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Symmetric group representations and \mathbb{Z} Représentations du groupe symétrique et \mathbb{Z} Anshul Adve^a, Alexander Yong^b^a University Laboratory High School, Urbana, IL 61801, USA^b Dept. of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA

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ABSTRACT

We discuss implications of the following statement about representation theory of symmetric groups: every integer appears infinitely often as an irreducible character evaluation and every nonnegative integer appears infinitely often as a Littlewood–Richardson coefficient and as a Kronecker coefficient.

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R É S U M É

Nous discutons les implications de l'énoncé suivant en théorie des représentations des groupes symétriques : tout entier apparaît une infinité de fois comme valeur d'un caractère irréductible, et tout entier positif ou nul apparaît une infinité de fois comme coefficient de Littlewood–Richardson et comme coefficient de Kronecker.

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Let \mathfrak{S}_n be the symmetric group of permutations of $\{1, 2, \dots, n\}$. A *representation* is a homomorphism $\rho : \mathfrak{S}_n \rightarrow \mathrm{GL}(V)$, where V is a vector space over \mathbb{C} . Equivalently, V is an \mathfrak{S}_n -module under the action defined by $\sigma \cdot v = \rho(\sigma)v$, for $\sigma \in \mathfrak{S}_n$ and $v \in V$. Then ρ is *irreducible* if there is no proper \mathfrak{S}_n -submodule of V . Conjugacy classes and hence irreducible representations of \mathfrak{S}_n biject with $\mathrm{Par}(n)$, the partitions of size n .

Consider three families of numbers from the theory.

(I) The *character* of ρ is

$$\chi^\rho : \mathfrak{S}_n \rightarrow \mathbb{C}; \quad \sigma \mapsto \mathrm{tr}(\rho(\sigma)).$$

Textbooks focus on the case where $V = V_\lambda$ is irreducible (because of Maschke's theorem). Since characters are constant on each conjugacy class μ , one needs only $\chi^\lambda(\mu)$. These are computed by the Murnaghan–Nakayama rule (see below). More recent results include bounds on (normalized) character evaluations [13,4].

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(II) If V_λ and V_μ are irreducible \mathfrak{S}_m and \mathfrak{S}_n -modules, respectively, then $V_\lambda \otimes V_\mu$ is an irreducible $\mathfrak{S}_m \times \mathfrak{S}_n$ -module. If V_ν is an irreducible \mathfrak{S}_{m+n} -representation, it restricts to an $\mathfrak{S}_m \times \mathfrak{S}_n$ -representation $V_\nu \downarrow_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}}$. The *Littlewood–Richardson coefficient* is

$$c_{\lambda,\mu}^\nu = \text{multiplicity of } V_\lambda \otimes V_\mu \text{ in } V_\nu \downarrow_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}}.$$

Many *Littlewood–Richardson rules* are available to count $c_{\lambda,\mu}^\nu$ [15].

(III) If V_λ, V_μ are \mathfrak{S}_n -modules then so is $V_\lambda \otimes V_\mu$. Hence we may write

$$V_\lambda \otimes V_\mu \cong \bigoplus_{\nu \in \text{Par}(n)} V_\nu^{\oplus g_{\lambda,\mu,\nu}}.$$

Here, $g_{\lambda,\mu,\nu}$ is the *Kronecker coefficient*. One has an \mathfrak{S}_3 -symmetric but cancellative formula $g_{\lambda,\mu,\nu} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \chi^\lambda(\sigma) \times \chi^\mu(\sigma) \chi^\nu(\sigma)$; it is an old open problem to give a manifestly nonnegative combinatorial rule. The study of Kronecker coefficients has been given new impetus from *Geometric Complexity Theory*, an approach to the P vs. NP problem; see [2] and the references therein.

This note visits a rudimentary point. While for finite groups, character evaluations are algebraic integers, for \mathfrak{S}_n , in fact $\chi^\lambda(\mu) \in \mathbb{Z}$. Moreover, by definition, $c_{\lambda,\mu}^\nu, g_{\lambda,\mu,\nu} \in \mathbb{Z}_{\geq 0}$. We remark that the three converses hold.¹ The proof uses standard facts, but we are unaware of any specific reference in the textbooks [6,5,15,14], or elsewhere.

Theorem. *Every integer is infinitely often an irreducible \mathfrak{S}_n -character evaluation. Every nonnegative integer is infinitely often a Littlewood–Richardson coefficient, and a Kronecker coefficient.*

Corollary A. *There exists a value-preserving multiset bijection between the Littlewood–Richardson and Kronecker coefficients.*

Proof. Clearly, the Theorem implies that for each $k \in \mathbb{Z}_{\geq 0}$, the sets

$$\text{LR}_k = \{(\lambda, \mu, \nu) : c_{\lambda,\mu}^\nu = k\} \text{ and } \text{Kron}_k = \{(\lambda, \mu, \nu) : g_{\lambda,\mu,\nu} = k\}$$

are countably infinite and thus in bijection. \square

Desirable would be a construction of an injection $\text{Kron}_k \hookrightarrow \text{LR}_k$ for each $k \in \mathbb{Z}_{\geq 0}$ (avoiding the countable axiom of choice). That should solve the Kronecker problem in (III), by reduction to (II). This we cannot do. However, there has been success in this vein [8] on another counting problem. See the Remark at the end of this paper.²

Proof of the Theorem. The *Murnaghan–Nakayama rule* states $\chi^\lambda(\mu) = \sum_T (-1)^{\text{ht}(T)}$, where T is a tableau of shape λ with μ_i many labels i ; the entries are weakly increasing along rows and columns, and the labels i form a connected skew shape T_i with no 2×2 subsquare; $\text{ht}(T)$ is the sum of the heights of each T_i , i.e. one less than the number of rows of T_i .

We sharpen the assertion about $\chi^\lambda(\mu)$. In particular, for a given n , we consider the intervals of consecutive integers achievable as character evaluations for \mathfrak{S}_n . From the rule, the character of the *defining representation* satisfies $\chi^{(n-1,1)}(\mu) = \#(1\text{'s in } \mu) - 1$ (see also [6, Lemma 6.9]). Hence, $\chi^{(n-1,1)}$ takes the values $[0, n-2]$. Similarly, $\chi^{(2,1^{n-2})}$ achieves an interval of negative integers: take $k \in [1, n-5] \cup \{n-3\}$. If $k \not\equiv n \pmod 2$, let $\mu = (n-k-1, 1^{k+1})$. Otherwise, if $k \equiv n \pmod 2$, let $\mu = (n-k-4, 3, 1^{k+1})$. Note that if $k = n-6$, let μ be these parts in decreasing order. In either case, the rule shows $\chi^{(2,1^{n-2})}(\mu) = -k$. Thus, for $n \geq 5$, $[-(n-5), n-2] \subseteq \{\chi^\lambda(\mu) : \lambda, \mu \in \text{Par}(n)\}$. Taking $n \rightarrow \infty$ implies the statement regarding character evaluations.

The *Kostka coefficient* $K_{\lambda,\mu}$ is the number of *semistandard Young tableaux* of shape λ with content μ , i.e., fillings of λ with μ_i many i 's such that rows are weakly increasing and columns are strictly increasing.

Lemma. *Every nonnegative integer is infinitely often a Kostka coefficient.*

Proof. Clearly, $K_{(1+j, 1^{k-1}), (j, 1^k)} = k$ for $j \geq 1$. The lemma then follows. \square

¹ Inspired by P. Polo [12]: every $f \in 1 + q\mathbb{Z}_{\geq 0}[q]$ is a Kazhdan–Lusztig polynomial for some \mathfrak{S}_n .

² There is debate about the idiomatic meaning of *counting rule* or *manifestly nonnegative combinatorial rule*, etc. Consider the (adjusted) Fibonacci numbers $(1, 1, 2, 3, 5, 8, 13, \dots)$. A counting rule is that F_n counts the number of $(1, 2)$ -lists whose sum is n . The recursive (and computationally efficient) description is $F_n = F_{n-1} + F_{n-2}$ ($n \geq 2$) where $F_0 = F_1 = 1$. Construct a binary tree \mathcal{T}_n with root labeled F_n ; each node of label F_i has a left child F_{i-1} and right child F_{i-2} . Leaves of \mathcal{T}_n are labeled F_1 or F_0 . F_n counts the number of leaves of \mathcal{T}_n . The latter description restates the recurrence and is not, *per se*, a counting rule.

The Littlewood–Richardson coefficient claim holds since it is long known that Kostka coefficients are a special case. To be specific, $K_{\lambda, \mu} = c_{\sigma, \lambda}^{\tau}$ where

$$\tau_i = \mu_i + \mu_{i+1} + \dots, \quad i = 1, 2, \dots, \ell(\mu), \text{ and}$$

$$\sigma_i = \mu_{i+1} + \mu_{i+2} + \dots, \quad i = 1, 2, \dots, \ell(\mu) - 1.$$

This reduction is used by H. Narayanan [11] to show computing $c_{\lambda, \mu}^{\nu}$ is a #P-complete problem.

For $\lambda = (\lambda_1, \lambda_2, \dots)$, let $\lambda[N] := (N - |\lambda|, \lambda_1, \lambda_2, \dots)$. F.D. Murnaghan [10] proved that, for an integer $N \gg 0$, $\chi^{\lambda[N]} \otimes \chi^{\mu[N]} = \sum_{\nu} \overline{g_{\lambda, \mu, \nu}} \chi^{\nu[N]}$. The $\overline{g_{\lambda, \mu, \nu}}$ are called *stable Kronecker coefficients* and are evidently a special case of Kronecker coefficients. When $|\lambda| + |\mu| = |\nu|$, one has $\overline{g_{\lambda, \mu, \nu}} = c_{\lambda, \mu}^{\nu}$. Hence one infers the Kronecker coefficient assertion. \square

When, e.g., $n = 25$, all of $[-853, 949]$ appear as some $\chi^{\lambda}(\mu)$, but the proof merely guarantees $[-20, 23]$. Let ℓ_n be the maximum size of an interval of consecutive character evaluations for \mathfrak{S}_n . Trivially, the results of [13,4] imply upper bounds for ℓ_n . Can one prove better upper or lower bounds for ℓ_n ?

Let A_n be the *alternating group* of even permutations in \mathfrak{S}_n . Sources about the representation theory of A_n include [7, Section 2.5] and [5, Section 5.1]. Character evaluations of A_n are not always integral, however.

Corollary B. *Every integer appears infinitely often as an A_n -irreducible character evaluation.*

Proof. Let $\psi^{\lambda} = \chi^{\lambda} \downarrow_{A_n}^{\mathfrak{S}_n}$ be the character of the restriction of the \mathfrak{S}_n -irreducible V_{λ} . If μ is not a partition with distinct odd parts, then the conjugacy class in \mathfrak{S}_n of cycle type μ is also a conjugacy class of A_n . If λ is not a self-conjugate partition, the restriction is an A_n -irreducible, and also $\psi^{\lambda}(\mu) = \chi^{\lambda}(\mu)$. Repeat the Theorem’s character argument, since for $n \geq 4$, neither the λ used is self-conjugate, and since for $k \geq 1$, μ has equal parts. \square

Definition. For a countable indexing set A , a family of nonnegative integers $(a_{\alpha})_{\alpha \in A}$ is *entire* if every $k \in \mathbb{Z}_{\geq 0}$ appears infinitely often.

Many of the nonnegative integers arising in algebraic combinatorics are entire. For example, this is true for the theory of *Schubert polynomials* (we refer to [9] for references). If $w_0 \in \mathfrak{S}_n$ is the longest permutation then $\mathbb{S}_{w_0}(x_1, \dots, x_n) = x_1^{n-1} x_2^{n-2} \dots x_{n-1}$. If $w \neq w_0$, $w(i) < w(i+1)$ for some i . Then $\mathbb{S}_w(x_1, \dots, x_n) = \partial_i \mathbb{S}_{w s_i}(x_1, \dots, x_n)$ where $\partial_i = \frac{f - s_i(f)}{x_i - x_{i+1}}$ and s_i is the simple transposition interchanging $i, i+1$. Nontrivially, each $\mathbb{S}_w \in \mathbb{Z}_{\geq 0}[x_1, x_2, \dots]$. Moreover, $\mathbb{S}_w = \mathbb{S}_{w \times 1}$ where $w \times 1 \in \mathfrak{S}_{n+1}$ is the usual image of $w \in \mathfrak{S}_n$. Thus we can discuss \mathbb{S}_w for $w \in \mathfrak{S}_{\infty}$; these form a \mathbb{Z} -linear basis of $\mathbb{Z}[x_1, x_2, \dots]$. The *Schubert structure constants* $C_{u,v}^w := [\mathbb{S}_w] \mathbb{S}_u \mathbb{S}_v \in \mathbb{Z}_{\geq 0}$ for geometric reasons. The *Stanley symmetric function* is defined by $F_w = \lim_{m \rightarrow \infty} \mathbb{S}_{1^m \times w} \in \mathbb{Z}[[x_1, x_2, \dots]]$; here $1^m \times w \in \mathfrak{S}_{m+n}$ sets $1^m \times w(i)$ equal to i if $1 \leq i \leq m$ and equal to $w(i - m + 1) + m$ otherwise. F_w is Schur-nonnegative.

Corollary C. *These families of nonnegative integers are entire:*

- (a) *the coefficients of monomials in Schubert polynomials,*
- (b) *the Schubert structure constants,*
- (c) *the coefficients of Schur functions in Stanley symmetric functions.*

Proof. (a) is true by the Lemma since when w is *Grassmannian* (has at most one descent), $\mathbb{S}_w(x_1, \dots, x_n)$ is a Schur polynomial s_{λ} . When u, v and w are Grassmannian with descent position d , then $C_{u,v}^w$ is a Littlewood–Richardson coefficient, so the Theorem implies (b). Finally, when w is 321-avoiding (i.e. there does not exist indices $i < j < k$ such that $w(i) > w(j) > w(k)$), $F_w = s_{\nu/\lambda} = \sum_{\mu} c_{\lambda, \mu}^{\nu} s_{\mu}$ is a skew Schur function. Hence, here the coefficient (c) is $c_{\lambda, \mu}^{\nu}$, and we apply the Theorem. \square

Abstractly, all entire families are mutually in value-preserving bijection. However, for **Corollary C**, one can say more: (a) and (c) are a special cases of (b) (see [1] and [3]). Can one construct a “wrong-way map” (as in $\mathbb{Q} \leftrightarrow \mathbb{N}$) for either (b) \leftrightarrow (a) or (b) \leftrightarrow (c) (thereby finding a rule for $C_{u,v}^w$)? A special case indicating the difficulty is the following one.

Problem. *Construct an explicit value-preserving injection between Littlewood–Richardson and Kostka coefficients.*

Remark. Finding a wrong-way map has solved a significant counting rule problem concerning A. Buch–W. Fulton’s *quiver coefficients*. These arise in the study of degeneracy loci of vector bundles over a smooth projective algebraic variety. It was conjectured by those two authors that these integers are nonnegative, with a conjectural counting rule. Also, A. Buch showed that special cases of the quiver coefficients are the numbers from (c) above. The resolution of this problem, due to A. Knutson–E. Miller–M. Shimozono, came by establishing the *opposite*: quiver coefficients are special cases of the well-understood numbers (c). We refer to the solution [8] for background and references.

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