Functional analysis

Common solutions to a system of variational inequalities over the set of common fixed points of demi-contractive operators

Solutions communes d’inégalités variationnelles sur l’ensemble des points fixes communs d’opérateurs semi-contractants

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ABSTRACT

In this paper, we introduce an explicit parallel algorithm for finding common solutions to a system of variational inequalities over the set of common fixed points of a finite family of demi-contractive operators. Under suitable assumptions, we prove the strong convergence of this algorithm in the framework of a Hilbert space. The results obtained in this paper extend and improve the results of Tian and Jiang (2017), of Censor, Gibali and Reich (2012), and of many others.

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RÉSUMÉ


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1. Introduction

Let $C$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $F: \mathcal{H} \to \mathcal{H}$ be a monotone operator. The classical variational inequality is formulated as the following problem:

$$\text{finding a point } x^* \in C \text{ such that } \langle Fx^*, y - x^* \rangle \geq 0, \quad \forall y \in C.$$
The set of solutions to this problem is denoted by \( V(I(F, C)) \). In recent years, variational inequalities have been used to study a large variety of problems arising in structural analysis, economics, optimization, operations research, and engineering sciences (see, e.g., [20,28,19] and the references therein).

Observe that the feasible set \( C \) of the variational inequality problem can always be represented as the fixed point set of some operator, say, \( C = \text{Fix}(P_C) \) (\( P_C \) is the metric projection onto \( C \)). Following this idea, Yamada [26] considered the variational inequality problem \( V(I(F, \text{Fix}(T))) \), which calls for finding a point \( x^* \in \text{Fix}(T) \) such that
\[
\langle F(x^*), y - x^* \rangle \geq 0 \quad \text{for all} \quad y \in \text{Fix}(T).
\]

Yamada [26] considered the following hybrid steepest-descent iterative method:
\[
x_{n+1} = (I - \mu \alpha_n F)Tx_n,
\]
where \( F \) is a Lipschitzian continuous and strongly monotone operator and \( T \) is a nonexpansive operator. Under some appropriate conditions, the sequence \( \{x_n\} \) converges strongly to the unique point in \( V(I(F, \text{Fix}(T))) \).

The literature on variational inequalities is vast, and the hybrid steepest-descent method has received great attention from many authors, who improved it in various ways; see, e.g., [27,7,6,29,5,15] and references therein.

Based on the hybrid steepest-descent method, recently (2017) Tian and Jiang [25] proved the following weak convergence theorem for zero points of an inverse strongly monotone operator and fixed points of a nonexpansive operator in a Hilbert space (Theorem 1.1).

**Theorem 1.1.** Let \( \mathcal{H} \) be a real Hilbert space and \( T : \mathcal{H} \to \mathcal{H} \) be a nonexpansive operator with \( \text{Fix}(T) \neq \emptyset \). Let \( F : \mathcal{H} \to \mathcal{H} \) be a \( k \)-inverse strongly monotone operator. Assume that \( \text{Fix}(T) \cap F^{-1}(0) \neq \emptyset \). Let \( \{x_n\} \) and \( \{y_n\} \) be sequences generated by \( x_1 \in \mathcal{H} \) and
\[
\begin{align*}
    y_n &= (1 - \lambda_n)x_n + \lambda_nTx_n, \\
    x_{n+1} &= (I - \mu \alpha_n F)y_n,
\end{align*}
\]
for each \( n \in \mathbb{N} \), where \( \{\lambda_n\} \in [a, b] \) for some \( a, b \in (0, 1) \) and \( \mu \alpha_n \subset [c, d] \) for some \( c, d \in (0, 2k) \). Then the sequences \( \{x_n\} \) and \( \{y_n\} \) converge weakly to a point \( z \in \text{Fix}(T) \cap F^{-1}(0) \), where \( z = \lim_{n \to \infty} P_{\text{Fix}(T) \cap F^{-1}(0)}x_n \). \( z \) is also a point in \( V(I(F, \text{Fix}(T))) \).

On the other hand, Censor, Gibali and Reich [11] (see also [12]), introduced the **Common Solutions to Variational Inequality Problem (CSVIP)**, which consists in finding common solutions to unrelated variational inequalities. The general form of the CSVIP is the following problem.

Let \( \mathcal{H} \) be a Hilbert space. Let there be given, for each \( i = 1, 2, ..., m \), an operator \( F_i : \mathcal{H} \to \mathcal{H} \) and a nonempty, closed and convex subset \( C_i \subset \mathcal{H} \), with \( \bigcap_{i=1}^{m} C_i \neq \emptyset \). The CSVIP (for single-valued operators) is to find a point \( z \in \bigcap_{i=1}^{m} C_i \) such that, for each \( i = 1, 2, ..., m \),
\[
\langle F_i z, x - z \rangle \geq 0, \quad \forall x \in C_i, \quad 1 \leq i \leq m.
\]
We note that in CSVIP, if we choose all \( F_i = 0 \), then the problem reduces to that of finding a point \( z \in \bigcap_{i=1}^{m} C_i \) in the nonempty intersection of a finite family of closed and convex sets, which is the well-known **Convex Feasibility Problem (CFP)**.

Now, in this paper, we study the following problem.

Let \( \mathcal{H} \) be a Hilbert space. Let there be given, for each \( i = 1, 2, ..., m \), an operator \( F_i : \mathcal{H} \to \mathcal{H} \) and an operator \( T_i : \mathcal{H} \to \mathcal{H} \) with \( \bigcap_{i=1}^{m} \text{Fix}(T_i) \neq \emptyset \). We intend to find a point \( z \in \bigcap_{i=1}^{m} \text{Fix}(T_i) \) such that, for each \( i = 1, 2, ..., m \),
\[
\langle F_i z, x - z \rangle \geq 0, \quad \forall x \in \text{Fix}(T_i), \quad 1 \leq i \leq m.
\]
Recall that an operator \( U : \mathcal{H} \to \mathcal{H} \) is said to be demicontractive [18] if there exists \( \mu \in [0, 1) \) such that
\[
\|Ux - p\|^2 \leq \|x - p\|^2 + \mu\|x - Ux\|^2, \quad \forall x \in \mathcal{H}, \quad \forall p \in \text{Fix}(U).
\]
In particular, if \( \mu = 0 \) then \( U \) is called quasi-nonexpansive on \( \mathcal{H} \). An operator satisfying (4) will be referred to as a \( \mu \)-demicontractive operator. This class of operators is fundamental because it includes many types of nonlinear operators arising in applied mathematics and optimization, see for example [23] and references therein.

In this paper, to solve (3), we introduce an explicit parallel algorithm based on the Halpern iterative method and the hybrid steepest-descent method for finding common solutions to a system of variational inequalities over the set of common fixed points of a finite family of quasi-nonexpansive operators. We prove the strong convergence of the algorithm for a family of inverse strongly monotone operators in the framework of a Hilbert space. Finally, some applications of our main results have been obtained. Our results generalize and improve the results of Tian and Jiang [25], Censor, Gibali and Reich [11], and of many others.
2. Preliminaries

We use the following notation in the sequel:

- \( \rightarrow \) for weak convergence and \( \rightarrow \) for strong convergence.

  Given a nonempty, closed convex set, \( C \subset \mathcal{H} \), the mapping that assigns every point, \( x \in \mathcal{H} \), to its unique nearest point in \( C \) is called the metric projection onto \( C \) and is denoted by \( P_C \); i.e., \( P_C(x) \in C \) and \( \|x - P_C(x)\| = \inf_{y \in C} \|x - y\| \). The metric projection \( P_C \) is characterized by the fact that \( P_C(x) \in C \) and

  \[
  \langle y - P_C(x), x - P_C(x) \rangle \leq 0, \quad \forall x \in \mathcal{H}, y \in C
  \]

(see for example, [[16], Section 3]).

We recall the following definitions concerning operator \( F : \mathcal{H} \to \mathcal{H} \). The operator \( F \) is called:

- Lipschitz continuous with constant \( L > 0 \) if

  \[
  \|F(x) - F(y)\| \leq L \|x - y\|, \quad \forall x, y \in \mathcal{H};
  \]

- monotone if

  \[
  \langle F(x) - F(y), x - y \rangle \geq 0, \quad \forall x, y \in \mathcal{H};
  \]

- strongly monotone with constant \( \beta > 0 \), if

  \[
  \langle F(x) - F(y), x - y \rangle \geq \beta \|x - y\|^2, \quad \forall x, y \in \mathcal{H};
  \]

- inverse strongly monotone with constant \( \beta > 0 \), \((\beta - ism)\) if

  \[
  \langle F(x) - F(y), x - y \rangle \geq \beta \|F(x) - F(y)\|^2, \quad \forall x, y \in \mathcal{H}.
  \]

It is known that every \( \beta \)-inverse strongly monotone operator is monotone and Lipschitz continuous. We note that there exist some operators that are inverse strongly monotone, but not strongly monotone [25].

We also have the following definitions concerning \( T : \mathcal{H} \to \mathcal{H} \). The operator \( T \) is called:

- nonexpansive, if

  \[
  \|T(x) - T(y)\| \leq \|x - y\|, \quad \forall x, y \in \mathcal{H};
  \]

- \( \beta \)-strict pseudo-contractive [2], if there exists a constant \( \beta \in [0, 1) \) such that

  \[
  \|Tx - Ty\|^2 \leq \|x - y\|^2 + \beta \|(x - Tx) - (y - Ty)\|^2, \quad \forall x, y \in \mathcal{H};
  \]

- generalized nonexpansive [14] if there exists a constant \( \mu \geq 0 \) such that

  \[
  \|Tx - Ty\| \leq \|x - y\| + \mu \|x - Tx\|, \quad \forall x, y \in \mathcal{H}.
  \]

We note that every generalized nonexpansive operator is quasi-nonexpansive. The class of demi-contractive operators contains the generalized nonexpansive operators and the strictly pseudo-contractive operators.

An operator \( T : \mathcal{H} \to \mathcal{H} \) is said to be an averaged operator [1] if there exists some number \( \alpha \in (0, 1) \) such that

\[
T = (1 - \alpha) I + \alpha S,
\]

where \( I : \mathcal{H} \to \mathcal{H} \) is the identity operator and \( S : \mathcal{H} \to \mathcal{H} \) is nonexpansive. More precisely, when (5) holds, we say that \( T \) is \( \alpha \)-averaged. It is not difficult to see that the averaged operator \( T \) is also nonexpansive and \( \text{Fix}(T) = \text{Fix}(S) \).

**Lemma 2.1.** [8] Let \( \mathcal{H} \) be a real Hilbert space. Let \( T : \mathcal{H} \to \mathcal{H} \) be an operator.

(i) \( T \) is nonexpansive if and only if the complement \( I - T \) is \( \frac{1}{2} \)-inverse strongly monotone.

(ii) If \( T \) is \( \kappa \)-inverse strongly monotone, then for \( \gamma > 0 \), \( \gamma T \) is \( \frac{\kappa}{\gamma^2} \)-inverse strongly monotone.

(iii) For \( \alpha \in (0, 1) \), \( T \) is \( \alpha \)-averaged if and only if \( I - T \) is \( \frac{1}{2\alpha} \)-inverse strongly monotone.

**Definition 2.2.** Let \( U : C \to C \) be an operator, then \( I - U \) is said to be demiclosed at zero if for any sequence \( \{x_n\} \) in \( C \), the conditions \( x_n \rightharpoonup x \) and \( \lim_{n \to \infty} \|x_n - Ux_n\| = 0 \), imply \( x = Ux \).

**Lemma 2.3.** [13] Let \( C \) be nonempty closed convex subset of a real Hilbert space \( \mathcal{H} \), and \( U : C \to C \) be \( \beta \)-demicontactive operator. Then the fixed point set \( \text{Fix}(U) \) of \( U \) is closed and convex.
Lemma 2.4. ([22]) Let C be nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, and let $U : C \to C$ be $\beta$-strict pseudo-contractive. Then $I - U$ is demiclosed at 0.

Lemma 2.5. ([14]) Let C be nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, and let $U : C \to C$ be a generalized nonexpansive operator. Then $I - U$ is demiclosed at 0.

Lemma 2.6. For all $x, y \in \mathcal{H}$ and $\alpha \in [0, 1]$, there holds the relation:

$$\|\alpha x + (1 - \alpha) y\|^2 = \alpha \|x\|^2 + (1 - \alpha) \|y\|^2 - \alpha (1 - \alpha) \|x - y\|^2.$$  

Lemma 2.7. ([17]) Assume $\{s_n\}$ is a sequence of nonnegative real numbers such that

$$\begin{align*}
s_{n+1} &\leq (1 - \lambda_n)s_n + \lambda_n \delta_n, \quad n \geq 0, \\
\sum_{n=1}^{\infty} \lambda_n &\leq \infty,
\end{align*}$$

where $\{\lambda_n\}$ is a sequence in $(0, 1)$, $\{\eta_n\}$ is a sequence of nonnegative real numbers and $\{\delta_n\}$ and $\{\mu_n\}$ are two sequences in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \lambda_n = \infty$,
(ii) $\lim_{n \to \infty} \mu_n = 0$,
(iii) $\lim_{n \to \infty} \eta_n = 0$, implies $\limsup_{n \to \infty} \delta_n \leq 0$ for any subsequence $(n_k) \subset (n)$.

Then $\lim_{n \to \infty} s_n = 0$.

3. An algorithm and its convergence analysis

In this section, we introduce an explicit parallel algorithm for finding common solutions to a system of variational inequalities over the set of common fixed points of a finite family of demi-contractive operators.

Theorem 3.1. Let $\mathcal{H}$ be a real Hilbert space. Let for each $i \in \{1, 2, \ldots, m\}$, $F_i : \mathcal{H} \to \mathcal{H}$ be a $\kappa_i$-inverse strongly monotone operator and $T_i : \mathcal{H} \to \mathcal{H}$ be a $\lambda_i$-demi-contractive operator such that $I - T_i$ is demiclosed at 0. Assume that $\mathcal{F} = \bigcap_{i=1}^{m} \text{Fix}(T_i) \cap \bigcap_{i=1}^{m} F_i^{-1}(0) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_0, v \in \mathcal{H}$ and by

$$\begin{align*}
y_n &\left\{ \begin{array}{ll}
(\forall i \in \{1, 2, \ldots, m\}) &\left\{ \begin{array}{ll}
y_n = (1 - \mu_n^i \rho_n^i F_i^i)^i T_i^i x_0, \\
\lambda_n = \lambda_n^i y_n + \sum_{i=1}^{m} \lambda_n^i y_n^i, \\
\end{array} \right.
\end{array} \right. \\
\end{align*}$$

where $T_i^i = \alpha_i^0 i + (1 - \alpha_i^0) T_i$. Let the sequences $\{\alpha_n^i\}, \{\beta_n^i\}$ and $\{\gamma_n^i\}$ satisfy the following conditions:

(i) $\{\gamma_n^i\} \subset [a_i, b_i] \subset (0, 1)$ and $\sum_{n=0}^{\infty} \gamma_n^i = 1$,
(ii) $\lim_{n \to \infty} \gamma_n^i = 0$ and $\sum_{n=0}^{\infty} \gamma_n^i = \infty$,
(iii) $\{\mu_n^i \beta_n^i\} \subset [c_i, d_i] \subset (0, 2 \kappa_i)$,
(iv) $\lambda_i < \alpha_i^0 \leq c_i < 1$.

Then the sequence $\{x_n\}$ converges strongly to $P_{\mathcal{F}} v \in \mathcal{F}$ which is also a point in $\bigcap_{i=1}^{m} \text{VI}(F_i, \text{Fix}(T_i))$.

Proof. First we show that $\{x_n\}$ is bounded. Take $x^* \in \mathcal{F}$. Since for each $i \in \{1, 2, \ldots, m\}$, $T_i$ is $\lambda_i$-demi-contractive, using Lemma 2.6 we arrive at

$$\begin{align*}
\|T_i^i x_0 - x^*\|^2 &= \|\alpha_i^0 x_0 + (1 - \alpha_i^0) T_i x_0 - x^*\|^2 \\
&\leq \alpha_i^0 \|x_0 - x^*\|^2 + (1 - \alpha_i^0) \|T_i x_0 - x^*\|^2 \leq \alpha_i^0 (1 - \alpha_i^0) \|T_i x_0 - x_0\|^2 \\
&\leq \alpha_i^0 \|x_0 - x^*\|^2 + (1 - \alpha_i^0) \|x_0 - x^*\|^2 + \lambda_i \|T_i x_0 - x_0\|^2)
\end{align*}$$

(7)

Put $w_n^i = T_i^i x_0$. From condition (iii) and by the assumption that $F_i$ is a $\kappa_i$-inverse strongly monotone operator, we get that

$$\begin{align*}
\|y_n^i - x^*\|^2 &= \|(1 - \mu_n^i \beta_n^i F_i^i) w_n^i - (1 - \mu_n^i \beta_n^i F_i^i) x^*\|^2 \\
&\leq \|w_n^i - x^*\|^2 + (\mu_n^i \beta_n^i)^2 \|F_i w_n^i - F_i x^*\|^2 - 2 \mu_n^i \beta_n^i (F_i w_n^i - F_i x^*, w_n^i - x^*)
\end{align*}$$
By the convexity of $\|\cdot\|^2$, we have:

$$\|x_{n+1} - x^*\|^2 = \|y_n^{(0)} v + \sum_{i=1}^{m} y_n^{(i)} y_n^{(i)} - x^*\|^2$$

$$\leq y_n^{(0)} \|v - x^*\|^2 + \sum_{i=1}^{m} y_n^{(i)} \|y_n^{(i)} - x^*\|^2$$

$$\leq y_n^{(0)} \|v - x^*\|^2 + (1 - y_n^{(0)}) \|x_n - x^*\|^2$$

$$\leq \max\{\|v - x^*\|^2, \|x_n - x^*\|^2\}$$

$$\leq \ldots \leq \max\{\|v - x^*\|^2, \|x_0 - x^*\|^2\}. \tag{8}$$

This yields that the sequence $\{x_n\}$ is bounded. Furthermore, the sequence $\{y_n^{(0)}\}$ is bounded. Next we prove that the sequences $\{x_n\}$ converge strongly to $v^* = P_{\mathcal{H}} v$. From the inequality, $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x \rangle (\forall x, y \in \mathcal{H})$, we find that

$$\|x_{n+1} - v^*\|^2 \leq \|\sum_{i=1}^{m} y_n^{(0)} y_n^{(0)} - (1 - y_n^{(0)}) v^*\|^2$$

$$+ 2 y_n^{(0)} \langle v - v^*, x_{n+1} - v^* \rangle$$

$$= \|1 - y_n^{(0)}\| \sum_{i=1}^{m} \frac{y_n^{(i)}}{1 - y_n^{(0)}} y_n^{(i)} - v^*\|^2$$

$$+ 2 y_n^{(0)} \langle v - v^*, x_{n+1} - v^* \rangle$$

$$= (1 - y_n^{(0)})^2 \sum_{i=1}^{m} \frac{y_n^{(i)}}{1 - y_n^{(0)}} \|y_n^{(i)} - v^*\|^2$$

$$+ 2 y_n^{(0)} \langle v - v^*, x_{n+1} - v^* \rangle$$

$$\leq (1 - y_n^{(0)}) \sum_{i=1}^{m} y_n^{(i)} \|y_n^{(i)} - v^*\|^2$$

$$+ 2 y_n^{(0)} \langle v - v^*, x_{n+1} - v^* \rangle$$

$$\leq (1 - y_n^{(0)})^2 \|x_n - v^*\|^2$$

$$+ 2 y_n^{(0)} \langle v - v^*, x_{n+1} - v^* \rangle. \tag{9}$$

It immediately follows that

$$\Gamma_{n+1} \leq (1 - y_n^{(0)})^2 \Gamma_n + 2 y_n^{(0)} \eta_n$$

$$= (1 - 2 y_n^{(0)}) \Gamma_n + (y_n^{(0)})^2 \Gamma_n + 2 y_n^{(0)} \eta_n$$

$$\leq (1 - 2 y_n^{(0)}) \Gamma_n + 2 y_n^{(0)} \left(\frac{y_n^{(0)} N}{2} + \eta_n\right)$$

$$\leq (1 - \rho_0) \Gamma_n + \rho_0 \delta_n. \tag{9}$$

where $\Gamma_n = \|x_n - \delta^*\|^2$, $\eta_n = \langle v - v^*, x_{n+1} - v^* \rangle$, $N = \sup\{\|x_n - v^*\|^2 : n \geq 0\}$, $\rho_0 = 2 y_n^{(0)}$ and $\delta_n = \frac{y_n^{(0)} N}{2} + \eta_n$. We observe that $\rho_0 \to 0$, $\sum_{n=1}^{\infty} \rho_n = \infty$.

Since $F_i$ is $k_i$-inverse strongly monotone, we can rewrite $y_n^{(i)}$ as

$$y_n^{(i)} = (1 - \xi_n^{(i)}) w_n^{(i)} + \xi_n^{(i)} \tilde{y}_n^{(i)} w_n^{(i)},$$

by using Lemma 2.1, where $\xi_n^{(i)} = \frac{\mu_n^{(i)} \rho_n^{(i)}}{2k_i}$ and $\tilde{y}_n^{(i)}$ are nonexpansive operators of $\mathcal{H}$ into $\mathcal{H}$. Utilizing Lemma 2.6, we have:
\[ \| y_n^{(i)} - v^* \|^2 = \| (1 - \xi_n^{(i)}) w_n^{(i)} + \xi_n^{(i)} S_n^{(i)} w_n^{(i)} - v^* \|^2 \]
\[ \leq (1 - \xi_n^{(i)}) \| w_n^{(i)} - v^* \|^2 + \xi_n^{(i)} \| S_n^{(i)} w_n^{(i)} - v^* \|^2 - \xi_n^{(i)} (1 - \xi_n^{(i)}) \| S_n^{(i)} w_n^{(i)} - w_n^{(i)} \|^2 \]
\[ \leq \| w_n^{(i)} - v^* \|^2 - \xi_n^{(i)} (1 - \xi_n^{(i)}) \| S_n^{(i)} w_n^{(i)} - w_n^{(i)} \|^2. \]

This implies that
\[ \| x_{n+1} - v^* \|^2 = \| y_n^{(0)} v + \sum_{i=1}^{m} y_n^{(i)} y_n^{(i)} - v^* \|^2 \]
\[ \leq \| y_n^{(0)} v - v^* \|^2 + \sum_{i=1}^{m} \| y_n^{(i)} y_n^{(i)} - v^* \|^2 \]
\[ \leq \| y_n^{(0)} v - v^* \|^2 + (1 - \| y_n^{(0)} \|) \| x_n^{(0)} - v^* \|^2 \]
\[ - \sum_{i=1}^{m} \| y_n^{(i)} (1 - \xi_n^{(i)}) \| S_n^{(i)} w_n^{(i)} - w_n^{(i)} \|^2 \]
\[ - \sum_{i=1}^{m} \| y_n^{(i)} (1 - \alpha_n^{(i)}) (\alpha_n^{(i)} - \lambda_i) \| T_i x_n - x_n \|^2. \]

Now by setting
\[ \xi_n = \sum_{i=1}^{m} \| y_n^{(i)} \| \xi_n^{(i)} (1 - \xi_n^{(i)}) \| S_n^{(i)} w_n^{(i)} - w_n^{(i)} \|^2 \]
\[ + \sum_{i=1}^{m} \| y_n^{(i)} (1 - \alpha_n^{(i)}) (\alpha_n^{(i)} - \lambda_i) \| T_i x_n - x_n \|^2, \]

and
\[ \rho_n = \| y_n^{(0)} v - v^* \|^2, \]
the inequality (10) can be rewritten in the following form:
\[ \Gamma_{n+1} \leq \Gamma_n - \xi_n + \rho_n. \]

To use Lemma 2.7 (considering inequalities (9) and (13)), it suffices to verify that, for all subsequences \( n_k \subset \{n\}, \lim_{k \to \infty} \xi_{n_k} = 0 \) implies
\[ \lim_{k \to \infty} \delta_{n_k} \leq 0. \]

We assume that \( \lim_{k \to \infty} \xi_{n_k} = 0 \). From (11) and by our assumptions on \( \{y_n^{(i)}\}, \{\alpha_n^{(i)}\}, \) and \( \{\xi_n^{(i)}\} \), we have:
\[ \lim_{k \to \infty} \| T_i x_{n_k} - x_{n_k} \| = \lim_{k \to \infty} \| S_n^{(i)} w_{n_k} - w_{n_k}^{(i)} \| = 0. \]

Since \( \{x_k\} \) is bounded, there exists a subsequence \( \{x_{n_k}\} \) of \( \{x_k\} \) that converges weakly to \( \hat{x} \). Without loss of generality, we can assume that \( x_{n_k} \to \hat{x} \). Since \( \lim_{n \to \infty} \| x_n - w_n \| = 0 \), we have \( w_{n_k} \to \hat{x} \). Since \( \{\beta_{n_k}^{(i)}\} \) is bounded, we can find a subsequence \( \{\beta_{n_k}^{(i)}\} \) converging to \( \beta^{(i)} \) such that \( \mu^{(i)} \beta^{(i)} \subset [c, d] \). Since \( \{w_{n_k}^{(i)}\} \) is bounded and \( F_i \) is inverse strongly monotone, we know that \( F_i w_{n_k}^{(i)} \) is bounded. Hence, we have:
\[ \| (I - \mu^{(i)} \beta_{n_k}^{(i)} F_i) w_{n_k}^{(i)} - (I - \mu^{(i)} \beta^{(i)} F_i) w_{n_k}^{(i)} \| \leq \| \mu^{(i)} \beta_{n_k}^{(i)} - \mu^{(i)} \beta^{(i)} \| \| F_i w_{n_k}^{(i)} \| \to 0. \]

From (14) we have \( \lim_{n \to \infty} \| y_n^{(i)} - w_{n_k}^{(i)} \| = 0 \), hence
\[ \| (I - \mu^{(i)} \beta_{n_k}^{(i)} F_i) w_{n_k}^{(i)} - w_{n_k}^{(i)} \| \to 0. \]

Therefore, we get
\[ \| (I - \mu^{(i)} \beta_{n_k}^{(i)} F_i) w_{n_k}^{(i)} - w_{n_k}^{(i)} \| \leq \| (I - \mu^{(i)} \beta_{n_k}^{(i)} F_i) w_{n_k}^{(i)} - (I - \mu^{(i)} \beta^{(i)} F_i) w_{n_k}^{(i)} \|
+ \| (I - \mu^{(i)} \beta_{n_k}^{(i)} F_i) w_{n_k}^{(i)} - w_{n_k}^{(i)} \| \to 0. \]
From the demiclosedness of \( I - \mu^{(i)} p^{(i)} F_i \), we obtain that
\[
\tilde{x} \in \text{Fix}(I - \mu^{(i)} p^{(i)} F_i) = F_i^{-1}(0), \quad i \in \{1, 2, \ldots, m\}.
\]
From the demiclosedness of \( I - T_i \) and using (14), we get that \( \tilde{x} \in \bigcap_{i=1}^m \text{Fix}(T_i) \). Thus \( \tilde{x} \in \mathcal{F} \). Now we show that
\[
\lim \sup_{k \to \infty} \delta_{n_k} = \lim \sup_{k \to \infty} (\vartheta - \vartheta^*, x_{n_k} - \vartheta^*) \leq 0. \tag{15}
\]
To show this inequality, we choose a subsequence \( \{x_{n_k}\} \) of \( \{x_{n_j}\} \) such that
\[
\lim_{j \to \infty} (v - v^*, x_{n_k} - v^*) = \lim \sup_{k \to \infty} (v - v^*, x_{n_k} - v^*) = (v - P_{\mathcal{F}}(v), \tilde{x} - P_{\mathcal{F}}(v)) \leq 0. \tag{16}
\]
Hence, all conditions of Lemma 2.7 are satisfied. Therefore, we immediately deduce that \( \lim_{n \to \infty} \Gamma_n = \lim_{n \to \infty} \|x_n - v^*\| = 0 \), that is \( \{x_n\} \) converges strongly to \( v^* = P_{\mathcal{F}}(v) \), which completes the proof. \( \square \)

**Theorem 3.2.** Let \( \mathcal{H} \) be a real Hilbert space. Let, for each \( i \in \{1, 2, \ldots, m\} \), \( F_i : \mathcal{H} \to \mathcal{H} \) be a \( k_i \)-inverse strongly monotone operator and \( T_i : \mathcal{H} \to \mathcal{H} \) be a \( k_i \)-strict pseudo-contractive operator. Assume that \( \mathcal{F} = \bigcap_{i=1}^m \text{Fix}(T_i) \cap \bigcap_{i=1}^m F_i^{-1}(0) \neq \emptyset \). Let \( \{x_n\} \) be the sequence generated by \( x_0 \in \mathcal{H} \) and
\[
x_{n+1} = \sum_{i=1}^m \gamma_n^{(i)} (I - \mu^{(i)} \rho_n^{(i)} F_i) T_i^n x_n, \quad \forall n \geq 0, \tag{17}
\]
where \( T_i^n = \alpha_n^1 I + (1 - \alpha_n^1) T_i \). Let the sequences \( \{\alpha_n^{(i)}\}, \{\rho_n^{(i)}\} \) and \( \{\gamma_n^{(i)}\} \) satisfy the following conditions:

(i) \( \gamma_n^{(i)} \subset [a_i, b_i] \subset (0, 1) \),

(ii) \( \sum_{i=1}^m \gamma_n^{(i)} = 1 - \gamma_n^{(0)} \) where \( \gamma_n^{(0)} \in (0, 1) \), \( \lim_{n \to \infty} \gamma_n^{(0)} = 0 \) and \( \sum_{n=1}^\infty \gamma_n^{(0)} = \infty \).

(iii) \( \mu^{(i)} \rho_n^{(i)} \subset [c_i, d_i] \subset (0, 2k_i) \).

(iv) \( \lambda_1 < \alpha_n^{(i)} \leq \varepsilon_1 < 1 \).

Then, the sequence \( \{x_n\} \) converges strongly to \( x^* \in \mathcal{F} \), which satisfies \( \|x^*\| = \min\{\|x\| : x \in \mathcal{F}\} \).

**Proof.** We note that every strict pseudo-contractive mapping is demi-contractive. Also, from Lemma 2.4 we know that \( I - T_i \) are demiclosed at 0. Now setting \( v = 0 \) in Theorem 3.1 we obtain the desired result. \( \square \)

Now we consider an algorithm similar to algorithm (6) for finding common solutions to a system of variational inequalities over the set of common fixed points of a finite family of strongly quasi-nonexpansive operators. Recall that an operator \( U : \mathcal{H} \to \mathcal{H} \) is said to be \( \rho \)-strongly quasi-nonexpansive, where \( \rho \geq 0 \), if
\[
\|UX - p\|^2 \leq \|X - p\|^2 - \rho \|X - UX\|^2, \quad \forall X \in \mathcal{H}, \quad \forall p \in \text{Fix}(U). \tag{18}
\]
More information on strongly quasi-nonexpansive operators can be found in Section 2.2 of [21].

**Theorem 3.3.** Let \( \mathcal{H} \) be a real Hilbert space. Let for each \( i \in \{1, 2, \ldots, m\} \), \( F_i : \mathcal{H} \to \mathcal{H} \) be a \( k_i \)-inverse strongly monotone operator and \( U_i : \mathcal{H} \to \mathcal{H} \) be a \( \rho_i \)-strongly quasi-nonexpansive operator where \( \rho_i > 0 \) and such that \( I - U_i \) is demiclosed at 0. Assume that \( \mathcal{F} = (\bigcap_{i=1}^m \text{Fix}(U_i)) \cap \bigcap_{i=1}^m F_i^{-1}(0) \neq \emptyset \). Let \( \{x_n\} \) be the sequence generated by \( x_0 \in \mathcal{H} \) and by
\[
\begin{align*}
\gamma_n^{(i)} = (I - \mu^{(i)} \rho_n^{(i)} F_i) U_i x_n & , \quad i = 1, 2, \ldots, m \\
x_{n+1} = \gamma_n^{(0)} v + \sum_{i=1}^m \gamma_n^{(i)} y_n^{(i)} & , \quad \forall n \geq 0. \tag{19}
\end{align*}
\]
Let the sequences \( \{\rho_n^{(i)}\} \) and \( \{\gamma_n^{(i)}\} \) satisfy the following conditions:

(i) \( \gamma_n^{(i)} \subset [a_i, b_i] \subset (0, 1) \) and \( \sum_{i=0}^m \gamma_n^{(i)} = 1 \).

(ii) \( \lim_{n \to \infty} \gamma_n^{(0)} = 0 \) and \( \sum_{n=1}^\infty \gamma_n^{(0)} = \infty \).

(iii) \( \mu^{(i)} \rho_n^{(i)} \subset [c_i, d_i] \subset (0, 2k_i) \).

Then, the sequence \( \{x_n\} \) converges strongly to \( P_{\mathcal{F}} v \in \mathcal{F} \) which is also a point in \( \bigcap_{i=1}^m \text{V Fix}(F_i) \).
Proof. Since $U_i$ is $\rho_i$-strongly quasi-nonexpansive operator with $\rho_i > 0$, for each $x^* \in F$, we have:
\[
\|U_i x_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \rho_i \|x_n - U_i x_n\|^2.
\] (20)
On substituting inequality (20) into inequality (7) in Theorem 3.1 and by similar arguments, we obtain the desired result. \qed

Remark 3.4. In [25], Tian and Jiang proved a weak convergence theorem (see Theorem 1.1) for finding zero points of an inverse strongly monotone operator and fixed points of a nonexpansive operator in a Hilbert space. In this paper, we generalized the result for finding common fixed points of a finite family of demi-contractive operators (as a general class of operators) and of zero points of a family of inverse strongly monotone operators. We also proved a strong convergence theorem, which is more desirable than weak convergence.

4. Applications

In this section, we present some application of our main result.

4.1. The multiple-set split feasibility problem

Let $\mathcal{H}$ and $\mathcal{K}$ be real Hilbert spaces, $A : \mathcal{H} \to \mathcal{K}$ be a bounded linear operator and let $\{C_i\}_{i=1}^p$ be a family of nonempty closed convex subsets in $\mathcal{H}$ and $\{Q_i\}_{i=1}^r$ be a family of nonempty closed convex subsets in $\mathcal{K}$. The multiple-set split feasibility problem (MSSFP) was introduced by Censor et al. (2005) [10], and is formulated as finding a point $x^*$ with the property:
\[
\bigcap_{i=1}^p C_i \quad \text{and} \quad A x^* \in \bigcap_{i=1}^r Q_i.
\]
Masad and Reich [24] studied the constrained multiple-set split convex feasibility problem (CMSSCFP). Let $A_i : \mathcal{H} \to \mathcal{K}$, $i = 1, 2, \ldots, r$, be $r$ bounded linear operators and let $\Omega$ be another closed and convex subset of $\mathcal{H}$. The CMSSCFP is formulated as follows:

find a point $x^* \in \Omega$ such that $x^* \in \bigcap_{i=1}^p C_i$ and $A_i(x^*) \in Q_i$ for each $i = 1, 2, \ldots, r$.

The multiple-set split feasibility problem with $p = r = 1$ is known as the split feasibility problem (SFP), which is formulated as finding a point $x^*$ with the property:
\[
x^* \in C \quad \text{and} \quad A x^* \in Q,
\]
where $C$ and $Q$ are nonempty closed convex subsets of $\mathcal{H}$ and $\mathcal{K}$, respectively. The split feasibility problem was introduced by Censor and Elfving (1994) ([9]). It has attracted many authors attention due to its application in optimization problem and signal processing. To solve the SFP, Byrne [3,4] proposed his CQ algorithm, which generates a sequence $\{x_n\}$ by
\[
x_{n+1} = P_C \left( I - \lambda A^* (I - P_Q) A \right) x_n
\]
where $\lambda \in (0, \frac{2}{\|A\|^2})$, $A^*$ is the adjoint of $A$.

Now we present an algorithm for solving the multiple-set split feasibility problem when $C_i$ are the fixed point set of nonlinear operators.

Theorem 4.1. Let $\mathcal{H}$ and $\mathcal{K}$ be two real Hilbert spaces. Let for each $i \in \{1, 2, \ldots, m\}$, $A_i : \mathcal{H} \to \mathcal{K}$ be a bounded linear operator and $T_i : \mathcal{H} \to \mathcal{H}$ be a generalized nonexpansive mapping. Let $\{Q_i\}_{i=1}^m$ be a family of nonempty closed convex subsets in $\mathcal{K}$. Assume that $\mathcal{F} = \{x^* \in \bigcap_{i=1}^m \text{Fix}(T_i) : A_i(x^*) \in Q_i, \quad i = 1, 2, \ldots, m\} \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_0, v \in \mathcal{H}$ and by
\[
\begin{align*}
\frac{y_n^{(i)}}{\gamma_n^{(i)}} & = (1 - \mu_n^{(i)}) \frac{\beta_n^{(i)}}{\gamma_n^{(i)}} A_i^* (I - P_Q) A_i T_i^n x_n, \quad i = 1, 2, \ldots, m
\end{align*}
\]
\[
\begin{align*}
x_{n+1} & = \frac{\gamma_n^{(0)}}{\gamma_n} v + \sum_{i=1}^m \frac{\gamma_n^{(i)}}{\gamma_n} y_n^{(i)}, \quad \forall n \geq 0,
\end{align*}
\] (21)
where $T_i^n = \alpha_n^{(i)} I + (1 - \alpha_n^{(i)}) T_i$. Let the sequences $\{\alpha_n^{(i)}\}, \{\beta_n^{(i)}\}$ and $\{\gamma_n^{(i)}\}$ satisfy the following conditions:
\begin{enumerate}
\item[(i)] $\{\gamma_n^{(i)}\} \in [a_i, b_i] \subset (0, 1)$ and $\sum_{n=0}^m \gamma_n^{(i)} = 1$,
\item[(ii)] $\lim_{n \to \infty} \gamma_n^{(i)} = 0$ and $\sum_{n=1}^m \gamma_n^{(i)} = \infty$,
\item[(iii)] $\{\mu_n^{(i)} \beta_n^{(i)}\} \subset [c_i, d_i] \subset (0, \frac{2}{\|A\|^2})$,
\item[(iv)] $\{\alpha_n^{(i)}\} \subset [e_i, l_i] \subset (0, 1)$.
\end{enumerate}
Then, the sequence $\{x_n\}$ converges strongly to $P_F v \in \mathcal{F}$.
Proof. Notice that $A_ix^* \in Q_i$ if and only if $x^* \in (A_i^*(I - P_{Q_i})A_i)^{-1}(0)$. Putting $F_i = A_i^*(I - P_{Q_i})A_i$ we see that $F_i$ is $\frac{1}{|A_i|^2}$-inverse strongly monotone operator (see [25] for details). We note that every generalized nonexpansive operator is 0-demi-contractive. Also, from Lemma 2.5 we know that $I - T_i$ are demiclosed at 0. Now, utilizing Theorem 3.1, we obtain the desired result. □

4.2. Common solutions to a system of variational inequalities

Now, we present a strong convergence theorem for finding common solutions to a system of variational inequalities that generalizes the result of Censor, Gibali, and Reich [11].

Theorem 4.2. Let $\mathcal{H}$ be a real Hilbert space. Let for each $i \in \{1, 2, \ldots, m\}$, $C_i$ be a nonempty, closed convex subset of $\mathcal{H}$ and $F_i : \mathcal{H} \to \mathcal{H}$ be a $k_i$-inverse strongly monotone operator. Assume that $\mathcal{F} = \bigcap_{i=1}^{m} V(I(F_i, C_i)) \neq \emptyset$. Let $\{x_0\}$ be the sequence generated by $x_0, v \in \mathcal{H}$ and by

\[
\begin{align*}
  y_n^{(i)} &= (I - \mu_n^{(i)} F_i^\beta_n^{(i)}) P_{C_i} x_n, \quad i = 1, 2, \ldots, m \\
  x_{n+1} &= y_n^{(0)} v + \sum_{i=1}^{m} y_n^{(i)} y_n^{(i)}, \quad \forall n \geq 0.
\end{align*}
\]

Let the sequences $\{\beta_n^{(i)}\}$ and $\{y_n^{(i)}\}$ satisfy the following conditions:

(i) $\{y_n^{(i)}\} \subset [a, b] \subset (0, 1)$ and $\sum_{i=1}^{m} y_n^{(i)} = 1$,

(ii) $\lim_{n \to \infty} y_n^{(0)} = 0$ and $\sum_{i=1}^{\infty} y_n^{(i)} = \infty$,

(iii) $\{\mu_n^{(i)}\} \subset [c, d] \subset (0, 2k_i)$.

Then, the sequence $\{x_n\}$ converges strongly to $P_{\mathcal{F}} v \in \bigcap_{i=1}^{m} V(I(F_i, C_i))$.

Proof. We note that the metric projection $P_C$ is a 1-strongly quasi-nonexpansive operator. Now utilizing Theorem 3.3, we obtain the desired result. □

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References


