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Complex analysis

A counterexample of a normality criterion for families of meromorphic functions ☆

Un contre-exemple au critère de normalité pour les familles de fonctions méromorphes

Caiyun Fang, Yan Xu

Institute of Mathematics, School of Mathematical Science, Nanjing Normal University, Nanjing 210023, PR China

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A B S T R A C T

Let $A > 1$ be a constant, and let $\mathcal{F}$ be a family of meromorphic functions in a domain D. If, for every function $f \in \mathcal{F}$, f has only zeros of multiplicity at least 2 and satisfies the following conditions: (1) $f(z) = 0 \Rightarrow |f''(z)| \le A|z|$, (2) $f''(z) \ne z$, (3) all poles of f have multiplicity at least 4, then $\mathcal{F}$ is normal in D. In this paper, we first give an example to show that condition (3) is sharp, and prove that our counterexample is unique in some sense.

R É S U M É

Soit $A > 1$ une constante et $\mathcal{F}$ une famille de fonctions méromorphes dans un domaine D. Si toute fonction $f \in \mathcal{F}$ n'a que des zéros de multiplicité au moins 2 et satisfait les conditions suivantes : (1) $f(z) = 0 \Rightarrow |f''(z)| \le A|z|$, (2) $f''(z) \ne z$, (3) tous les pôles de f ont multiplicité au moins 4, alors $\mathcal{F}$ est normale dans D. Dans cette Note, nous donnons un exemple montrant que la condition (3) est précise. Nous montrons ensuite que notre exemple est, en quelque sorte, unique.

1. Introduction and main results

Let $D \subseteq \mathbb{C}$ be a domain, and $\mathcal{F}$ be a family of meromorphic functions defined on D. $\mathcal{F}$ is said to be normal on D, in the sense of Montel, if for each sequence $\{f_n\} \subset \mathcal{F}$ there exists a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}\}$ converges spherically locally uniformly on D to a meromorphic function or ∞ (see [1,3,5]).

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E-mail addresses: 05325@njnu.edu.cn (C. Fang), xuyan@njnu.edu.cn (Y. Xu).

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In 2009, Zhang–Pang–Zalcman [6] proved the following result.

Theorem A. *Let $k \geq 2$ be a positive integer. Let $\mathcal{F}$ be a family of meromorphic functions defined on a domain D, all of whose zeros have multiplicity at least $k+1$ and whose poles are multiple. Let $h(z)(\not\equiv 0)$ be a holomorphic function on D. If, for each $f \in \mathcal{F}$, $f^{(k)}(z) \neq h(z)$, then $\mathcal{F}$ is normal in D.*

They [6] indicated that the multiplicity $k+1$ of the zeros of functions in $\mathcal{F}$ can not be reduced to k, by considering the following example.

Example 1. (see [6]) Let $\Delta = \{z : |z| < 1\}$, $h(z) = z$, and let

$$\mathcal{F} = \left\{ f_n(z) = nz^k \right\}.$$

Clearly, all zeros of f_n are of multiplicity k, and $f_n^{(k)}(z) = nk! \neq z$ on Δ. However, $\mathcal{F}$ fails to be equicontinuous at 0, and then $\mathcal{F}$ is not normal in Δ.

Recently, Xu [4] proved that the multiplicity of the zeros of functions in $\mathcal{F}$ can be reduced from $k+1$ to k for the case $h(z) = z$, but restricting the values $f^{(k)}$ can take at the zeros of f, as follows.

Theorem B. *Let $k \geq 4$ be a positive integer, $A > 1$ be a constant. Let $\mathcal{F}$ be a family of meromorphic functions in a domain D. If, for every function $f \in \mathcal{F}$, f has only zeros of multiplicity at least k and satisfies the following conditions:*

(a) $f(z) = 0 \Rightarrow |f^{(k)}(z)| \leq A|z|$,
(b) $f^{(k)}(z) \neq z$,
(c) all poles of f are multiple,

then $\mathcal{F}$ is normal in D.

Theorem C. *Let $k = 2$ or 3, $A > 1$ be a constant. Let $\mathcal{F}$ be a family of meromorphic functions in a domain D. If, for every function $f \in \mathcal{F}$, f has only zeros of multiplicity at least k and satisfies the following conditions:*

(a) $f(z) = 0 \Rightarrow |f^{(k)}(z)| \leq A|z|$,
(b) $f^{(k)}(z) \neq z$,
(c) all poles of f have multiplicity at least 3,

then $\mathcal{F}$ is normal in D.

We remark that for $k = 2$ condition (c) in Theorem C is insufficient. For the case $k = 2$, the multiplicities of poles of $f \in \mathcal{F}$ need be larger.

Theorem C′. *Let $A > 1$ be a constant. Let $\mathcal{F}$ be a family of meromorphic functions in a domain D. If, for every function $f \in \mathcal{F}$, f has only zeros of multiplicity at least 2 and satisfies the following conditions:*

(a) $f(z) = 0 \Rightarrow |f''(z)| \leq A|z|$,
(b) $f''(z) \neq z$,
(c′) all poles of f have multiplicity at least 4,

then $\mathcal{F}$ is normal in D.

In fact, case (a) in the proof (case 1) of Lemma 9 in [4, p. 478] can not be ruled out, since c_1, c_2, c_3 are complex numbers, so that f has another possible form

$$f(z) = \frac{(z-c_1)^2(z-c_2)^2(z-c_3)^2}{6\,(z-b)^3}$$

for $k = 2$, where c_1, c_2, c_3, and b are distinct constants. Now since the multiplicities of poles of $f \in \mathcal{F}$ are at least 4 for $k = 2$, as the proof of Theorem 1 in [4, p. 483], we can also have the form (17) in [4], and hence Theorem C′ holds (for details, see [4]).

Remark. For $k = 1$, the above theorems are no longer true, even if the multiplicities of poles of $f \in \mathcal{F}$ are large enough, which is shown by Example 2 in [4]. The following example shows that the number "4" in condition (c′) of Theorem C′ is sharp.

Example 2. Let $\Delta = \{z : |z| < 1\}$, and let

$$\mathcal{F} = \left\{ f_n(z) = \frac{(z-1/n)^2(z-\mathrm{e}^{\frac{2\pi i}{3}}/n)^2(z-\mathrm{e}^{\frac{4\pi i}{3}}/n)^2}{6z^3} \right\}.$$

Clearly,

$$f_n''(z) = z + \frac{2}{n^6 z^5} \neq z.$$

For each n, f_n has three zeros $z_1 = 1/n, z_2 = \mathrm{e}^{\frac{2\pi i}{3}}/n$, and $z_3 = \mathrm{e}^{\frac{4\pi i}{3}}/n$ of multiplicity 2,

$$|f_n''(z_i)| = \frac{3}{n} \leq 3|z_i|, \ (i = 1, 2, 3).$$

Since $f_n(1/n) = 0$ and $f_n(0) = \infty$, $\mathcal{F}$ fails to be equicontinuous at 0, and then $\mathcal{F}$ is not normal at 0.

Furthermore, we prove the following result, which illustrates that the above counterexample is unique in some sense.

Theorem 1. *Let $A > 1$ be a constant, and let $\mathcal{F}$ be a family of meromorphic functions defined in D, all of whose zeros are multiple and whose poles all have multiplicity at least 3, such that for every $f \in \mathcal{F}$, $f(z) = 0 \Rightarrow |f''(z)| \leq A|z|$, and $f''(z) \neq z$. If $\mathcal{F}$ is not normal at $z_0 \in D$, then $z_0 = 0$, and there exist $r > 0$ and $\{f_n\} \subset \mathcal{F}$ such that*

$$f_n(z) = \frac{\prod_{i=1}^{3}(z-\xi_{ni})^2}{(z-\eta_n)^3}\hat{f}_n(z)$$

on $\Delta_r = \{z : |z| < r\}$, where $\xi_{ni}/\rho_n \to c_i$ $(i = 1, 2, 3)$ and $\eta_n/\rho_n \to (c_1 + c_2 + c_3)/3$ for some sequence of positive numbers $\rho_n \to 0$ and distinct constants c_1, c_2, and c_3. Moreover, $\hat{f}_n(z)$ is holomorphic and non-vanishing on Δ_r, so that $\hat{f}_n(z) \to \hat{f}(z) \equiv 1/6$ locally uniformly on Δ_r.

In this paper, we denote by $\Delta_r = \{z : |z| < r\}$ and $\Delta_r' = \{z : 0 < |z| < r\}$, and the number r may be different in different places. When $r = 1$, we drop the subscript.

2. Lemmas

To prove our results, we need the following lemmas.

Lemma 1. *([2, Lemma 2]) Let k be a positive integer and let $\mathcal{F}$ be a family of meromorphic functions in a domain D, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, $f \in \mathcal{F}$. If $\mathcal{F}$ is not normal at $z_0 \in D$, then for each α, $0 \leq \alpha \leq k$, there exist a sequence of complex numbers $z_n \in D$, $z_n \to z_0$, a sequence of positive numbers $\rho_n \to 0$, and a sequence of functions $f_n \in \mathcal{F}$ such that*

$$g_n(\zeta) = \frac{f_n(z_n + \rho_n\zeta)}{\rho_n^\alpha} \to g(\zeta)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$, all of whose zeros have multiplicity at least k, so that $g^\#(\zeta) \leq g^\#(0) = kA + 1$. Moreover, $g(\zeta)$ has order at most 2.

Here, as usual, $g^\#(\xi) = |g'(\xi)|/(1 + |g(\xi)|^2)$ is the spherical derivative of g.

Lemma 2. *([4, Lemma 6]) Let f be a transcendental meromorphic function of finite order ρ, and let $k(\geq 2)$ be a positive integer. If f has only zeros of multiplicity at least k, and there exists $A > 1$ such that $f(z) = 0 \Rightarrow |f^{(k)}(z)| \leq A|z|$, then $f^{(k)}$ has infinitely many fix-points.*

The next lemma is Lemma 9 in [4], but the form (4) is ruled out by mistake (since c_1, c_2, c_3 are complex numbers, $(c_1 - c_2)^2 + (c_1 - c_3)^2 + (c_2 - c_3)^2 = 0$ does not imply $c_1 = c_2 = c_3$. For details, see [4, p. 478]).

Lemma 3. *(cf. [4, Lemma 9]) Let f be a rational function, all of whose zeros are multiple. If $f''(z) \neq z$, then one of the following cases must occur:*

(i)

$$f(z)=\frac{(z+c)^3}{6}; \tag{1}$$

(ii)

$$f(z)=\frac{(z-c_1)^4}{6\,(z-b)}; \tag{2}$$

(iii)

$$f(z)=\frac{(z-c_1)^2(z-c_2)^3}{6\,(z-b)^2}; \tag{3}$$

(iv)

$$f(z)=\frac{\prod_{i=1}^{3}(z-c_i)^2}{6\,[z-(c_1+c_2+c_3)/3]^3}, \tag{4}$$

where c is a nonzero constant, c_1, c_2, c_3 and b are distinct constants.

Lemma 4. *([4, Lemma 11]) Let $\mathcal{F}$ be a family of meromorphic functions in a domain D, $A>1$ be a constant. Suppose that, for every $f\in\mathcal{F}$, f has only zeros of multiplicity at least 2, and satisfies the following conditions:*

(a) $f(z)=0\Rightarrow|f''(z)|\le A|z|$,
(b) $f''(z)\ne z$,
(c) all poles of f are of multiplicity at least 3,

then $\mathcal{F}$ is normal in $D\backslash\{0\}$.

3. Proof of Theorem 1

Since $\mathcal{F}$ is not normal at z_0, by Lemma 4, $z_0=0$. Without loss of generality, we assume $D=\Delta=\{z:|z|<1\}$. Again by Lemma 4, $\mathcal{F}$ is normal on Δ'.

Consider the family

$$\mathcal{G}=\left\{g(z)=\frac{f(z)}{z}:f\in\mathcal{F}\right\}.$$

We claim that $f(0)\ne 0$ for every $f\in\mathcal{F}$. Otherwise, if $f(0)=0$, by the assumption of Theorem 1, $|f''(0)|\le 0$, and then $f''(0)=0$. But $f''(z)\ne z$, which is a contradiction. Thus, for each $g\in\mathcal{G}$, $g(0)=\infty$. Furthermore, all zeros of $g(z)$ are multiple. On the other hand, by a simple calculation, we have:

$$g''(z)=\frac{f''(z)}{z}-\frac{2g''(z)}{z}.$$

Since $f(z)=0\Rightarrow|f''(z)|\le A|z|$, we deduce that $g(z)=0\Rightarrow|g''(z)|\le A$.

Clearly, $\mathcal{G}$ is normal on Δ'. We claim that $\mathcal{G}$ is not normal at $z=0$. Indeed, if $\mathcal{G}$ is normal at $z=0$, then $\mathcal{G}$ is normal on the whole disk Δ and hence equicontinuous on Δ with respect to the spherical distance. On the other hand, $g(0)=\infty$ for each $g\in\mathcal{G}$, so there exists $\epsilon>0$ such that for every $g\in\mathcal{G}$ and every $z\in\Delta_\epsilon$, $|g(z)|\ge 1$. Then $f(z)$ is non-vanishing, and thus $1/f$ is holomorphic on Δ_ϵ for all $f\in\mathcal{F}$. Since $\mathcal{F}$ is normal on Δ' but not normal on Δ, the family $\mathcal{F}_1=\{1/f,\,f\in\mathcal{F}\}$ is holomorphic on Δ_ϵ and normal on Δ'_ϵ, but it is not normal at $z=0$. Therefore, there exists a sequence $\{1/f_n\}\subset\mathcal{F}_1$ that converges locally uniformly on Δ'_ϵ, but not in Δ_ϵ. Hence, by the maximum modulus principle, $1/f_n\to\infty$ on Δ'_ϵ. Thus $f_n\to 0$ converges locally uniformly on Δ'_ϵ, and so does $\{g_n\}\subset\mathcal{G}$, where $g_n=f_n/z$. But $|g_n(z)|\ge 1$ for $z\in\Delta_\epsilon$, which is a contradiction.

Then, by Lemma 1, there exist functions $g_n\in\mathcal{G}$, points $z_n\to 0$ and positive numbers $\rho_n\to 0$ such that

$$G_n(\zeta)=\frac{g_n(z_n+\rho_n\zeta)}{\rho_n^2}\to G(\zeta)$$

converges spherically uniformly on compact subsets of $\mathbb{C}$, where G is a non-constant meromorphic function on $\mathbb{C}$ and of finite order, all zeros of G are multiple, and $G^\#(\zeta)\le G^\#(0)=2A+1$ for all $\zeta\in\mathbb{C}$.

By [4, pages 481–482], we can assume that $z_n/\rho_n\to\alpha$ (a finite complex number). Then

$$\frac{g_n(\rho_n\zeta)}{\rho_n^2}=\frac{g_n(z_n+\rho_n(\zeta-z_n/\rho_n))}{\rho_n^2}=G_n(\zeta-z_n/\rho_n)\to G(\zeta-\alpha)=\widetilde{G}(\zeta)$$

spherically uniformly on compact subsets of $\mathbb{C}$. Clearly, $\widetilde{G}(0)=\infty$.

Set

$$H_n(\zeta)=\frac{f_n(\rho_n\zeta)}{\rho_n^3}. \tag{5}$$

Then

$$H_n(\zeta)=\frac{f_n(\rho_n\zeta)}{\rho_n^3}=\zeta\,\frac{g_n(\rho_n\zeta)}{\rho_n^2}\to\zeta\widetilde{G}(\zeta)=H(\zeta) \tag{6}$$

spherically uniformly on compact subsets of $\mathbb{C}$, and

$$H_n''(\zeta)=\frac{f_n''(\rho_n\zeta)}{\rho_n}\to H''(\zeta) \tag{7}$$

locally uniformly on $\mathbb{C}\setminus H^{-1}(\infty)$. By the assumption of Theorem 1 and (6), all zeros of H are multiple, and all poles of H have multiplicity at least 3. Since $\widetilde{G}(0)=\infty$, $H(0)\neq 0$.

Claim: (I) $H(\zeta)=0\Rightarrow|H''(\zeta)|\leq A|\zeta|$; (II)$H''(\zeta)\neq\zeta$.

If $H(\zeta_0)=0$, by Hurwitz's theorem and (6), there exist $\zeta_n\to\zeta_0$ such that $f_n(\rho_n\zeta_n)=0$ for n sufficiently large. By the assumption, $|f_n''(\rho_n\zeta_n)|\leq A|\rho_n\zeta_n|$. Then, it follows from (7) that $|H''(\zeta_0)|\leq A|\zeta_0|$. Claim (I) is proved.

Suppose that there exists ζ_0 such that $H''(\zeta_0)=\zeta_0$. By (7),

$$0\neq\frac{f_n''(\rho_n\zeta)-\rho_n\zeta}{\rho_n}=H_n''(\zeta)-\zeta\to H''(\zeta)-\zeta,$$

uniformly on compact subsets of $\mathbb{C}\setminus H^{-1}(\infty)$. Hurwitz's theorem implies that $H''(\zeta)\equiv\zeta$ on $\mathbb{C}\setminus H^{-1}(\infty)$, and then on $\mathbb{C}$. Hence H is a polynomial of degree 3. In view of the fact that all zeros of H are multiple, we know that H has only one zero, say ζ_1, with multiplicity 3, so that $H''(\zeta_1)=0$, and thus $\zeta_1=0$ since $H''(\zeta)\equiv\zeta$. But $H(0)\neq 0$, which is a contradiction. This proves claim (II).

Noting that H is of finite order, Lemma 2 implies that H must be a rational function. Since all poles of H have multiplicity at least 3, it follows from Lemma 3 that

$$H(\zeta)=\frac{1}{6}(\zeta+c)^3$$

or

$$H(\zeta)=\frac{\prod_{i=1}^{3}(\zeta-c_i)^2}{6\,[\zeta-(c_1+c_2+c_3)/3]^3},$$

where $c_1,\ c_2$, and c_3 are distinct constants, and c is a nonzero constant. The former case can be ruled out as the form (17) in [4, pp. 483–485]. So this, together with (5) and (6), gives that

$$\frac{f_n(\rho_n\zeta)}{\rho_n^3}\to\frac{\prod_{i=1}^{3}(\zeta-c_i)^2}{6\,[\zeta-(c_1+c_2+c_3)/3]^3}. \tag{8}$$

Noting that all zeros of f_n are multiple, there exist $\zeta_{ni}\to c_i(i=1,2,3)$ and $\lambda_n\to(c_1+c_2+c_3)/3$ such that $\xi_{ni}=\rho_n\zeta_{ni}(i=1,2,3)$ are zeros of f_n with exact multiplicity 2, and $\eta_n=\rho_n\lambda_n$ is the pole of f_n with exact multiplicity 3.

Now write

$$f_n(z)=\frac{\prod_{i=1}^{3}(z-\xi_{ni})^2}{(z-\eta_n)^3}\hat{f}_n(z). \tag{9}$$

Then, by (8) and (9), we obtain

$$\hat{f}_n(\rho_n\zeta)\to\frac{1}{6} \tag{10}$$

on $\mathbb{C}$.

Next we complete our proof in three steps.

Step 1. We first prove that there exists $\delta>0$ such that $\hat{f}_n(z)\neq 0$ on Δ_δ.

Suppose not, taking a sequence and renumbering if necessary, that $\hat{f}_n$ has zeros tending to 0. Assume that $\hat{z}_n\to 0$ is the zero of $\hat{f}_n$ with the smallest modulus. Then, by (10), it is easy to see that $\hat{z}_n/\rho_n\to\infty$.

Set

$$\widehat{f}_n^*(z)=\hat{f}_n(\hat{z}_n z). \tag{11}$$

Clearly, $\widehat{f}_n^*(z)$ is well defined on $\mathbb{C}$ and not vanishing on Δ. Moreover, $\widehat{f}_n^*(1)=0$.

Now let

$$M_n(z)=\frac{\prod_{i=1}^{3}(z-\xi_{ni}/\hat{z}_n)^2}{(z-\eta_n/\hat{z}_n)^3}\widehat{f}_n^*(z). \tag{12}$$

It follows from (9), (11), and (12) that

$$M_n(z)=\frac{\prod_{i=1}^{3}(z\hat{z}_n-\xi_{ni})^2}{(z\hat{z}_n-\eta_n)^3}\frac{\hat{f}_n(\hat{z}_n z)}{\hat{z}_n^3}=\frac{f_n(\hat{z}_n z)}{\hat{z}_n^3}.$$

Obviously, all zeros of $M_n(z)$ have multiplicity at least 2 and all poles of $M_n(z)$ have multiplicity at least 3. Since $f_n(z)=0\Rightarrow|f_n''(z)|\le A|z|$, it follows that $M_n(z)=0\Rightarrow|M_n''(z)|\le A|z|$. In view of $f_n''(z)\ne z$, we have

$$M_n''(z)-z=\frac{f_n''(\hat{z}_n z)-\hat{z}_n z}{\hat{z}_n}\ne 0. \tag{13}$$

Thus Lemma 4 implies that $\{M_n(z)\}$ is normal on $\mathbb{C}^*=\mathbb{C}\backslash\{0\}$.

Since $\xi_{ni}/\rho_n=\zeta_{ni}\to c_i$ for $i=1,2,3$, $\eta_n/\rho_n=\lambda_n\to(c_1+c_2+c_3)/3$ and $\hat{z}_n/\rho_n\to\infty$, we have

$$\frac{\xi_{ni}}{\hat{z}_n}=\frac{\xi_{ni}}{\rho_n}\frac{\rho_n}{\hat{z}_n}\to 0\ (i=1,2,3);\quad \frac{\eta_n}{\hat{z}_n}=\frac{\eta_n}{\rho_n}\frac{\rho_n}{\hat{z}_n}\to 0.$$

We now see from (12) that $\{\widehat{f}_n^*\}$ is also normal on $\mathbb{C}^*$.

Then, by taking a subsequence, we assume that $\widehat{f}_n^*\to\widehat{f}^*$ spherically locally uniformly on $\mathbb{C}^*$. Moreover, since $\widehat{f}_n^*(1)=\hat{f}_n(\hat{z}_n)=0$, we know that $\widehat{f}^*(1)=0$ with multiplicity at least 2.

Set

$$L_n(z)=M_n''(z)-z. \tag{14}$$

From (13), we have $L_n\ne 0$.

Now we show that $\hat{f}^*(z)\not\equiv 0$. Otherwise $\hat{f}_n^*(z)\to 0$, thus $L_n(z)\to -z$ and $L_n'(z)\to -1$ locally uniformly on $\mathbb{C}^*$. By the argument principle, we get

$$\left|n(1,L_n)-n(1,\frac{1}{L_n})\right|=\frac{1}{2\pi}\left|\int_{|z|=1}\frac{L_n'}{L_n}\mathrm{d}z\right|\to\frac{1}{2\pi}\left|\int_{|z|=1}\frac{1}{z}\mathrm{d}z\right|=1,$$

where $n(r,f)$ denotes the number of poles of f in Δ_r, counting multiplicity. It follows that $n(1,L_n)=1$. On the other hand, the poles of $L_n(z)=M_n''(z)-z$ have multiplicity at least 5, which is a contradiction.

Then $1/\hat{f}_n^*\to 1/\hat{f}^*\not\equiv\infty$ spherically locally uniformly on $\mathbb{C}^*$. Recalling that $\hat{f}_n^*\ne 0$ on Δ, then $1/\hat{f}_n^*$ is holomorphic on Δ. The maximum modulus principle implies that $1/\hat{f}_n^*\to 1/\hat{f}^*$, and then $\hat{f}_n^*\to\hat{f}^*$ on Δ. Hence, $\hat{f}_n^*\to\hat{f}^*$ spherically locally uniformly on $\mathbb{C}$. In particular, $\hat{f}_n^*(0)=\hat{f}_n(0)\to 1/6=\hat{f}^*(0)$.

Then, we obtain from (12) and (14) that

$$L_n(z)\to L(z)=(z^3\widehat{f}^*(z))''-z$$

locally uniformly on $\mathbb{C}^*\setminus\{(\widehat{f}^*)^{-1}(\infty)\}$. Note that $L_n(z)\ne 0$, then each $1/L_n(z)$ is holomorphic on $\mathbb{C}$, and thus $1/L_n(z)\to 1/L(z)$ locally uniformly on $\mathbb{C}^*\setminus\{(\widehat{f}^*)^{-1}(\infty)\}$. By Hurwitz's theorem, $1/L(z)\equiv\infty$ or $1/L(z)$ is holomorphic on $\mathbb{C}^*\setminus\{(\widehat{f}^*)^{-1}(\infty)\}$. If $1/L(z)\equiv\infty$, then $L(z)\equiv 0$ on $\mathbb{C}^*\setminus\{(\widehat{f}^*)^{-1}(\infty)\}$, and hence on $\mathbb{C}$, that is,

$$(z^3\widehat{f}^*(z))''-z\equiv 0.$$

It follows that

$$\widehat{f}^*(z)=\frac{z^3+c_1z+c_2}{6z^3},$$

where c_1,c_2 are constants. Since $\widehat{f}^*(1)=0$ and all zeros of $\widehat{f}^*$ are multiple, we have

$$\widehat{f}^*(z)=\frac{(z-1)^3}{6z^3},$$

which is impossible, since $z^3+c_1z+c_2\ne(z-1)^3$. Thus $1/L(z)$ is holomorphic on $\mathbb{C}^*\setminus\{(\widehat{f}^*)^{-1}(\infty)\}$. The maximum modulus principle implies that $L_n(z)\to L(z)$ locally uniformly on $\mathbb{C}$. Since $L_n(z)\ne 0$, we have $L(z)\ne 0$ or $L(z)\equiv 0$. As before, $L(z)\equiv 0$ is impossible. Then we have $L(z)\ne 0$. But $L(0)=0$ since $\hat{f}^*(0)=1/6$, which is a contradiction. Thus our claim is proved.

Step 2. We now show that $\hat{f}_n \to \hat{f}$ spherically locally uniformly on Δ, and each $\hat{f}_n(z)$ is holomorphic on $\Delta_{\delta'}$ for some $\delta' > 0$.

Since $\{f_n\}$, and hence $\{\hat{f}_n\}$ is normal on Δ', taking a subsequence and renumbering, we have $\hat{f}_n \to \hat{f}$ spherically locally uniformly on Δ'.

We claim that $\hat{f}(z) \not\equiv 0$ on Δ'. Otherwise, we have $f_n''(z) \to 0$ and $f_n'''(z) \to 0$ locally uniformly on Δ'. Then the argument principle yields that

$$\left| n(\frac{1}{2}, f_n'' - z) - n(\frac{1}{2}, \frac{1}{f_n'' - z}) \right| = \frac{1}{2\pi} \left| \int_{|z|=\frac{1}{2}} \frac{f_n''' - 1}{f_n'' - z} dz \right| \to \frac{1}{2\pi} \left| \int_{|z|=\frac{1}{2}} \frac{1}{z} dz \right| = 1.$$

Now that $f_n''(z) \neq z$, it follows that $n(\frac{1}{2}, f_n'') = n(\frac{1}{2}, f_n'' - z) = 1$, which is impossible.

Recalling that $\hat{f}_n(z) \neq 0$, as before, the maximum modulus principle implies that $\hat{f}_n \to \hat{f}$ spherically locally uniformly on Δ. Since $\hat{f}_n(0) \to 1/6$, we have $\hat{f}(0) = 1/6$. Hence $\hat{f}$ is holomorphic at 0. Moreover, there exists a positive number δ' such that each $\hat{f}_n$ is holomorphic on $\Delta_{\delta'}$.

Step 3. Finally, we prove that $\hat{f}(z) \equiv 1/6$.

By (9), we get $f_n(z) \to z^3 \hat{f}(z)$ on Δ'. Thus

$$f_n''(z) - z \to [z^3 \hat{f}(z)]'' - z, \tag{15}$$

on $\Delta' \backslash \{\hat{f}^{-1}(\infty)\}$. If $[z^3 \hat{f}(z)]'' - z \not\equiv 0$, noting that $f_n''(z) \neq z$, the maximum modulus principle implies that (15) still holds on Δ. Then, Hurwitz's theorem yields that $[z^3 \hat{f}(z)]'' - z \neq 0$, violating the fact that $([z^3 \hat{f}(z)]'' - z)|_{z=0} = 0$. Hence, $[z^3 \hat{f}(z)]'' - z \equiv 0$. This, together with $\hat{f}(0) = 1/6$, gives $\hat{f}(z) \equiv 1/6$.

Letting $r = \min\{\delta, \delta'\}$, the proof of Theorem 1 is completed. □

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References

[1] W.K. Hayman, Meromorphic Functions, Clarendon Press, Oxford, UK, 1964.
[2] X.C. Pang, L. Zalcman, Normal families and shared values, Bull. Lond. Math. Soc. 32 (2000) 325–331.
[3] J. Schiff, Normal Families, Springer-Verlag, New York/Berlin, 1993.
[4] Y. Xu, Normal families and fixed-points of meromorphic functions, Monatshefte Math. 179 (2016) 471–485.
[5] L. Yang, Value Distribution Theory, Springer-Verlag & Science Press, Berlin, 1993.
[6] G.M. Zhang, X.C. Pang, L. Zalcman, Normal families and omitted functions II, Bull. Lond. Math. Soc. 41 (2009) 63–71.