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On the ordinariness of coverings of stable curves

De l'ordinarité des revêtements de courbes stables

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A R T I C L E   I N F O

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A B S T R A C T

In the present paper, we study the ordinariness of coverings of stable curves. Let $f : Y \to X$ be a morphism of stable curves over a discrete valuation ring R with algebraically closed residue field of characteristic $p > 0$. Write S for $\operatorname{Spec} R$ and η (resp. s) for the generic point (resp. closed point) of S. Suppose that the generic fiber X_η of X is smooth over η, that the morphism $f_\eta : Y_\eta \to X_\eta$ over η on the generic fiber induced by f is a Galois étale covering (hence Y_η is smooth over η too) whose Galois group is a solvable group G, that the genus of the normalization of each irreducible component of the special fiber X_s is ≥ 2, and that Y_s is ordinary. Then we have that the morphism $f_s : Y_s \to X_s$ over s induced by f is an admissible covering. This result extends a result of M. Raynaud concerning the ordinariness of coverings to the case where X_s is a stable curve. If, moreover, we suppose that G is a p-group, and that the p-rank of the normalization of each irreducible component of X_s is ≥ 2, we can give a numerical criterion for the admissibility of f_s.

R É S U M É

Dans la présente Note, nous étudions l'ordinarité des revêtements de courbes stables. Soit $f : Y \to X$ un morphisme de courbes stables sur un anneau de valuation discrète R, dont le corps résiduel est algébriquement clos, de caractéristique $p > 0$. Notons S pour $\operatorname{Spec}(R)$ et η (resp. s) le point générique (resp. le point fermé) de S. Supposons que la fibre générique X_η de X est lisse au-dessus de η, que le morphisme $f_\eta : Y_\eta \to X_\eta$ des fibres génériques induit par f au-dessus de η soit un revêtement étale galoisien (et donc Y_η est aussi lisse au-dessus de η), dont le groupe de Galois G est résoluble, que le genre des normalisations des composantes irréductibles de la fibre spéciale X_s soit au moins 2 et que Y_s soit ordinaire. Alors, le morphisme $f_s : Y_s \to X_s$ induit par f au-dessus de s est un revêtement admissible. Ce résultat étend un énoncé de M. Raynaud sur l'ordinarité des revêtements lorsque X_s est une courbe stable. Si, de plus, on suppose que G est un p-groupe et que le p-rang de la normalisation de chaque composante irréductible de X_s est au moins 2, nous pouvons donner un critère numérique pour l'admissibilité de f_s.

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0. Introduction

Let R be a discrete valuation ring with an algebraically closed residue field k of characteristic $p > 0$, and K the quotient field. We use the notation S to denote $\operatorname{Spec} R$. Write η and s for the generic point of S and the closed point of S corresponding to the natural morphisms $\operatorname{Spec} K \to S$ and $\operatorname{Spec} k \to S$, respectively. Let G be a finite group, and let X be a stable curve of genus $g(X)$ (in the present paper, the genus of a curve means the arithmetic genus of the curve) over S. Write X_η and X_s for the generic fiber of X and the special fiber of X, respectively. Moreover, we suppose that X_η is smooth over η.

We are interested in understanding the reduction of an étale covering of X_η. Let Y_η be a smooth, geometrically connected curve over η, and $f_\eta : Y_\eta \to X_\eta$ a Galois étale covering over η whose Galois group is G. By replacing S by a finite extension of S, we have that Y_η admits a stable model over S, and f_η extends to a unique G-stable covering $f : Y \to X$ over S (cf. Definition 1.5 and Remark 1.5.1). In the present paper, we focus on a geometric invariant $\sigma(Y_s)$ of the special fiber Y_s, which is called the p-rank of Y_s (cf. Definition 1.2).

Let us recall some known results concerning the p-rank of the special fiber Y_s. Let x be a closed point of X_s, and G an arbitrary p-group. M. Raynaud (cf. [7, Théorème 1]) proved that, if x is a smooth point, the p-rank of $f^{-1}(x)$ is equal to 0 (note that $f^{-1}(x)$ is not a finite set in general). Afterwards, M. Saïdi (cf. [10, Theorem 1 and Proposition 1]) treated the case where x is a singular point of X_s. Saïdi obtained an explicit formula and a bound for the p-rank of $f^{-1}(x)$ under the assumption that G is a cyclic p-group. Recently, the author generalized the formula for the p-rank of $f^{-1}(x)$ to the case where G is an arbitrary p-group and obtained a bound for the p-rank of $f^{-1}(x)$ in the case where G is an arbitrary abelian p-group (cf. [14, Theorem 4.8], [15, Theorem 3.4]). On the other hand, if G is an arbitrary finite group, and X_s is smooth over s, Raynaud proved that, if the morphism f_s on special fibers induced by f is not an étale covering, then Y_s is not ordinary (cf. [8, Proposition 3]).

In the present paper, we study the ordinariness of stable coverings. Our main theorem is as follows, see also Theorem 2.6.

Theorem 0.1. *Let Y be a stable curve over S and $f : Y \to X$ a $\mathbb{Z}/p\mathbb{Z}$-stable covering over S. Suppose that the genus of the normalization of each irreducible component of X_s is ≥ 2, and the morphism $f_s : Y_s \to X_s$ over s induced by f is p-new-ordinary (cf. Definition 2.4). Then f_s is an admissible covering (cf. Definition 1.1). If, moreover, we suppose that the p-rank of the normalization of each irreducible component of X_s is ≥ 2, then f_s is an admissible covering if and only if*

$$\sigma(Y_s) - 1 = p(\sigma(X_s) - 1).$$

As a corollary, we generalize the main result of [8] to the case where X_s is a stable curve, and G is a solvable group; moreover, if G is a p-group, we obtain a numerical criterion for the admissibility of G-stable coverings as follows, see also Corollary 2.7.

Corollary 0.2. *Let G be a finite solvable group, Y a stable curve over S, and $f : Y \to X$ a G-stable covering over S. Suppose that the genus of the normalization of each irreducible component of X_s is ≥ 2, and that Y_s is ordinary (i.e. $\sigma(Y_s) = g(Y_s) = (\#G)(g(X_s) - 1) + 1$). Then the morphism $f_s : Y_s \to X_s$ over s induced by f is an admissible covering. Moreover, suppose that the p-rank of the normalization of each irreducible component of X_s is ≥ 2, and that G is a p-group. Then the morphism $f_s : Y_s \to X_s$ over s induced by f is an admissible covering if and only if*

$$\sigma(Y_s) - 1 = (\#G)(\sigma(X_s) - 1).$$

Remark 0.2.1. Suppose that X_s is ordinary, and that f_s is an admissible covering over s. If G is not a p-group, then Y_s is not ordinary in general.

Finally, we would like to mention that Saïdi extended the main result of [8] to the case where $f_\eta : Y_\eta \to X_\eta$ is a Galois covering over η (cf. [11, Theorem]). More precisely, Saïdi proved the following result: let X be a smooth stable curve over S and $f : Y \to X$ a morphism of stable curves over S; suppose that $\operatorname{char}(k) = p > 0$, and $\eta : Y_\eta \to X_\eta$ is a Galois covering whose Galois group is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ (i.e. the extension of function fields $K(Y_\eta)/K(X_\eta)$ induced by f_η is a Galois extension whose Galois group is isomorphic to $\mathbb{Z}/p\mathbb{Z}$). Saïdi proved that, if $f_s : Y_s \to X_s$ is not generically étale, then Y_s is not ordinary. Note that, if $\operatorname{char}(K) = 0$ and $\operatorname{char}(k) = p > 0$, then this result follows immediately from [7, Théorème 1′] (i.e. a tame version of [7, Théorème 1]).

1. Preliminaries

In this section, we give some definitions that will be used in the present paper.

Definition 1.1. Let C_1 and C_2 be two semi-stable curves over an algebraically closed field l and $\phi : C_2 \to C_1$ a morphism of semi-stable curves over $\operatorname{Spec} l$.

We shall call ϕ a **Galois admissible covering** over Spec l (or Galois admissible covering for short) if the following conditions hold: (i) there exists a finite group $G \subseteq \mathrm{Aut}_k(C_2)$ such that $C_2/G = C_1$, and ϕ is equal to the quotient morphism $C_2 \to C_2/G$; (ii) for each $c_2 \in C_2^{\mathrm{sm}}$, ϕ is étale at c_2, where $(-)^{\mathrm{sm}}$ denotes the smooth locus of $(-)$; (iii) for any $c_2 \in C_2^{\mathrm{sing}}$, the image $\phi(c_2)$ is contained in C_1^{sing}, where $(-)^{\mathrm{sing}}$ denotes the singular locus of $(-)$; (iv) for each $c_2 \in C_2^{\mathrm{sing}}$, the local morphism between two nodes (cf. (iii)) induced by ϕ may be described as follows:

$$\begin{array}{ccc} \hat{\mathcal{O}}_{C_1,\phi(c_2)} \cong l[[u,v]]/uv & \to & \hat{\mathcal{O}}_{C_2,c_2} \cong l[[s,t]]/st \\ u & \mapsto & s^n \\ v & \mapsto & t^n, \end{array}$$

where $(n, \mathrm{char}(l)) = 1$ if $\mathrm{char}(l) = p > 0$; moreover, write $D_{c_2} \subseteq G$ for the decomposition group of c_2; then $\tau(s) = \zeta_{\#D_{c_2}} s$ and $\tau(t) = \zeta_{\#D_{c_2}}^{-1} t$ for each $\tau \in D_{c_2}$, where $\zeta_{\#D_{c_2}}$ is a primitive $\#D_{c_2}$-th root of unit.

We shall call ϕ an **admissible covering** if there exists a morphism of stable curves $\phi' : C_2' \to C_2$ over Spec l such that the composite morphism $\phi \circ \phi' : C_2' \to C_1$ is a Galois admissible covering over Spec l.

For more details on admissible coverings and the admissible fundamental groups for (pointed) semi-stable curves, see [5,6].

Remark 1.1.1. Note that, if C_2 is smooth over l, then the definition of admissible coverings implies that ϕ is an étale covering.

Definition 1.2. Let C be a proper algebraic curve over an algebraically closed field of characteristic $p > 0$. We define the ***p*-rank** $\sigma(C)$ of C to be

$$\sigma(C) := \dim_{\mathbb{F}_p} \mathrm{H}^1_{\text{ét}}(C, \mathbb{F}_p).$$

Moreover, let C' be a Noetherian scheme of dimension 0 over an algebraically closed field of characteristic $p > 0$. Then we define the p-rank of C' to be $\sigma(C') = 0$.

Remark 1.2.1. Suppose that C is a semi-stable curve over an algebraically closed field of characteristic $p > 0$. Write Γ_C for the dual graph of C, $v(\Gamma_C)$ for the set of vertices of Γ_C, C_v for the irreducible component of C corresponding to $v \in v(\Gamma_C)$, and $\widetilde{C_v}$ for the normalization of C_v, respectively. Then it is easy to prove that the p-rank $\sigma(C)$ of C is equal to

$$\sum_{v \in v(\Gamma_C)} \sigma(\widetilde{C_v}) + \mathrm{rank}(\mathrm{H}^1(\Gamma_C, \mathbb{Z})),$$

where $\mathrm{rank}(-)$ denotes the rank of $(-)$ as a free $\mathbb{Z}$-module.

Definition 1.3. Let C be a semi-stable curve of genus $g(C)$ over an algebraically closed field of characteristic $p > 0$. We shall call C **ordinary** if $\sigma(C) = g(C)$. Note that Remark 1.2.1 implies that C is ordinary if and only if $\widetilde{C_v}$ is ordinary for each $v \in v(\Gamma_C)$.

Definition 1.4. Let $\psi : C_2 \to C_1$ be a Galois covering (possibly ramified) of smooth projective curves over an algebraically closed field of characteristic $p > 0$, whose Galois group is a finite p-group G. Write $g(C_1)$ and $g(C_2)$ for the genera of C_1 and C_2, respectively. We shall call ψ ***p*-new-ordinary** if $g(C_2) - \sigma(C_2) = (\#G)(g(C_1) - \sigma(C_1))$, where $\#(-)$ denotes the cardinality of $(-)$.

Remark 1.4.1. Note that, if C_1 is ordinary, then ψ is p-new-ordinary if and only if C_2 is ordinary.

Remark 1.4.2. For any closed point $c_2 \in C_2$, write e_{c_2} for the ramification index of ψ at c_2 and δ_{c_2} for the degree of the different of ψ at c_2. Then the genus and the p-rank of C_2 can be calculated by using the Riemann–Hurwitz formula

$$2g(C_2) - 2 = (\#G)(2g(C_1) - 2) + \sum_{c_2} \delta_{c_2}$$

and the Deuring–Shafarevich formula (cf. [2, p35], [1, Theorem 3.1])

$$\sigma(C_2) - 1 = (\#G)(\sigma(C_1) - 1) + \sum_{c_2} e_{c_2},$$

respectively. Thus, we have

$$g(C_2) - \sigma(C_2) - (\#G)(g(C_1) - \sigma(C_1)) = \sum_{c_2} (\delta_{c_2} - 2(e_{c_2} - 1))/2.$$

Let $I_{c_2} \subseteq G$ be the inertia group of c_2 and $I_{c_2,j}$ the j-th ramification group of c_2. Since G is a p-group, we obtain that $I_{c_2} = I_{c_2,0} = I_{c_2,1}$. Moreover, we have

$$\delta_{c_2} = \sum_{j \geq 0} (\#I_{c_2,j} - 1) = 2(\#I_{c_2} - 1) + \sum_{j \geq 2} (\#I_{c_2,j} - 1).$$

Thus, ψ is p-new-ordinary if and only if $\delta_{c_2} = 2(e_{c_2} - 1)$ (i.e. $I_{c_2,j}$ are *trivial* for all $j \geq 2$ and for all $c_2 \in C_2$).

From now on, we fix some notations. Let R be a discrete valuation ring with algebraically closed residue field k of characteristic $p > 0$, K the quotient field of R, and $\overline{K}$ an algebraic closure of K. We use the notation S to denote the spectrum of R. Write η, $\overline{\eta}$ and s for the generic point of S, the geometric generic point of S, and the closed point of S corresponding to the natural morphisms $\mathrm{Spec}\, K \to S$, $\mathrm{Spec}\, \overline{K} \to S$, and $\mathrm{Spec}\, k \to S$, respectively. Let X be a semi-stable curve over S of genus $g_X \geq 2$. Write $X_\eta := X \times_S \eta$ for the generic fiber of X, $X_{\overline{\eta}} := X \times_S \overline{\eta}$ for the geometric generic fiber of X, and $X_s := X \times_S s$ for the special fiber of X, respectively. Moreover, we suppose that X_η is smooth over η.

Definition 1.5. Let Y be a *stable curve* over S, $f : Y \to X$ a morphism of semi-stable curves over S, and G a finite group. We shall call f a G**-semi-stable covering** over S if the morphism $f_\eta : Y_\eta \to X_\eta$ over η induced by f on generic fibers is a Galois étale covering whose Galois group is isomorphic to G. We shall call f a G**-stable covering** over S if f is a G-semi-stable covering over S, and X is a stable curve over S.

Remark 1.5.1. Suppose that X is a stable curve over S. Let $W_\eta \to X_\eta$ be any geometrically connected Galois étale covering over η whose Galois group is G. [4, Proposition 4.4 (a)] implies that, by replacing S by a finite extension of S, the morphism $W_\eta \to X_\eta$ may extend to a G-stable covering over S.

Remark 1.5.2. Let Y be a stable curve over S, $f : Y \to X$ a G-semi-stable covering over S, and y any closed point of Y. Then f induces a morphism $f_y : \mathrm{Spec}\, \widehat{\mathcal{O}}_{Y,y} \to \mathrm{Spec}\, \widehat{\mathcal{O}}_{X,f(y)}$ over S. Suppose that $f_s : Y_s \to X_s$ over s induced by f is generically étale. We claim that f is an *admissible covering*.

First, we prove that f is a finite morphism. Let x be any closed point of X. If x is a smooth point, then Zariski–Nagata's purity theorem implies f_s is étale over x. If x is a singular point of X_s, then Zariski–Nagata's purity theorem and [12, Lemma 2.1 (iii)] imply that $f^{-1}(x)$ is a set of singular points of Y_s. Thus, f is a finite morphism.

Second, we prove that f_s is an admissible covering. If y is a smooth point, then $f(y) \in X$ is a smooth point too (cf. [9, Lemme 6.3.5] or [13, Lemma 2.1]). Then Zariski–Nagata's purity theorem implies that the morphism f_y is étale. If y is a singular point of Y_s, then $f(y) \in X$ is a singular point of X_s too (cf. [9, Lemme 6.3.5] or [13, Lemma 2.1]). Then Zariski–Nagata's purity theorem and [12, Lemma 2.1 (iii)] also imply that the morphism of local rings $\widehat{\mathcal{O}}_{X_s,f(y)} \to \widehat{\mathcal{O}}_{Y_s,y}$ induced by f_y satisfies the condition (iv) of Definition 1.1.

Thus, we have f_s is a Galois admissible covering over s if and only if f_s is generically étale.

Definition 1.6. Let Y be a stable curve over S and $f : Y \to X$ a G-semi-stable covering over S. Suppose that the morphism $f_s : Y_s \to X_s$ on special fibers induced by f is not finite. A closed point $x \in X$ is called a **vertical point** associated with f, or for simplicity, a vertical point when there is no fear of confusion, if $\dim(f^{-1}(x)) = 1$. The inverse image $f^{-1}(x)$ is called the **vertical fiber** associated with x.

Remark 1.6.1. Suppose that R has mixed characteristic, and k is an algebraic closure of a finite field. Moreover, suppose that X is a stable curve over R. Then A. Tamagawa proved that, for any closed point x, after replacing S by a finite extension of S, there exists a finite group G and a G-stable covering $f : Y \to X$ over S such that x is a vertical point associated with f (cf. [12, Theorem 0.2 (v)]).

Next, we recall some results concerning the p-ranks of vertical fibers. First, in the case of smooth points, the following result was proved by Raynaud (cf. [7, Théorème 1]).

Proposition 1.7. *Let G be a finite p-group, Y a stable curve over S, $f : Y \to X$ a G-semi-stable covering over S, and x a vertical point associated with f. Suppose that x is a smooth point of X_s. Then the p-rank of each connected component of the vertical fiber $f^{-1}(x)$ associated with x is equal to* 0.

In the remainder of this section, let Y be a stable curve over S, $f : Y \to X$ a $\mathbb{Z}/p\mathbb{Z}$*-stable covering* over S and x a *vertical point* associated with f; moreover, we suppose that x is a *singular point* of X_s. Then there are two irreducible components X_1 and X_2 (which may be equal) of X_s such that $x \in X_1 \cap X_2$. Write Y_1 (resp. Y_2) for an irreducible component of Y_s such that $f_s(Y_1) = X_1$ (resp. $f_s(Y_2) = X_2$). Since Y is a stable curve over S, the action of $\mathbb{Z}/p\mathbb{Z}$ on the generic fiber Y_η induces an action of $\mathbb{Z}/p\mathbb{Z}$ on the special fiber Y_s. Write I_1 (resp. I_2) for the inertia group of Y_1 (resp. Y_2) (note that I_1 (resp. I_2) does not depend on the choices of Y_1 (resp. Y_2)).

Write Y' for the normalization of X in the function field $K(Y)$ induced by f and $f': Y' \to X$ for the normalization morphism. Let $y' \in Y'$ be the closed point such that $f'(y') = x$. Since x is a vertical point associated with f, the closed point y' is not a node of the special fiber Y'_s of Y'. We consider the morphism $\operatorname{Spec}\mathcal{O}_{Y',y'} \to \operatorname{Spec}\mathcal{O}_{X,x}$ induced by f'. Since $\mathbb{Z}/p\mathbb{Z}$ is a p-group, the Zariski–Nagata's purity theorem and [12, Lemma 2.1 (iii)] imply that, if $I_1 = I_2 = \{1\}$, the morphism $\operatorname{Spec}\mathcal{O}_{Y',y'} \to \operatorname{Spec}\mathcal{O}_{X,x}$ is étale. This means that y' is a node. Thus, either $I_1 \cong \mathbb{Z}/p\mathbb{Z}$ or $I_2 \cong \mathbb{Z}/p\mathbb{Z}$ holds. Without loss of generality, we may assume that $I_1 \cong \mathbb{Z}/p\mathbb{Z}$. Note that $f^{-1}(x)$ is connected. For the p-rank of $f^{-1}(x)$, we have the following lemma.

Lemma 1.8. *Write Γ_x for the dual graph of the semi-stable curve $f^{-1}(x)_{\mathrm{red}} \subset Y_s$ over s, where $(-)_{\mathrm{red}}$ denotes the reduced induced closed subscheme of $(-)$.*

(a) If $I_1 \cong \mathbb{Z}/p\mathbb{Z}$, and I_2 is trivial, then $\sigma(f^{-1}(x)) = 0$.

(b) If $I_1 = I_2 \cong \mathbb{Z}/p\mathbb{Z}$, then one of the following conditions holds: (i) $\sigma(f^{-1}(x))$ is equal to 0; (ii) $\sigma(f^{-1}(x)) = \operatorname{rank}(\mathrm{H}^1(\Gamma_x, \mathbb{Z})) = p-1$; (iii) $\sigma(f^{-1}(x)) = p-1$ and Γ_x is a tree.

Proof. The lemma follows immediately from [10, Proposition 1] or [14, Theorem 4.8 and Corollary 4.10] when $G = \mathbb{Z}/p\mathbb{Z}$. □

Remark 1.8.1. In fact, Saïdi obtained a p-rank formula for vertical fibers in the case where G is a cyclic p-group (cf. [10, Proposition 1]). Moreover, the author generalizes the p-rank formula to the case where G is an arbitrary p-group (cf. [14, Theorem 4.8 and Corollary 4.10]).

Remark 1.8.2. We can construct some $\mathbb{Z}/p\mathbb{Z}$-stable coverings that satisfy the conditions of Lemma 1.8 (a) and Lemma 1.8 (b)-(ii). However, the author does not know how to construct a $\mathbb{Z}/p\mathbb{Z}$-stable covering that satisfies the conditions of Lemma 1.8 (b)-(i) or of Lemma 1.8 (b)-(iii).

Remark 1.8.3. Y. Hoshi obtained an anabelian pro-p good reduction criterion for a smooth proper ordinary hyperbolic curve (i.e. the reduction is an ordinary stable curve) over a p-adic field (cf. [3]). It is very interesting for the author to know whether or not the pro-p good reduction criterion of Hoshi can be extended to arbitrary proper hyperbolic curves. One of the main technical difficulties is how to construct a p-covering of a given proper hyperbolic curve such that there exist two irreducible components whose p-ranks are positive. We have the following question.

Question: Suppose that $\dim_{\mathbb{F}_p}(\mathrm{H}^1_{\mathrm{\acute{e}t}}(X_{\overline{\eta}}, \mathbb{F}_p)) - \sigma(X_s) > 0$ (note that, if $\operatorname{char}(K) = 0$, the inequality always holds). After replacing S by a finite extension of S, does there exist a $\mathbb{Z}/p\mathbb{Z}$-stable covering over S such that, for some vertical point x, the vertical fiber associated with x satisfies the conditions of Lemma 1.8 (b)-(iii)?

Proposition 1.9. *Suppose that the semi-stable curve $f^{-1}(x)_{\mathrm{red}}$ over s is ordinary. If $I_1 = I_2 \cong \mathbb{Z}/p\mathbb{Z}$, then $\sigma(f^{-1}(x)) = p-1$.*

Proof. We maintain the notations introduced in the proof of Lemma 1.8. If $\sigma(f^{-1}(x)) = 0$, then for each $1 \le i \le n$, $I_{P_i} \cong \mathbb{Z}/p\mathbb{Z}$. This means that $V_i \subset Y_s$ is a projective line for each $1 \le i \le n$. Since Y_s is a stable curve over s, we have $V_i \cap h^{-1}(B)_{\mathrm{red}} \neq \emptyset$ for each $1 \le i \le n$. Thus, $h^{-1}(B)_{\mathrm{red}} \neq \emptyset$. On the other hand, since Y_s is a stable curve over s, Proposition 1.7 implies that $h^{-1}(B)_{\mathrm{red}}$ is not ordinary. This is a contradiction. Then the proposition follows from Lemma 1.8 (b). □

2. Ordinariness of stable coverings

In this section, we prove the main theorem of the present paper.

Definition 2.1. Let C_1 and C_2 be two semi-stable curves over an algebraically closed field l of characteristic $p > 0$, $\psi: C_2 \to C_1$ a finite surjective morphism over l, and $G \subseteq \operatorname{Aut}(C_2/C_1)$ a finite p-group. We shall call ψ a Galois covering with Galois group G if G acts generically freely on C_2, G acts freely at the nodes of C_2, and ψ is equal to the quotient morphism $C_2 \to C_2/G$.

Lemma 2.2. *Let G be a p-group, C_1 and C_2 two semi-stable curves over an algebraically closed field l of characteristic $p > 0$, and $\psi: C_2 \to C_1$ a Galois covering with Galois group G. Then we have*

$$\sigma(C_2) - 1 = (\#G)(\sigma(C_1) - 1) + \sum_{c_2 \in C_2^{\mathrm{cl}}} (e_{c_2} - 1),$$

where C_2^{cl} denotes the set of closed points of C_2, and e_{c_2} denotes the ramification index of ψ at c_2.

Proof. There exist many proofs of the lemma. For example, it is easy to see that the proof of the Deuring–Shafarevich formula given in [1, Theorem 3.1] can be extended to the case where ψ is a Galois covering of semi-stable curves. □

Remark 2.2.1. Lemma 2.2 extends the Deuring–Shafarevich formula to Galois coverings of semi-stable curves. Moreover, the author also extended the Deuring–Shafarevich formula to a more general case by using the theory of *semi-graphs with p-rank* (cf. [14, Theorem 4.5]).

Definition 2.3. Let Γ be a finite graph. We use the notation $v(\Gamma)$ to denote the set of vertices of Γ and $e(\Gamma)$ to denote the set of edges of Γ. For an edge $e \in e(\Gamma)$, we use the notation $v(e)$ to denote the set of vertices that are abutted by e. We define an equivalence relation "$\sim$" on $e(\Gamma)$ as follows: $e_1 \sim e_2$ if $v(e_1) = v(e_2)$. Then we obtain a new finite graph $\Gamma^{\mathrm{ind}} := \Gamma/\sim$. We shall call Γ^{ind} the **induced graph** of Γ. Note that $v(\Gamma^{\mathrm{ind}}) = v(\Gamma)$ and $e(\Gamma^{\mathrm{ind}}) = e(\Gamma)/\sim$.

Definition 2.4. Let Y be a stable curve over S and $f : Y \to X$ a $\mathbb{Z}/p\mathbb{Z}$-stable covering over S. For each irreducible component Y_v of the special fiber Y_s of Y, write X_v for $f(Y_v)$. We shall call f_s p**-new-ordinary** if, for each irreducible component $Y_v \subseteq Y_s$, one of the following conditions holds: (i) if $f_s|_{Y_v}$ is a constant morphism (i.e. $f(Y_v)$ is a point), then Y_v is ordinary; (ii) if the restriction morphism $f_s|_{Y_v}$ is generically étale, then $\widetilde{f_s|_{Y_v}} : \widetilde{Y_v} \to \widetilde{X_v}$ induced by $f_s|_{Y_v}$ is p-new-ordinary (cf. Definition 1.4), where $\widetilde{(-)}$ denotes the normalization of $(-)$.

Remark 2.4.1. Note that, if X_s is ordinary, then f_s is p-new-ordinary if and only if Y_s is ordinary.

Definition 2.5. Let Z be a stable curve over an algebraically closed field. We shall call Z **sturdy** if the genus of the normalization of each irreducible component of Z is ≥ 2.

Now, let us prove the main theorem.

Theorem 2.6. *Let $f : Y \to X$ be a $\mathbb{Z}/p\mathbb{Z}$-stable covering over S. Suppose that X_s is sturdy, and the morphism $f_s : Y_s \to X_s$ over s induced by f is p-new-ordinary. Then f_s is an admissible covering. If, moreover, we suppose that the p-rank of the normalization of each irreducible component of X_s is ≥ 2, then f_s is an admissible covering if and only if*

$$\sigma(Y_s) = p(\sigma(X_s) - 1) + 1.$$

Proof. Write $\{X_i^{\text{ét}}\}_{i\in I}$ (resp. $\{X_j^{\text{in}}\}_{j\in J}$) for the set of stable subcurves of X_s such that the following conditions hold: (i) for each $i \in I$ (resp. $j \in J$), f_s is generically étale over $X_i^{\text{ét}}$ (resp. purely inseparable over X_j^{in}); (ii) for each $i \in I$ (resp. $j \in J$) and each irreducible component $X_v \subseteq X_s$, if $X_v \cap X_i^{\text{ét}} \neq \emptyset$ and $X_v \not\subseteq X_i^{\text{ét}}$ (resp. $X_v \cap X_j^{\text{in}} \neq \emptyset$ and $X_v \not\subseteq X_j^{\text{in}}$), then f_s is purely inseparable (resp. f_s is generically étale) over X_v. Then we have

$$X_s = (\cup_{i\in I} X_i^{\text{ét}}) \cup (\cup_{j\in J} X_j^{\text{in}}).$$

For each $i \in I$ (resp. $j \in J$), we write $\Gamma_{X_i^{\text{ét}}}$ (resp. $\Gamma_{X_j^{\text{in}}}$) for the dual graph of $X_i^{\text{ét}}$ (resp. X_j^{in}) and $g(X_i^{\text{ét}})$ (resp. $g(X_j^{\text{in}})$) for the genus of $X_i^{\text{ét}}$ (resp. X_j^{in}).

Write $\mathcal{V}$ for the set of vertical points associated with f. For each vertical point $x \in \mathcal{V}$, write E_x for the vertical fiber associated with x (note that E_x is connected) and $g(E_x)$ for the genus of E_x. If $\mathcal{V}$ contains a smooth point of X_s, then Proposition 1.7 and Definition 2.4 imply that f_s is not p-new-ordinary. Thus, $\mathcal{V}$ is contained in the singular locus of X_s. For each singular point x' of X_s, Remark 1.5.2 implies that f_s is étale over x'. Thus, we have $\mathcal{V} \subseteq \cup_{j\in J} X_j^{\text{in}}$. This means that, for each $x \in \mathcal{V}$, we have either $x \in \cup_{j\in J} X_j^{\text{in}} \setminus \cup_{i\in I} X^{\text{ét}}$ or $x \in (\cup_{j\in J} X_j^{\text{in}}) \cap (\cup_{i\in I} X^{\text{ét}})$.

In order to prove the theorem we will calculate the p-rank of Y_s by using the Deuring–Shafarevich formula. By applying Lemma 2.2, *we may assume that $X_i^{\text{ét}}$ is irreducible for each $i \in I$.* Let $L := \cup_j e(\Gamma_{X_j^{\text{in}}}) \subseteq e(\Gamma_{X_s})$ (cf. Definition 2.3). We have the following claim.

Claim 1. *We may deform the stable curve X_s along L to obtain a new stable curve over $\overline{\eta} := \operatorname{Spec} \overline{K}$ such that the set of edges of the dual graph of the new stable curve may be naturally identified with $e(\Gamma_{X_s}) \setminus L$.*

Let us prove Claim 1. Suppose that $\phi_s : s \to \overline{\mathcal{M}}_{g(X),S} := \overline{\mathcal{M}}_{g(X)} \times_{\operatorname{Spec}\mathbb{Z}} S$ is the classifying morphism determined by $X_s \to s$. Thus the completion of the local ring of the moduli stack at ϕ_s is isomorphic to $R[\![t_1, \ldots, t_{3g(X)-3}]\!]$, where $t_1, \ldots, t_{3g(X)-3}$ are indeterminates. Furthermore, the indeterminates $t_1, \ldots, t_m$ may be chosen so as to correspond to the deformations of the nodes of X_s. Suppose that $\{t_1, \ldots, t_d\}$ is the subset of $\{t_1, \ldots, t_m\}$ corresponding to the subset $L \subseteq e(\Gamma_{X_s})$. Now fix a morphism $S \to \operatorname{Spec} R[\![t_1, \ldots, t_{3g(X)-3}]\!]$ such that $t_{d+1}, \ldots, t_m \mapsto 0 \in R$, but $t_1, \ldots, t_d$ map to nonzero elements of R. Then the composite morphism $\phi : S \to \operatorname{Spec} R[\![t_1, \ldots, t_{3g(X)-3}]\!] \to \overline{\mathcal{M}}_{g(X),S}$ determines a stable curve $\mathcal{X}$ over S. Moreover, the special fiber of $\mathcal{X}$ is naturally isomorphic to X_s over s. Write X_s^* for the geometric generic fiber $\mathcal{X} \times_\eta \overline{\eta}$ over $\overline{\eta}$ and $\Gamma_{X_s^*}$ for the dual graph of X_s^*. It follows from the construction of X_s^* that we have two natural maps

$$v(\Gamma_{X_s}) \to v(\Gamma_{X_s^*}), \quad e(\Gamma_{X_s}) \setminus L \xrightarrow{\sim} e(\Gamma_{X_s^*})$$

(the latter of which is a bijection). This completes the proof of Claim 1.

Note that

$$\#v(\Gamma_{X_s^*}) = \#I + \#J.$$

Write n_i for $\#(X_i^{\text{ét}} \cap (\cup_{j\in J} X_j^{\text{in}}))$, r_{X_s} for $\text{rank}(\text{H}^1(\Gamma_{X_s}, \mathbb{Z}))$, $r_{X_s^*}^{\text{ind}}$ for $\text{rank}(\text{H}^1(\Gamma_{X_s^*}^{\text{ind}}, \mathbb{Z}))$, $r_{X_j^{\text{in}}}$ for $\text{rank}(\text{H}^1(\Gamma_{X_j^{\text{in}}}, \mathbb{Z}))$, and $r_{X_s}^{\text{in}}$ for $\sum_{j\in J} r_{X_j^{\text{in}}}$, respectively, where $\Gamma_{X_s^*}^{\text{ind}}$ denotes the induced graph of $\Gamma_{X_s^*}$ (cf. Definition 2.3). Then we have

$$r_{X_s} = r_{X_s^*}^{\text{ind}} + r_{X_s}^{\text{in}} + \sum_{i\in I} n_i - \#e(\Gamma_{X_s^*}^{\text{ind}}).$$

For each $i \in I$ (resp. $j \in J$), write $Y_i^{\text{ét}}$ (resp. Y_j^{in}) for the closed subscheme $f_s^{-1}(X_i^{\text{ét}})_{\text{red}}$ of Y_s (resp. $\overline{\{f_s^{-1}(X_j^{\text{in}} \setminus \cup_{i\in I} X_i^{\text{ét}})_{\text{red}}\}}$ of Y_s, where $\overline{\{-\}}$ denotes the closure of $\{-\}$), and $g(Y_i^{\text{ét}})$ (resp. $g(Y_j^{\text{in}})$) for the genus of $Y_i^{\text{ét}}$ (resp. Y_j^{in}). Then we have

$$Y_i^{\text{ét}} = F_i^{\text{ét}} \cup (\cup_{x\in\mathcal{V}\cap X_i^{\text{ét}}} E_x)$$
$$(\text{resp. } Y_j^{\text{in}} = F_j^{\text{in}} \cup (\cup_{x\in X_j^{\text{in}}\cap(\mathcal{V}\setminus X_i^{\text{ét}})} E_x)),$$

where $F_i^{\text{ét}}$ (resp. F_j^{in}) denotes the closed subscheme of $Y_i^{\text{ét}}$ (resp. Y_j^{in}), which is generically étale over $X_i^{\text{ét}}$ (resp. purely inseparable over X_j^{in}). Next, we start to prove the theorem.

Step 1. For any $i \in I$ (resp. $j \in J$), let us calculate $g(Y_i^{\text{ét}})$ and $\sigma(Y_i^{\text{ét}})$ (resp. $g(Y_j^{\text{in}})$ and $\sigma(Y_j^{\text{in}})$) under the assumption that f_s is p-new-ordinary, respectively.

If $F_i^{\text{ét}}$ is irreducible, by the Riemann–Hurwitz formula and Lemma 1.8 (a), we have

$$g(Y_i^{\text{ét}}) = p(g(X_i^{\text{ét}}) - 1) + \frac{1}{2} \cdot \deg(\mathcal{R}_i) + 1 + (p-1)\#(\mathcal{V} \cap X_i^{\text{ét}}),$$

where $\mathcal{R}_i$ denotes the ramification divisor of $f_s|_{F_i^{\text{ét}}} : F_i^{\text{ét}} \to X_i^{\text{ét}}$. Note that we have

$$\#\text{Supp}(\mathcal{R}_i) + \#(\mathcal{V} \cap X_i^{\text{ét}}) = n_i.$$

Moreover, since we assume that f_s is p-new-ordinary, Remark 1.4.2 and Definition 2.4 imply that $\deg(\mathcal{R}_i) = 2\#\text{Supp}(\mathcal{R}_i)(p-1)$. Thus, we obtain

$$g(Y_i^{\text{ét}}) = p(g(X_i^{\text{ét}}) - 1) + n_i(p-1) + 1.$$

For the p-rank of $Y_i^{\text{ét}}$, we have

$$\sigma(Y_i^{\text{ét}}) = p(\sigma(X_i^{\text{ét}}) - 1) + (p-1)(\#\deg(\mathcal{R}_i) + \#(\mathcal{V} \cap X_i^{\text{ét}})) + 1 = p(\sigma(X_i^{\text{ét}}) - 1) + n_i(p-1) + 1.$$

If $F_i^{\text{ét}}$ is disconnected, then we have $\mathcal{V} \cap X_i^{\text{ét}} = X_i^{\text{ét}} \cap (\cup_j X_j^{\text{in}})$. Since we assume that f_s is p-new-ordinary, Lemma 1.8 (a) and Definition 2.4 imply that $F_i^{\text{ét}} \cong X_i^{\text{ét}}$, and for any $x \in \mathcal{V} \cap X_i^{\text{ét}}$, all the irreducible components of E_x are isomorphic to $\mathbb{P}^1$. Note that $\text{rank}(\text{H}^1(\Gamma_{Y_i^{\text{ét}}}, \mathbb{Z}))$ is equal to $(n_i - 1)(p-1)$. Thus, we have

$$g(Y_i^{\text{ét}}) = pg(X_i^{\text{ét}}) + (n_i - 1)(p-1) = p(g(X_i^{\text{ét}}) - 1) + n_i(p-1) + 1$$

and

$$\sigma(Y_i^{\text{ét}}) = p\sigma(X_i^{\text{ét}}) + (n_i - 1)(p-1) = p(\sigma(X_i^{\text{ét}}) - 1) + n_i(p-1) + 1.$$

On the other hand, since we assume that f_s is p-new-ordinary, by Proposition 1.9, for each $x \in X_j^{\text{in}} \cap (\mathcal{V} \setminus \cup_i X_i^{\text{ét}})$, we have $\sigma(E_x) = g(E_x) = p-1$. Then we obtain

$$g(Y_j^{\text{in}}) = g(F_j^{\text{in}}) + \sum_{x\in X_j^{\text{in}}\cap(\mathcal{V}\setminus\cup_i X_i^{\text{ét}})} g(E_x) = g(X_j^{\text{in}}) + (p-1)\#(X_j^{\text{in}} \cap (\mathcal{V} \setminus \cup_i X_i^{\text{ét}}))$$

and

$$\sigma(Y_j^{\text{in}}) = \sigma(F_j^{\text{in}}) + \sum_{x \in X_j^{\text{in}} \cap (\mathcal{V} \setminus \cup_i X_i^{\text{ét}})} \sigma(E_x) = \sigma(X_j^{\text{in}}) + (p-1)\#(X_j^{\text{in}} \cap (\mathcal{V} \setminus \cup_i X_i^{\text{ét}}))$$

where $g(F_j^{\text{in}})$ denotes the genus of F_j^{in}.

Step 2. Let us prove the first part of the theorem (i.e. f_s is an admissible covering under the assumption that f_s is p-new-ordinary). The idea of the proof of the first part of the theorem is to compare the genus of generic fiber Y_η with the genus of special fiber Y_s. We will compute the genus of generic fiber Y_η by applying the Riemann–Hurwitz formula, and compute the genus of special fiber Y_s by applying the properties of p-new-ordinary and the results obtained in Step 1.

Write m_j for $\#(X_j^{\text{in}} \cap (\mathcal{V} \setminus \cup_i X_i^{\text{ét}}))$. Then we have

$$\begin{aligned} g(Y_s) &= \sum_i g(Y_i^{\text{ét}}) + \sum_j g(Y_j^{\text{in}}) + r_{X_s} - r_{X_s}^{\text{in}} \\ &= \sum_i (p(g(X_i^{\text{ét}}) - 1) + n_i(p-1) + 1) + \sum_j (g(X_j^{\text{in}}) + m_j(p-1)) + r_{X_s} - r_{X_s}^{\text{in}}. \end{aligned}$$

On the other hand, by applying the Riemann–Hurwitz formula to $f_\eta : Y_\eta \to X_\eta$, we obtain that the genus $g(Y_\eta)$ of the generic fiber Y_η is equal to

$$p((\sum_i g(X_i^{\text{ét}}) + \sum_j g(X_j^{\text{in}}) + r_{X_s} - r_{X_s}^{\text{in}}) - 1) + 1.$$

Since $g(Y_\eta)$ is equal to $g(Y_s)$, we obtain

$$(1-p)(\sum_j (g(X_j^{\text{in}}) - m_j) - 1 + r_{X_s} - r_{X_s}^{\text{in}} - \sum_i (n_i - 1)) = 0.$$

Then we have

$$\begin{aligned} 0 &= \sum_j (g(X_j^{\text{in}}) - m_j) - 1 + r_{X_s} - r_{X_s}^{\text{in}} - \sum_i (n_i - 1) \\ &= \sum_j (g(X_j^{\text{in}}) - m_j) - 1 + r_{X_s^*}^{\text{ind}} + \sum_i n_i - \#e(\Gamma_{X_s^*}^{\text{ind}}) - \sum_i (n_i - 1) \\ &= \sum_j (g(X_j^{\text{in}}) - m_j) - 1 + r_{X_s^*}^{\text{ind}} - \#e(\Gamma_{X_s^*}^{\text{ind}}) + \#I \end{aligned}$$

By applying Euler–Poincaré characteristic formula for the graph $\Gamma_{X_s^*}^{\text{ind}}$, we obtain

$$r_{X_s^*}^{\text{ind}} - \#e(\Gamma_{X_s^*}^{\text{ind}}) + \#I - 1 = -\#v(\Gamma_{X_s^*}^{\text{ind}}) + \#I = -\#J.$$

Then we have

$$0 = \sum_j (g(X_j^{\text{in}}) - m_j) - \#J = \sum_j (g(X_j^{\text{in}}) - m_j - 1).$$

On the other hand, by the assumptions that X_s is sturdy, we have

$$\begin{aligned} g(X_j^{\text{in}}) &= \sum_{v \in v(\Gamma_{X_j^{\text{in}}})} g(\widetilde{X}_v) + r_{X_j^{\text{in}}} \\ &\geq 2 \cdot \#v(\Gamma_{X_j^{\text{in}}}) + r_{X_j^{\text{in}}} = \#v(\Gamma_{X_j^{\text{in}}}) + \#e(\Gamma_{X_j^{\text{in}}}) + 1, \end{aligned}$$

where $\widetilde{X}_v$ denotes the genus of the normalization of X_v, and $g(\widetilde{X}_v)$ denotes the genus of $\widetilde{X}_v$. If $\{X_j^{\text{in}}\}_{j \in J}$ is not empty, since $\#e(\Gamma_{X_j^{\text{in}}}) \geq m_j$, we have $\sum_j (g(X_j^{\text{in}}) - m_j - 1) > 0$. Then we obtain a contradiction. Thus, $\{X_j^{\text{in}}\}_{j \in J}$ is empty. This means that f_s is generically étale. Then, by Remark 1.5.2, we have that f_s is an admissible covering.

Step 3. Let us prove the "moreover" part of the theorem. The idea of the proof of the "moreover" part is to compare the p-rank of Y_s with the p-rank of Y_s when f_s is p-new-ordinary. We will compute the p-rank of Y_s by applying the Deuring–Shafarevich formula, the properties of p-new ordinary, and the results obtained in Step 1.

If f_s is an admissible covering, then the "moreover" part follows from Lemma 2.2. Thus, we suppose that $\sigma(Y_s) = p(\sigma(X_s) - 1) + 1$. Then we have

$$\begin{aligned}\sigma(Y_s) &= p(\sigma(X_s) - 1) + 1\\ &= p((\sum_i \sigma(X_i^{\text{ét}}) + \sum_j \sigma(X_j^{\text{in}}) + r_{X_s} - r_{X_s}^{\text{in}}) - 1) + 1.\end{aligned}$$

Write m_j for $\#(X_j^{\text{in}} \cap (\mathcal{V} \setminus \cup_i X_i^{\text{ét}}))$. On the other hand, $\sigma(Y_s)$ attains its maximum if and only if f_s is p-new-ordinary. Moreover, if f_s is p-new-ordinary, the p-rank of Y_s is

$$\begin{aligned}&\sum_i \sigma(Y_i^{\text{ét}}) + \sum_j \sigma(Y_j^{\text{in}}) + r_{X_s} - r_{X_s}^{\text{in}}\\ &= \sum_i (p(\sigma(X_i^{\text{ét}}) - 1) + n_i(p-1) + 1) + \sum_j (\sigma(X_j^{\text{in}}) + m_j(p-1)) + r_{X_s} - r_{X_s}^{\text{in}}.\end{aligned}$$

Thus, we have

$$\begin{aligned}\sigma(Y_s) &= p((\sum_i \sigma(X_i^{\text{ét}}) + \sum_j \sigma(X_j^{\text{in}}) + r_{X_s} - r_{X_s}^{\text{in}}) - 1) + 1\\ &\leq \sum_i (p(\sigma(X_i^{\text{ét}}) - 1) + n_i(p-1) + 1) + \sum_j (\sigma(X_j^{\text{in}}) + m_j(p-1)) + r_{X_s} - r_{X_s}^{\text{in}}.\end{aligned}$$

Arguments similar to those given in the proof above imply that

$$\sum_j (\sigma(X_j^{\text{in}}) - m_j - 1) \leq 0.$$

On the other hand, since $\sigma(\widetilde{X}_v) \geq 2$ for each $v \in v(\Gamma_{X_j^{\text{in}}})$, we have

$$\begin{aligned}\sigma(X_j^{\text{in}}) &= \sum_{v \in v(\Gamma_{X_j^{\text{in}}})} \sigma(\widetilde{X}_v) + r_{X_j^{\text{in}}}\\ &\geq 2 \cdot \# v(\Gamma_{X_j^{\text{in}}}) + r_{X_j^{\text{in}}} = \# v(\Gamma_{X_j^{\text{in}}}) + \# e(\Gamma_{X_j^{\text{in}}}) + 1.\end{aligned}$$

If $\{X_j^{\text{in}}\}_{j \in J}$ is not empty, since $\# e(\Gamma_{X_j^{\text{in}}}) \geq m_j$, we have $\sum_j (\sigma(X_j^{\text{in}}) - m_j - 1) > 0$. Then we obtain a contradiction. Thus, $\{X_j^{\text{in}}\}_{j \in J}$ is empty. This means that f_s is generically étale. Then, by Remark 1.5.2, we have that f_s is an admissible covering. We complete the proof of the theorem. □

By applying Theorem 2.6, we generalize the main result of [8] as follows. Moreover, we obtain a numerical criterion for the admissibility of G-stable coverings if G is a p-group.

Corollary 2.7. *Let G be a finite solvable group, Y a stable curve over S, and $f : Y \to X$ a G-stable covering over S. Suppose that X_s is sturdy, and that Y_s is ordinary (i.e. $\sigma(Y_s) = g(Y_s) = (\#G)(g(X_s) - 1) + 1$). Then the morphism $f_s : Y_s \to X_s$ over s induced by f is an admissible covering. Moreover, suppose that the p-rank of the normalization of each irreducible component of X_s is ≥ 2, and that G is a p-group. Then the morphism $f_s : Y_s \to X_s$ over s induced by f is an admissible covering if and only if*

$$\sigma(Y_s) - 1 = (\#G)(\sigma(X_s) - 1).$$

Proof. If X_s is not ordinary, then Y_s is not ordinary. Thus, we may assume that X_s is ordinary. Since G is a finite solvable group, we have a series of subgroups

$$\{1\} =: G_{m+1} \subset G_m \subset G_{m-1} \subset \ldots\ldots \subset G_0 := G$$

such that G_i/G_{i+1}, $i = 0, \ldots, m$, is a cyclic group of prime order. Note that $Y_i := Y/G_{m+1-i}, i = 0, \ldots, m$, is a semi-stable curve over S. Then the series of subgroups of G induces a sequence of morphisms of semi-stable curves

$$Y =: Y_0 \xrightarrow{f_0} Y_1 \xrightarrow{f_1} \ldots\ldots \xrightarrow{f_{m-1}} Y_m \xrightarrow{f_m} X$$

such that $f_m \circ \ldots\ldots \circ f_0 = f$.

Suppose that f_s is not an admissible covering. Then there exists $0 \le w \le m$ such that $(f_j)_s$ is an admissible covering for each $j \ge w+1$ and $(f_w)_s$ is not an admissible covering. Note that since an admissible covering of a sturdy stable curve is sturdy, Y_{w+1} is sturdy. Moreover, Y_j, $j \ge w$, is a stable curve over S, and f_j, $j \ge w$ is a G_j/G_{j+1}-stable covering over S.

If $(Y_{w+1})_s$ is not ordinary, then Y_s is not ordinary. Thus, we may assume $(Y_{w+1})_s$ is ordinary. Since $(f_w)_s$ is not an admissible covering, G_w/G_{w+1} is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Then the corollary follows from Theorem 2.6.

The "moreover" part follows immediately from the "moreover" part of Theorem 2.6 and Lemma 2.2. □

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