \bar{L} \left(\frac{H(\rho)}{\rho_w^3} \right) \right\}.
 \end{aligned}$$

Furthermore, exchanging $z \longleftrightarrow w$, there is an exact formal coincidence:

$$I_{[z]} = I_{[w]},$$

although this is not needed here.

3. Pullback to an exceptional curve on an ellipsoid

To prove [Theorem 4](#), it suffices to verify that:

$$0 \stackrel{?}{=} \gamma^*(I_{[w]})(\theta). \tag{\forall \theta \in \mathbb{R}}$$

Drop the factor $12(\rho_w)^9 \doteq 1$, and call $T_1, T_2, T_3, T_4, T_5, T_6, T_7$ the seven concerned terms, so that the goal becomes:

$$0 \stackrel{?}{=} \gamma^*(T_1) + \gamma^*(T_2) + \gamma^*(T_3) + \gamma^*(T_4) + \gamma^*(T_5) + \gamma^*(T_6) + \gamma^*(T_7).$$

Hand computations provide formulas of the shape:

$$T_1 = \frac{1}{8} \sqrt{-1} (a - 1) \frac{N_1}{D},$$

$$T_2 = \frac{3}{4} \sqrt{-1} (a - 1) \frac{N_2}{D},$$

$$T_3 = \frac{1}{2} \sqrt{-1} (a - 1) \frac{N_3}{D},$$

$$T_4 = \frac{1}{8} \sqrt{-1} (a - 1) \frac{N_4}{D},$$

$$T_5 = \frac{15}{8} \sqrt{-1} (a - 1) \frac{N_5}{D},$$

$$T_6 = \frac{5}{4} \sqrt{-1} (a - 1) \frac{N_6}{D},$$

$$T_7 = \frac{15}{8} \sqrt{-1} (a - 1) \frac{N_7}{D},$$

with, in denominator place:

$$D := \left(\sqrt{a} \cos \theta - \sqrt{-1} \sqrt{b} \sin \theta \right)^8 (ab - 1) \left(\frac{b - 1}{ab - 1} \right)^{\frac{11}{2}},$$

with numerator 1:

$$\begin{aligned}
 N_1 := & \cos^7 \theta \left[499 a^{9/2} b^3 + 625 a^{9/2} b^2 - 233 a^{7/2} b^3 + 205 a^{9/2} b - 631 a^{7/2} b^2 + 15 a^{9/2} - 415 a^{7/2} b - 65 a^{7/2} \right] \\
 & + \sqrt{-1} \cos^6 \theta \sin \theta \left[2887 a^4 b^{7/2} + 4401 a^4 b^{5/2} - 1297 a^3 b^{7/2} + 1905 a^4 b^{3/2} - 4059 a^3 b^{5/2} + 215 a^4 b^{1/2} - 3327 a^3 b^{3/2} - 725 a^3 b^{1/2} \right] \\
 & + \cos^5 \theta \sin^2 \theta \left[-7023 a^{7/2} b^4 - 13021 a^{7/2} b^3 + 3013 a^{5/2} b^4 - 7105 a^{7/2} b^2 + 11011 a^{5/2} b^3 - 1075 a^{7/2} b + 11059 a^{5/2} b^2 + 3141 a^{5/2} b \right] \\
 & + \sqrt{-1} \cos^4 \theta \sin^3 \theta \left[-9267 a^3 b^{9/2} - 20989 a^3 b^{7/2} + 3757 a^2 b^{9/2} - 14101 a^3 b^{5/2} + 16279 a^2 b^{7/2} - 2683 a^3 b^{3/2} + 19891 a^2 b^{5/2} + 7113 a^2 b^{3/2} \right] \\
 & + \cos^3 \theta \sin^4 \theta \left[7113 a^{5/2} b^5 + 19891 a^{5/2} b^4 - 2683 a^{3/2} b^5 + 16279 a^{5/2} b^3 - 14101 a^{3/2} b^4 + 3757 a^{5/2} b^2 - 20989 a^{3/2} b^3 - 9267 a^{3/2} b^2 \right] \\
 & + \sqrt{-1} \cos^2 \theta \sin^5 \theta \left[3141 a^2 b^{11/2} + 11059 a^2 b^{9/2} - 1075 a b^{11/2} + 11011 a^2 b^{7/2} - 7105 a b^{9/2} + 3013 a^2 b^{5/2} - 13021 a b^{7/2} - 7023 a b^{5/2} \right] \\
 & + \cos^1 \theta \sin^6 \theta \left[-725 a^3 b^6 - 3327 a^3 b^5 + 215 a^{1/2} b^6 - 4059 a^{3/2} b^4 + 1905 a^{1/2} b^5 - 1297 a^{3/2} b^3 + 4401 a^{1/2} b^4 + 2287 a^{1/2} b^3 \right] \\
 & + \sqrt{-1} \sin^7 \theta \left[-65 a b^{13/2} - 415 a b^{11/2} + 15 b^{13/2} - 631 a b^{9/2} + 205 b^{11/2} - 233 a b^{7/2} + 625 b^{9/2} + 499 b^{7/2} \right],
 \end{aligned}$$

and with similar $N_2, N_3, N_4, N_5, N_6, N_7$, the expressions of which can be found in the announcement arxiv.org/abs/ of this publication.

End of proof of Theorem 4. The sum:

$$\frac{1}{8} N_1(\theta) + \frac{3}{4} N_2(\theta) + \frac{1}{2} N_3(\theta) + \frac{1}{8} N_4(\theta) + \frac{15}{8} N_5(\theta) + \frac{5}{4} N_6(\theta) + \frac{15}{8} N_7(\theta) = 0,$$

is checked on a computer to be identically null.

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