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Lifshitz tails for Schrödinger operators with random breather potential





Inégalité de Lifchitz pour la densité d'états intégrée pour des opérateurs de Schrödinger avec potentiel aléatoire de breather

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ABSTRACT

We prove a Lifshitz tail bound on the integrated density of states of random breather Schrödinger operators. The potential is composed of translated single-site potentials. The single-site potential is an indicator function of the set tA where t is from the unit interval and A is a measurable set contained in the unit cell. The challenges of this model are that, since A is not assumed to be star-shaped, the dependence of the potential on the parameter t is not monotone. It is also non-linear and not differentiable.

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RÉSUMÉ

Nous prouvons une inégalité de Lifchitz pour la densité d'états intégrée pour des opérateurs de Schrödinger avec potentiel aléatoire de *breather*. Plus précisément, le potentiel est composé de translations d'un potentiel simple site, qui est une fonction caractéristique de l'ensemble tA, où $t \in [0, 1]$ et $A \subset [-1/2, 1/2]^d$ est mesurable. L'enjeu de ce modèle réside dans le fait que, puisque nous n'assumons pas que la partie A soit étoilée, le potentiel est une fonction non monotone de la variable t. De plus, la dépendance est non linéaire et non différentiable.

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1. Model and result

We prove a Lifshitz tail bound on the integrated density of states (IDS) for a random Schrödinger operator with breather potential. In comparison to other models, in particular the well-studied alloy-type potential, the major challenge in our model is that it is neither monotone nor linear as a function of the random parameter(s). This is a feature shared with the random displacement model [4] and with random quantum waveguides [1], to name just two problems that have been

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studied recently in the literature. Moreover, the operator family under consideration here is not analytic in the sense of Kato. In fact, its derivative does not exist as a bounded operator. This is due to the fact that the most natural single-site potential is a characteristic function of a measurable set.

The direct predecessor of our work is [3]. Below we will compare the results of [3] with ours.

Model. Let $A \subset \mathcal{D} := [-1/2, 1/2]^d \subset \mathbb{R}^d$ be measurable with Lebesgue measure $|A| \in (0, 1/2]$, $tA := \{tx \in \mathbb{R}^d \mid x \in A\}$, $u(t, x) := \chi_{tA}(x)$ the indicator function, and $\lambda_j : \Omega \to [0, 1]$, $j \in \mathbb{Z}^d$, an i.i.d. sequence of random variables satisfying

$$\mathbb{P}(\lambda_0 = 0) < 1, \qquad \forall \varepsilon > 0 \colon \mathbb{P}(\lambda_0 \in [0, \varepsilon]) > 0.$$

For $V_{\text{per}} \in L^{\infty}(\mathbb{R}^d)$ periodic with respect to \mathbb{Z}^d , we define the unperturbed background operator

 $H_{\text{per}} := -\Delta + V_{\text{per}}$ with domain $W^{2,2}(\mathbb{R}^d)$

and its random perturbation

$$H_{\omega} := H_{\text{per}} + W_{\omega} := H_{\text{per}} + \sum_{j \in \mathbb{Z}^d} u(\lambda_j(\omega), \cdot - j) = H_{\text{per}} + \sum_{j \in \mathbb{Z}^d} \chi_{\lambda_j(\omega)A}(\cdot - j).$$
(1)

A Borel-Cantelli argument shows that

$$E_0 := \inf \sigma(H_{per}) = \inf \sigma(H_{\omega})$$
 a.s.

Consequently, the IDS

$$N \colon \mathbb{R} \to [0, \infty), \quad N(E) := \mathbb{E}[\operatorname{Tr} \chi_{\mathcal{D}} \chi_{(-\infty, E]}(H_{\omega})]$$

vanishes below E_0 and is positive above E_0 .

For Schrödinger operators with "truly" random potential, one expects that N is very thin near E_0 . In fact, we prove, for the above model, the following theorem.

Theorem. There exist $C_1, C_2 \in (0, \infty)$ and $E' > E_0$ such that for all $E \in (E_0, E']$

$$N(E) \le C_1 (E - E_0)^{d/2} \exp(-C_2 (E - E_0)^{-d/2}).$$
(2)

Remark 1. In a longer companion paper [6], we will discuss more details, in particular:

- more general breather models than (1), in fact an abstract non-linear model incorporating the usual breather and alloy-type models,
- a lower bound complementary to (2),
- applications, in particular initial length scale estimates and its team work with recent Wegner estimates [9–11] to yield Anderson localization,
- the history of the problem and previous literature.

In contrast, in the present paper we want to keep the presentation simple and concentrate on the main idea of our proof for a very intuitive model.

Remark 2. We compare our result to its direct predecessor in [3]. This is also the easiest way to point out the differences between the two proofs.

In [3], a Lipschitz or differentiability condition was required for the single-site potential, namely

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}u(\lambda,\,\cdot\,)\in L^{\infty}(\mathbb{R}^{d}).$$
(3)

For our choice $u(\lambda, x) = \chi_{\lambda A}(x)$, the derivative $\frac{d}{d\lambda}u(\lambda, \cdot)$ is not even a function, let alone an element of L^{∞} . Condition (3) was used in [3] to linearize the non-linear model and apply Temple's inequality in a similar fashion as in the case of the linear alloy-type model. Furthermore, in [3] it is assumed that $\lambda \mapsto u(\lambda, x)$ is isotone for almost every $x \in \mathbb{R}^d$. This is obviously *not the case* for $\lambda \mapsto \chi_{\lambda A}$ unless *A* is star shaped with center 0, a condition we *do not* impose. In our proof, we use merely monotonicity-on-average, roughly speaking the fact that

$$\int \chi_{tA} \, \mathrm{d}x = |tA|$$

is increasing in *t*. Finally, let us stress that we do not assume any topological properties of *A*, neither openness nor regularity of the boundary (see Fig. 1). In particular, *A* may be a fractal set.

To avoid the assumptions that have been necessary in [3], we do not use Temple's inequality, but Thirring's inequality [8] instead. Note that Thirring's inequality was used in the pioneering work [2] on Lifshitz Tails for random Schödinger operators of alloy type, but has been abandoned in favor of Temple's inequality in subsequent papers, starting with [7].

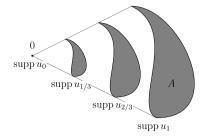


Fig. 1. Support of the single-site potential u_t for different values of t with arbitrary base set A.

2. Proof

The proof of the Theorem relies on the following perturbation bound, whose proof via the projection method can be found in [8], see the forthcoming paper [6] for details.

Thirring's inequality. Let *H* be a self-adjoint operator on a Hilbert space \mathcal{H} , such that $E_1(H) := \min \sigma(H)$ is a simple eigenvalue with normalized eigenstate $\psi \in \mathcal{H}$ and $E_2(H) := \inf(\sigma(H) \setminus \{E_1(H)\}) > E_1(H)$. Let *V* be an invertible, positive operator on \mathcal{H} , such that $\langle \psi, V^{-1}\psi \rangle > 0$. Then

$$\min\{E_1(H) + \langle \psi, V^{-1}\psi \rangle^{-1}, E_2(H)\} \le E_1(H+V).$$

Proof of the Theorem. We define the relevant index set $I_L := [-L, L)^d \cap \mathbb{Z}^d$ for $L \in \mathbb{N}$ and assemble the box $\Lambda_L := I_L + \mathcal{D}$. With $|\cdot|$ for Lebesgue measure and # for cardinality we have $|\Lambda_L| = #I_L$. We need to show (2) only for points *E* of continuity of *N*, since *N* is monotone and the right-hand side is continuous in *E*.

Let H^{L}_{ω} denote the restriction of H_{ω} to Λ_{L} with Neuman boundary conditions. Weyl's bound implies that for nonnegative single-site potentials

$$N(E) \le C_1 E^{d/2} \mathbb{P}\{\omega \mid E_1(H_{\omega}^L) \le E\}$$

for $L \in \mathbb{N}$ and points $E \in \mathbb{R}$ of continuity of *N*. Here we will for simplicity assume that $V_{per} = 0$, in particular $E_0 = 0$. (For $V_{per} \neq 0$, one needs to use in the following arguments Mezincescu's boundary conditions [5] instead, as we elaborate on it in detail in [6].) It is thus sufficient to derive an exponential bound on the probability that the first eigenvalue $E_1(H_{\omega}^L)$ of H_{ω}^L does not exceed *E* for a suitably chosen $L = L_E$.

In order to apply Thirring's inequality, we need the random potential to be strictly positive. We therefore regularize the potential by letting

$$H_0^L := -\Delta^L + -\gamma_L$$
 and $V_\omega := W_\omega + \gamma_L$

with $\gamma_L := C_3/(2L^2)$ and $C_3 := \pi^2/4$. This shift by γ_L scales like the gap between the first and the second eigenvalue of $-\Delta^L$. The normalized ground state Ψ_L of H_0^L is given by $\Psi_L = |\Lambda_L|^{-1/2} \chi_{\Lambda_L}$. Furthermore,

$$E_1(H_0^L) = -\gamma_L$$
 and $E_2(H_0^L) = \frac{C_3}{L^2} - \gamma_L = \gamma_L$ $(L \in \mathbb{N}).$

As V_{ω} does not vanish, V_{ω}^{-1} is well defined as a multiplication operator. By construction, we have:

$$\langle \Psi_L, V_{\omega}^{-1} \Psi_L \rangle = \int_{\Lambda_L} \frac{|\Psi_L(x)|^2}{V_{\omega}(x)} \, \mathrm{d}x = \frac{1}{|\Lambda_L|} \int_{\Lambda_L} \frac{\mathrm{d}x}{V_{\omega}(x)} = \frac{1}{\# I_L} \sum_{k \in I_L \mathcal{D} + k} \frac{\mathrm{d}x}{V_{\omega}(x)}.$$

The last integral is easily calculated:

$$\int_{\mathcal{D}+k} \frac{\mathrm{d}x}{V_{\omega}(x)} = \int_{\mathcal{D}} \frac{\mathrm{d}x}{u_{k,\omega}(x) + \gamma_L} = \frac{|\lambda_k(\omega)A|}{1 + \gamma_L} + \frac{1 - |\lambda_k(\omega)A|}{\gamma_L} = \frac{1 + \gamma_L - |\lambda_k(\omega)A|}{(1 + \gamma_L)\gamma_L}$$

With $S_L := \frac{1}{\# I_L} \sum_{k \in I_L} |\lambda_k A|$, we get

$$\langle \Psi_L, V_{\omega}^{-1} \Psi_L \rangle = \frac{1 + \gamma_L - S_L(\omega)}{(1 + \gamma_L)\gamma_L}$$

or

$$E_1(H_0^L) + \langle \Psi_L, V_\omega^{-1} \Psi_L \rangle^{-1} = \frac{\gamma_L S_L(\omega)}{1 + \gamma_L - S_L(\omega)}.$$

For all $L \ge L_0 := \sqrt{C_3/2}$, we have $\gamma_L \le 1$. Using this as well as $0 \le S_L \le 1/2$ a.s., we derive

$$\frac{\gamma_L S_L(\omega)}{2} \leq E_1(H_0^L) + \langle \Psi_L, V_{\omega}^{-1} \Psi_L \rangle^{-1} \leq \gamma_L = E_2(H_0^L).$$

Thus, Thirring's inequality implies for all $L \in \mathbb{N}$, $L \ge L_0$,

$$E_1(H_{\omega}^L) = E_1(H_0^L + V_{\omega}^L) \ge \min\{E_1(H_0^L) + \langle \Psi_L, V_{\omega}^{-1}\Psi_L \rangle^{-1}, E_2(H_0^L)\} \ge \frac{\gamma_L S_L(\omega)}{2}$$

From our assumptions on A and λ_0 , we have $\mathbb{E}[S_L] = \mathbb{E}[|\lambda_0 A|] > 0$. Let $L_E := \lfloor \sqrt{C_3 \mathbb{E}[S_1]/(8E)} \rfloor$. For E small enough, $L_E \ge L_0$. Hence, since $\mathbb{E}[S_1] = \mathbb{E}[S_{L_F}]$, we see

$$\mathbb{P}\{\omega \mid E_1(H_{\omega}^{L_E}) \le E\} \le \mathbb{P}\left\{\frac{\gamma_{L_E}}{2}S_{L_E} \le E\right\} \le \mathbb{P}\left\{S_{L_E} \le \frac{1}{2}\mathbb{E}[S_{L_E}]\right\}$$

Finally, observe that the random variables $|\lambda_k A|$, $k \in \mathbb{Z}$, are independent. Bernstein's inequality bounds the last probability by $\exp(-C_4(2L_E)^d)$ with some positive constant C_4 , since $\#I_{L_E} = (2L_E)^d$. Restricting *E* further to be smaller than $C_3\mathbb{E}[S_1]/32$, we see, from the definition of L_E ,

$$N(E) \le C_1 E^{d/2} \exp(-C_4 (2L_E)^d) \le C_1 E^{d/2} \exp(-C_2 E^{-d/2})$$

with $C_2 = C_4 (C_3 \mathbb{E}[|\lambda_0 A|]/8)^{d/2}$. \Box

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