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Number theory

# Powerful numbers in $(1^\ell + q^\ell)(2^\ell + q^\ell)\cdots(n^\ell + q^\ell)$

## *Nombres de la forme $(1^\ell + q^\ell)(2^\ell + q^\ell)\ldots(n^\ell + q^\ell)$ qui ne sont pas puissants*

Quan-Hui Yang [a], Qing-Qing Zhao [b]

[a] *School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China*
[b] *Jincheng College, Nanjing University of Aeronautics and Astronautics, Nanjing 211156, China*

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A B S T R A C T

Let $q$ be a positive integer. Recently, Niu and Liu proved that, if $n \geq \max\{q, 1198 - q\}$, then the product $(1^3+q^3)(2^3+q^3)\cdots(n^3+q^3)$ is not a powerful number. In this note, we prove (1) that, for any odd prime power $\ell$ and $n \geq \max\{q, 11 - q\}$, the product $(1^\ell+q^\ell)(2^\ell+q^\ell)\cdots(n^\ell+q^\ell)$ is not a powerful number, and (2) that, for any positive odd integer $\ell$, there exists an integer $N_{q,\ell}$ such that, for any positive integer $n \geq N_{q,\ell}$, the product $(1^\ell+q^\ell)(2^\ell+q^\ell)\cdots(n^\ell+q^\ell)$ is not a powerful number.



R É S U M É

Soit $q$ un entier positif. Récemment, Niu et Liu ont montré que, si $n \geq \max(q, 1198 - q)$, alors le produit $(1^3+q^3)(2^3+q^3)\ldots(n^3+q^3)$ n'est pas un nombre puissant. Dans cette Note, nous montrons : (1) que le produit $(1^\ell+q^\ell)(2^\ell+q^\ell)\ldots(n^\ell+q^\ell)$ n'est pas un nombre puissant pour toute puissance $\ell$ d'un nombre premier impair et $n \geq \max(q, 11-q)$ ; (2) que, pour tout nombre impair positif $\ell$, il existe un entier $N_{q,\ell}$ tel que pour tout entier $n \geq N_{q,\ell}$, le produit $(1^\ell+q^\ell)(2^\ell+q^\ell)\ldots(n^\ell+q^\ell)$ ne soit pas un nombre puissant.



## 1. Introduction

A positive integer $t$ is called a powerful number if $t > 1$ and $p^2 \mid t$ for every prime divisor $p$ of $t$ (see [8]). In 2008, Cilleruelo [4] proved that, for any integer $n > 3$, the product $(1^2+1)(2^2+1)\cdots(n^2+1)$ is not a square. Amdeberhan, Medina and Moll [1] claimed that, if $n > 12$ and $\ell$ is an odd prime, then $(1^\ell+1)(2^\ell+1)\cdots(n^\ell+1)$ is not a square. Gürel and Kisisel [11] confirmed the claim for $\ell = 3$, while Zhang and Wang [18] confirmed the claim for any prime $\ell \geq 5$. In fact, they proved that $(1^\ell+1)(2^\ell+1)\cdots(n^\ell+1)$ is not a powerful number. Later, Chen et al. [2,3] proved that, if $\ell$ is an odd

*E-mail addresses:* yangquanhui01@163.com (Q.-H. Yang), zhaoqingqing116@163.com (Q.-Q. Zhao).

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integer with at most two distinct prime factors, then $(1^\ell+1)(2^\ell+1)\cdots(n^\ell+1)$ is not a powerful number. There are many related results on this topic, one can refer to [5,7,9,10,12,15–17].

Recently, Niu and Liu [13] extended the work of Gürel and Kisisel and proved the following theorem.

**Theorem A.** *For any positive integers $q$ and $n \ge \max\{q, 1198-q\}$, the product $(1^3+q^3)(2^3+q^3)\cdots(n^3+q^3)$ is not a powerful number.*

In this paper, we generalize the results of Niu and Liu in the following theorem.

**Theorem 1.** *Let $q$ be a positive integer and $\ell$ be an odd prime power. For any integer $n \ge \max\{q, 11-q\}$, the product $(1^\ell+q^\ell)(2^\ell+q^\ell)\cdots(n^\ell+q^\ell)$ is not a powerful number.*

The next theorem is a generalization of Theorem 2 in [3].

**Theorem 2.** *For any positive integer $q$ and odd positive integer $\ell$, there exists an integer $N_{q,\ell}$ such that, for any positive integer $n \ge N_{q,\ell}$, the product $(1^\ell+q^\ell)(2^\ell+q^\ell)\cdots(n^\ell+q^\ell)$ is not a powerful number.*

## 2. Preliminary lemmas

**Lemma 1.** *Let $p$ be a prime and $q, \ell$ be positive integers with $2 \nmid \ell$ and $\gcd(\ell, p-1)=1$. Then the congruence equation $x^\ell+q^\ell \equiv 0 \pmod p$ has only one solution $x \equiv -q \pmod p$.*

**Proof.** If $p \mid q$, then the congruence equation has only one solution $x \equiv 0 \equiv -q \pmod p$; the result is true. Now we assume $p \nmid q$. Let $g$ be a primitive root modulo $p$. Then $g^{\frac{p-1}{2}} \equiv -1 \pmod p$. Let $x \equiv g^t \pmod p$, $q \equiv g^m \pmod p$, where $0 \le t, m \le p-2$. Then the congruence equation $x^\ell+q^\ell \equiv 0 \pmod p$ is equivalent to $g^{t\ell} \equiv g^{\frac{p-1}{2}+m\ell} \pmod p$, that is, $\ell t \equiv \frac{p-1}{2}+m\ell \pmod{p-1}$. Since $(\ell, p-1)=1$, it follows that $t$ has only one solution. Hence $x$ also has only one solution. By $2 \nmid \ell$, it is easy to see that $x \equiv -q \pmod p$ is the only solution. □

**Corollary 1.** *Let $q$ be a positive integer and $\ell = k^s$, where $k$ is an odd prime and $s$ is a positive integer. If $p$ is a prime with $k \nmid p-1$, then the congruence equation $x^\ell+q^\ell \equiv 0 \pmod p$ has only one solution $x \equiv -q \pmod p$.*

For a nonzero integer $m$ and a prime $p$, let $\nu_p(m)$ denote the smallest nonnegative integer $k$ such that $p^k \mid m$ and $p^{k+1} \nmid m$.

**Lemma 2.** *Let $\ell = k^s$ be an odd prime power, $p$ be a prime, and $q$ be a positive integer such that $p > q$, $p \ne k$ and $k \nmid p-1$. If $p-q \le n \le 2p-q-1$, then the product $(1^\ell+q^\ell)(2^\ell+q^\ell)\cdots(n^\ell+q^\ell)$ is not a powerful number.*

**Proof.** By Corollary 1, the smallest two positive integers $x$ satisfying $x^\ell+q^\ell \equiv 0 \pmod p$ are $p-q$ and $2p-q$. Noting that $p > q$ and $p \ne k$, we have $p^2 \nmid ((p-q)^\ell+q^\ell)$. Hence, if $p-q \le n \le 2p-q-1$, then

$$\nu_p((1^\ell+q^\ell)(2^\ell+q^\ell)\cdots(n^\ell+q^\ell)) = \nu_p((p-q)^\ell+q^\ell)=1,$$

and so the product $(1^\ell+q^\ell)(2^\ell+q^\ell)\cdots(n^\ell+q^\ell)$ is not a powerful number. □

For any positive integers $m$ and $k$, let

$$P(m) = \{p : \ p \text{ is a prime}, \ \frac{m+1}{2} < p \le m+1\},$$
$$P(m; k, 1) = \{p : \ p \text{ is a prime}, \ \frac{m+1}{2} < p \le m+1, \ p \equiv 1 \pmod k\}.$$

**Lemma 3.** *(See [18, Lemma 2.3].) If $m \ne 1, 3, 5$ or $9$, then $|P(m)| \ge 2$.*

**Lemma 4.** *(See [18, Lemma 2.4].) If $m \ge 4k$, where $k$ is an odd prime with $k \ge 5$, then $|P(m)| > |P(m; k, 1)|$.*

**Lemma 5.** *(See [3, Lemma 2].) Let $m$ be an integer with $m \ge 4$ and $m \ne 9$. Then, there is always an odd prime $p \in P(m)$ with $p \equiv 2 \pmod 3$.*

The following lemma is a powerful lemma for solving exponential Diophantine equations. It is pretty well known in the Olympiad folklore (see, e.g., [6]), though its origins are hard to trace.

**Lemma 6** *(Lifting the exponent lemma). Let $x$, $y$ be two integers, $\ell$ be an odd positive integer, and $p$ be an odd prime such that $p \mid x+y$ and none of $x$ and $y$ is divisible by $p$. We have:*

$$\nu_p(x^\ell + y^\ell) = \nu_p(x+y) + \nu_p(\ell).$$

## 3. Proofs of Theorems 1 and 2

**Proof of Theorem 1.** By Lemma 2, it is enough to prove that there exists a prime $p > q$ with $p \neq k$ and $k \nmid p-1$ such that $p-q \le n \le 2p-q-1$. It is easy to see that $p-q \le n \le 2p-q-1$ is equivalent to $\frac{n+q}{2} < p \le n+q$. Since $n \ge q$, it follows that $p > \frac{n+q}{2} \ge q$. Hence we need to prove that there exists a prime $p \neq k$ with $p \not\equiv 1 \pmod k$ such that $\frac{n+q}{2} < p \le n+q$.

By $n \ge 11-q$, we have $n+q-1 \ge 10$. Hence, by Lemma 3, we obtain

$$|P(n+q-1)| \ge 2. \tag{1}$$

Suppose that $k=3$. Since $n+q-1 \ge 10$, by Lemma 5, there exists an odd prime $p$ with $p \equiv 2 \pmod 3$ such that $\frac{n+q}{2} < p \le n+q$. It is clear that $p \neq 3$.

Now we assume $k \ge 5$.

Case 1. $n < 2k-q+1$. If $p \in P(n+q-1; k, 1)$, then $p \equiv 1 \pmod k$ and $p \ge 2k+1 > n+q$, which is a contradiction. Hence, $|P(n+q-1; k, 1)| = 0$ in this case. Therefore, by (1), there exists at least one prime $p \neq k$ with $p \not\equiv 1 \pmod k$ such that $\frac{n+q}{2} < p \le n+q$.

Case 2. $2k-q+1 \le n < 4k-q+1$. Suppose that $|P(n+q-1; k, 1)| = |P(n+q-1)|$. Then $|P(n+q-1; k, 1)| \ge 2$. Hence, there exist two primes $p_1$ and $p_2$ satisfying $p_1 < p_2 \le n+q < 4k+1$ and $p_1 \equiv p_2 \equiv 1 \pmod k$. It follows that $p_1 \ge 2k+1$ and $p_2 \ge 4k+1$, which is a contradiction. Hence $|P(n+q-1)| > |P(n+q-1; k, 1)|$. Therefore, there exists a prime $p$ with $p \not\equiv 1 \pmod k$ such that $\frac{n+q}{2} < p \le n+q$. Clearly, $p > \frac{n+q}{2} \ge \frac{2k+1}{2} > k$.

Case 3. $n \ge 4k-q+1$. It follows that $n+q-1 \ge 4k$. By Lemma 4, there exists a prime $p$ with $p \not\equiv 1 \pmod k$ such that $\frac{n+q}{2} < p \le n+q$. Clearly, $p > \frac{n+q}{2} \ge \frac{4k+1}{2} > k$.

By three cases above, there exists a prime $p \neq k$ with $p \not\equiv 1 \pmod k$ such that $\frac{n+q}{2} < p \le n+q$.

Therefore, the product $(1^\ell + q^\ell)(2^\ell + q^\ell) \cdots (n^\ell + q^\ell)$ is not a powerful number. □

**Proof of Theorem 2.** By Dirichlet's theorem on arithmetic progressions (see [14, p. 285]), there exists an integer $N_{q,\ell} > q$ such that for any integer $n \ge N_{q,\ell}$, there is an odd prime $p \in P(n+q-1)$ with $p \equiv 2 \pmod \ell$. Clearly, $\frac{n+q+1}{2} \le p \le n+q$ and $\gcd(p-1, \ell) = 1$. Suppose that the product $(1^\ell + q^\ell)(2^\ell + q^\ell) \cdots (n^\ell + q^\ell)$ is a powerful number. Noting that $\frac{n+q+1}{2} \le p \le n+q$ and $n \ge N_{q,\ell} > q$, we have $p \ge \frac{n+q+1}{2} \ge q+1$, and so $\nu_p(\prod_{a=1}^n (a+q)) = 1$. Hence, by

$$\prod_{a=1}^n (a^\ell + q^\ell) = \prod_{a=1}^n (a+q) \cdot \prod_{a=1}^n \frac{a^\ell + q^\ell}{a+q},$$

it follows that $p \mid \frac{a^\ell + q^\ell}{a+q}$ for some $1 \le a \le n$. Since $p \mid a^\ell + q^\ell$, $2 \nmid \ell$ and $\gcd(p-1, \ell) = 1$, by Lemma 1, we have $p \mid a+q$. On the other hand, by $p \equiv 2 \pmod \ell$ and $p \ge q+1$, we have $p \nmid \ell$ and $p \nmid q$, and so $p \nmid a$. Hence, by Lemma 6, we have

$$\nu_p(a^\ell + q^\ell) = \nu_p(a+q) + \nu_p(\ell) = \nu_p(a+q).$$

That is, $p \nmid \frac{a^\ell + q^\ell}{a+q}$, a contradiction.

This completes the proof of Theorem 2. □

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