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About the conditional value at risk of partial sums

Valeur à risque conditionnelle de sommes de variables aléatoires réelles

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ABSTRACT

In this note, we give normal approximation results for the conditional value at risk (CVaR) of partial sums of random variables satisfying moment assumptions. These results are based on Berry–Esseen-type bounds for transport costs in the central limit theorem and extensions of Cantelli's inequalities to the CVaR.

RÉSUMÉ

Dans cette note, nous donnons des résultats d'approximation normale pour la CVaR d'une somme de variables aléatoires réelles satisfaisant des hypothèses de moments. Ces résultats sont fondés sur des bornes de type Berry–Esseen pour des coûts de transport dans le théorème limite central ainsi que sur des extensions des inégalités de Cantelli à la CVaR.

1. Introduction

Let $(X_k)_{k>0}$ be a sequence of integrable and centered real-valued random variables. Set $S_n = X_1 + \cdots + X_n$ and $S_0 = 0$. In this note, we are interested in upper bounds on the conditional value at risk (CVaR) of S_n, as defined below.

Definition 1.1. Let X be a real-valued integrable random variable. The tail function H_X is defined by $H_X(t) = \mathbb{P}(X > t)$. In this note, the function Q_X is the cadlag inverse of H_X, which is defined by

$$Q_X(u) = \inf\{x : H_X(x) \le u\}.$$

We emphasize that $Q_X(u)$ is the value of the usual quantile function at point $1-u$. The function $\tilde{Q}_X$ is defined by

$$\tilde{Q}_X(u) = u^{-1}\int_0^u Q_X(s)\,ds \quad \text{for any } u \in\,]0,1]. \tag{1.1}$$

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With this definition, $\tilde{Q}_X$ is the conditional value at risk (CVaR) of X, also called "expected shortfall". It is worth noticing that $\tilde{Q}_X(u)$ is the expected value of X conditionally on the fact that X is larger than $Q_X(u)$.

Remark 1.1. The function $\tilde{Q}_X(u)$ is in fact the Hardy–Littlewood maximal function associated with the law of X, as defined by Hardy and Littlewood [6] in a well-known paper.

Throughout this note, we assume that $(S_n)_{n\geq 0}$ is a martingale in L^1. Our original motivation was to give deviation inequalities for $S_n^* = \max(S_0, S_1, \ldots, S_n)$. Recall now that

$$Q_{S_n^*}(u) \leq \tilde{Q}_{S_n}(u) \ \text{ for any } u \in]0,1], \tag{1.2}$$

which shows that $Q_{S_n} \leq Q_{S_n^*} \leq \tilde{Q}_{S_n}$. This result may be found in [5]. From (1.2), any upper bound on the CVaR of S_n yields immediately the same upper bound on the quantile of S_n^*. Accordingly, from now on we will focus on conditional values at risk.

Let p be any real in $]1,\infty[$. For random variables X_k in L^p for some p in $]1,2]$, Rio [9, Theorem 4.1] gives the following upper bound on the conditional value at risk of S_n:

$$\tilde{Q}_{S_n}(1/z) \leq \|S_n\|_p \, z^{1/p}\big(1+(z-1)^{1-p}\big)^{-1/p} \ \text{ for any } z > 1. \tag{1.3}$$

For example, when $p = 2$, the above upper bound is equal to $\|S_n\|_2\sqrt{z-1}$. Our aim in this note is to improve (1.3) in the case $p \geq 2$. A possible way to improve this bound is to use global versions of the central limit theorem. Let $V_n = \operatorname{Var} S_n$. For independent and identically distributed square integrable random variables X_k, Agnew [1, Theorem 3.1] proved that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \big|H_{S_n}\big(t\sqrt{V_n}\big) - H_Y(t)\big| \, \mathrm{d}t = 0, \tag{1.4}$$

where Y is a random variable with standard normal law $N(0,1)$. From the elementary identity

$$\lim_{n\to\infty} \int_{\mathbb{R}} \big|H_{S_n}\big(t\sqrt{V_n}\big) - H_Y(t)\big| \mathrm{d}t = \int_0^1 \big|V_n^{-1/2} Q_{S_n}(u) - Q_Y(u)\big| \mathrm{d}u, \tag{1.5}$$

we immediately infer that

$$\big|V_n^{-1/2}\tilde{Q}_{S_n}(1/z) - \tilde{Q}_Y(1/z)\big| \leq z \int_{\mathbb{R}} \big|H_{S_n}\big(t\sqrt{V_n}\big) - H_Y(t)\big| \mathrm{d}t \tag{1.6}$$

for any $z > 1$. It follows that $V_n^{-1/2}\tilde{Q}_{S_n}(1/z)$ converges to $\tilde{Q}_Y(1/z)$ as n tends to ∞. Since $\tilde{Q}_Y(1/z) \leq \sqrt{2\log z}$ (*see* Remark 3.1), (1.6) provides a much more efficient upper bound on $\tilde{Q}_{S_n}(1/z)$ than (1.3) for z large, at least for large values of n. However, the upper bound on the right-hand side in the above inequality is rapidly increasing as z tends to ∞. To improve (1.6), we will use Fréchet's L^r-distances or generalizations of these distances rather than the L^1-distance between the tail functions. For $r \geq 1$ and X and Y real-valued random variables in L^r with respective laws P_X and P_Y, the Fréchet distance W_r is defined by

$$W_r(P_X, P_Y) = \left(\int_0^1 |Q_X(u) - Q_Y(u)|^r \mathrm{d}u\right)^{1/r}. \tag{1.7}$$

As shown by Fréchet [4], $W_r(P_X, P_Y)$ is the minimal transport cost in L^r. In Section 2, we give upper bounds on $\tilde{Q}_X - \tilde{Q}_Y$ involving $W_r(P_X, P_Y)$ or generalizations of this distance. In Section 3, we apply the results of Section 2 to the conditional value at risk of sums of independent random variables in L^p for $p \geq 3$. Section 4 is devoted to stationary martingale differences sequences.

2. Conditional value at risk and power transport costs

Throughout this section, X and Y are real-valued random variables in L^r for some $r \geq 1$. Let ψ be a positive, measurable and integrable function on $]0,1[$, bounded from below by some positive constant. Let $L_{r,\psi}$ be the subspace of L^r of real-valued random variables Z such that

$$\int_0^1 |Q_Z(u)|^r \psi(u) \, \mathrm{d}u < \infty. \tag{2.1}$$

For $r \geq 1$ and X and Y real-valued random variables in $L_{r,\psi}$ with respective laws P_X and P_Y, the distance $W_{r,\psi}$ is defined by

$$W_{r,\psi}(P_X, P_Y) = \left(\int_0^1 |Q_X(u) - Q_Y(u)|^r \psi(u)\,\mathrm{d}u \right)^{1/r}. \tag{2.2}$$

For random variables in $L_{r,\psi}$, the following upper bound holds true.

Theorem 2.1. *Let $r > 1$ and let X and Y be real-valued random variables in $L_{r,\psi}$. Then, for any $z > 1$*

$$|\tilde{Q}_X(1/z) - \tilde{Q}_Y(1/z)| \leq z^{1/r} \eta(z) W_{r,\psi}(P_X, P_Y), \;\; \textit{where} \;\; \eta(z) = \left(z \int_0^{1/z} \big(\psi(u)\big)^{-1/(r-1)} \mathrm{d}u \right)^{(r-1)/r}.$$

Remark 2.1. When $\psi = 1$, Theorem 2.1 yields the elementary inequality

$$|\tilde{Q}_X(1/z) - \tilde{Q}_Y(1/z)| \leq z^{1/r} W_r(P_X, P_Y). \tag{2.3}$$

If $\psi(u) \to \infty$ as u tends to 0, then $\eta(z)$ converges to 0 as $z \uparrow \infty$. In that case, Theorem 2.1 provides sharper bounds than (2.3) for large values of z. If furthermore ψ is nonincreasing on $]0, 1/2]$, then $\eta(z) \leq (\psi(1/z))^{-1/r}$ for any $z \geq 2$ and, consequently,

$$|\tilde{Q}_X(1/z) - \tilde{Q}_Y(1/z)| \leq (z/\psi(1/z))^{1/r} W_{r,\psi}(P_X, P_Y), \;\; \text{for any } z \geq 2. \tag{2.4}$$

Proof of Theorem 2.1. From the definition of the CVaR,

$$|\tilde{Q}_X(1/z) - \tilde{Q}_Y(1/z)| \leq z \int_0^{1/z} f(u) g(u)\,\mathrm{d}u, \;\; \text{where } f(u) = |Q_X - Q_Y|\, \psi^{1/r} \;\text{ and } g = \psi^{-1/r}.$$

Applying now the Hölder inequality to f and g with respective exponents r and $r/(r-1)$, we get that

$$|\tilde{Q}_X(1/z) - \tilde{Q}_Y(1/z)| \leq \Big(z \int_0^{1/z} |Q_X(u) - Q_Y(u)|^r \psi(u)\,\mathrm{d}u \Big)^{1/r} \eta(z),$$

which implies Theorem 2.1. □

When $\psi = 1$, Theorem 2.1 can be further improved for random variables X and Y in L^r satisfying the additional condition $\mathbb{E}(X) = \mathbb{E}(Y)$, as shown by Theorem 2.2 below.

Theorem 2.2. *Let $r \geq 1$ and let X and Y be real-valued random variables in L^r. Assume furthermore that $\mathbb{E}(X) = \mathbb{E}(Y)$. Then*

$$|\tilde{Q}_X(1/z) - \tilde{Q}_Y(1/z)| \leq z^{1/r} (1 + (z-1)^{1-r})^{-1/r} W_r(P_X, P_Y) \;\; \textit{for any } z > 1. \tag{2.5}$$

Remark 2.2. From Theorem 2.2 applied with $z = 2$,

$$|\tilde{Q}_X(1/2) - \tilde{Q}_Y(1/2)| \leq W_r(P_X, P_Y) \;\; \text{for any } r \geq 1, \tag{2.6}$$

which cannot be derived from (2.3). Note also that, for $r = 2$, Theorem 2.2 yields the Cantelli-type inequality

$$|\tilde{Q}_X(1/z) - \tilde{Q}_Y(1/z)| \leq \sqrt{z-1}\; W_2(P_X, P_Y) \;\; \text{for any } z > 1. \tag{2.7}$$

Remark 2.3. For $r > 1$, the upper bound in (2.5) is equivalent to $z^{1/r} W_r(P_X, P_Y)$ as $z \uparrow \infty$. Now, from the elementary equality

$$z^{1/r} (1 + (z-1)^{1-r})^{-1/r} = z^{1/r} (z-1)^{(r-1)/r} (1 + (z-1)^{r-1})^{-1/r}, \tag{2.8}$$

the upper bound in (2.5) is equivalent to $(z-1)^{1-(1/r)} W_r(P_X, P_Y)$ as $z \downarrow 1$. Hence, for $r > 1$ this upper bound converges to 0 as $z \downarrow 1$.

Remark 2.4. Inequality (2.5) is equivalent to the symmetric inequality

$$|u \tilde{Q}_X(u) - u \tilde{Q}_Y(u)| \leq \big(u(1-u)\big)^{1-(1/r)} \big(u^{r-1} + (1-u)^{r-1}\big)^{-1/r} W_r(P_X, P_Y) \;\; \text{for any } u \in\,]0, 1[.$$

Proof of Theorem 2.2. We proceed as in Rio [9], Section 4. If U be a random variable with uniform law on $[0,1]$, then $Q_X(U)$ has the same law as X. Hence $\mathbb{E}(X)=\tilde{Q}_X(1)$ and $\mathbb{E}(Y)=\tilde{Q}_Y(1)$, which ensures that $\tilde{Q}_X(1)=\tilde{Q}_Y(1)$. It follows that, for any b in $[0,1]$,

$$|\tilde{Q}_X(1/z)-\tilde{Q}_Y(1/z)|=\Big|z\int_0^1 (Q_X(u)-Q_Y(u))(\mathbf{1}_{zu\le 1}-b)\,du\Big|.$$

Applying then the Hölder inequality with exponents r and $q=r/(r-1)$ to the integral on the right-hand side, we get that

$$|\tilde{Q}_X(1/z)-\tilde{Q}_Y(1/z)|\le z\,W_r(P_X,P_Y)\|\mathbf{1}_{zU\le 1}-b\|_q.$$

If $r=1$, then $q=\infty$. We then choose $b=1/2$, which gives (2.5) for $r=1$. If $r>1$, we choose $b=1/(1+(z-1)^{r-1})$ as in Rio [9], which yields (2.5) for $r>1$ (*see* the proof of Theorem 4.1 in Rio [9] for the calculations). □

3. Normal approximation of the CVaR in the independent case

Throughout Section 3, $(X_k)_{k>0}$ is a sequence of independent and centered random variables in L^{p+2} for some $p\ge 1$. Let

$$V_n=\operatorname{Var}S_n,\ \ W_n=V_n^{-1/2}S_n,\ \ \mu_n=P_{W_n}\ \text{ and }\ L_{s,n}=V_n^{-s/2}\sum_{k=1}^n \mathbb{E}|X_k|^s\ \text{ for any } s\ge 2. \tag{3.1}$$

Let γ denote the standard normal law. Then, according to Theorem 4.1 in Rio [8] for $p\le 2$ and Corollary 1.2 in Bobkov [2] for $p>2$, there exists a positive constant c_p depending only on p, such that

$$W_r(\mu_n,\gamma)\le c_p L_{p+2,n}^{1/r}\ \text{ for any } r\in[p,p+2]. \tag{3.2}$$

If $L_{p+2,n}$ converges to 0 as $n\uparrow\infty$, (3.2) provides a rate of convergence in the central limit theorem for the L^p-transport cost. Applying Theorem 2.2 and (3.2), we immediately get the normal approximation result below.

Theorem 3.1. *Let Y be a random variable with standard normal law. Then, for any $z>1$ and any r in $[p,p+2]$,*

$$|\tilde{Q}_{W_n}(1/z)-\tilde{Q}_Y(1/z)|\le c_p\varphi(r,z),\ \ \text{where } \varphi(r,z)=\big(L_{p+2,n}z\big)^{1/r}(1+(z-1)^{1-r})^{-1/r}$$

and c_p is the constant in Inequality (3.2).

Remark 3.1. According to Inequality (3.13) in Pinelis [7], $\tilde{Q}_Y(1/z)\le\sqrt{2\log z}$ for any $z>1$, which shows that $\tilde{Q}_Y(1/z)$ is much smaller than $\sqrt{z-1}$ for large values of z.

Remark 3.2. Assume that $L_{p+2,n}\le 1/2$. Then, for any z in $]2,1/L_{p+2,n}]$, the function φ is increasing with respect to r, since

$$\frac{\partial\log\varphi}{\partial r}=-r^{-2}\log(L_{p+2,n}z)+\frac{(z-1)\log(z-1)}{r\,(1+(z-1)^{1-r})}>0 \tag{3.3}$$

for any z in $]2,1/L_{p+2,n}]$. In that case, the best upper bound in Theorem 3.1 is obtained for $r=p$.

Remark 3.3. Suppose that the random variables X_k are identically distributed. Set $\beta_s=(\operatorname{Var}X_1)^{-s/2}\mathbb{E}|X_1|^s$. Then $L_{p+2,n}=n^{-p/2}\beta_{p+2}$ and consequently Theorem 3.2 applied with $r=p$ yields the non-uniform Berry–Esseen bound

$$\sqrt{n}\,|\tilde{Q}_{W_n}(1/z)-\tilde{Q}_Y(1/z)|\le c_p\beta_{p+2}^{1/p}z^{1/p}(1+(z-1)^{1-p})^{-1/p}\ \text{ for any } z>1, \tag{3.4}$$

where c_p is the constant in Inequality (3.2). This upper bound is equivalent to $c_p\beta_{p+2}^{1/p}z^{1/p}$ as $z\uparrow\infty$ and, using (2.8), equivalent to $c_p\beta_{p+2}^{1/p}(z-1)^{1-(1/p)}$ as $z\downarrow 1$.

We now give an improved version of Inequality (3.2) with $r=p$, valid for p in $]1,2]$.

Theorem 3.2. *Let Y be a random variable with standard normal law. Then, for any p in $]1,2]$ there exists a constant C_p depending only on p such that*

$$\int_0^1 |Q_{W_n}(u)-\tilde{Q}_Y(u)|^p|\log(\min(u,1-u))|\,du\le C_pL_{p+2,n},$$

provided that $L_{p+2,n}\le 1$.

Before proving Theorem 3.2, we give an application of this theorem to the normal approximation of the CVaR.

Corollary 3.1. *Let Y be a random variable with standard normal law. Then, for any p in $]1,2]$, there exists a constant c'_p depending only on p such that*

$$|\tilde{Q}_{W_n}(1/z) - \tilde{Q}_Y(1/z)| \le (C_p L_{p+2,n})^{1/p}(z/\log z)^{1/p} \text{ for any } z \ge 2,$$

provided that $L_{p+2,n} \le 1$.

Remark 3.4. It is worth noticing that Corollary 3.1 provides a better estimate as $z \uparrow \infty$ than Theorem 3.1. In particular, in the identically distributed case

$$\sqrt{n}\,|\tilde{Q}_{W_n}(1/z) - \tilde{Q}_Y(1/z)| \le (C_p \beta_{p+2})^{1/p}(z/\log z)^{1/p} \text{ for any } z \ge 2, \tag{3.5}$$

which improves (3.4) for p in $]1,2]$ and large values of z.

Proof of Theorem 3.2. For X and Y in L^{p+2}, define the cost function $\kappa_{p,2}$ by

$$\kappa_{p,2}(P_X, P_Y) = \int_0^1 |Q_X(u) - Q_Y(u)|^p \big(1 + Q_X^2(u) + Q_Y^2(u)\big)\,\mathrm{d}u. \tag{3.6}$$

Then, from Theorem 6.1 in Rio [8], for any p in $]1,2]$, there exists some constant c'_p depending only on p such that

$$\kappa_{p,2}(P_{W_n}, P_Y) \le c'_p L_{p+2,n}. \tag{3.7}$$

Now, recall that $\log H_Y(x) \sim -x^2/2$ as x tends to ∞. Hence there exists some positive x_0 such that

$$-\log H_Y(x) \ge x^2 \text{ for any } x \ge x_0.$$

It follows that $Q_Y^2(u) \ge \log(1/u)$ for any $u \le \exp(-x_0^2)$, which ensures that $1 + Q_Y^2(u) \ge c\log(1/u)$ for any u in $]0,1/2]$, for some positive universal constant c. Since $Q_Y(1-u) = -Q_Y(u)$, we finally get that

$$1 + Q_Y^2(u) \ge c|\log(\min(u, 1-u))| \text{ for any } u \in]0,1[. \tag{3.8}$$

Now (3.7) and (3.8) imply Theorem 3.2 with $C_p = c'_p/c$. □

Proof of Corollary 3.1. Set $\psi(u) = |\log(\min(u, 1-u))|$. Then ψ fulfills the conditions of Theorem 2.1 and Remark 2.1. Consequently (2.4) holds true. Hence, for any $z \ge 2$,

$$|\tilde{Q}_{W_n}(1/z) - \tilde{Q}_Y(1/z)| \le (z/\log z)^{1/p} W_{p,\psi}(P_{W_n}, Y) \le (C_p L_{p+2,n})^{1/p}(z/\log z)^{1/p}$$

by Theorem 3.2, which ends the proof of Corollary 3.1. □

We now discuss the optimality of (3.2), (3.4), and (3.5). Concerning (3.2), some lower bounds have been proved in Rio [8, Theorem 5.1] in the independent and identically distributed case. For any $r \ge 1$ and any $a \ge 1$, there exists a sequence $(X_i)_{i>0}$ of iid random variables with mean zero, satisfying $\mathbb{E}(X_1^2) = 1$ and $\mathbb{E}(|X_1|^{r+2}) = a^r$, such that

$$\liminf_{n\to\infty} \sqrt{n}\, W_r(\mu_n, \gamma_1) \ge b_r^{1/r} a \text{ with } b_r = 2^{-r}/(r+1). \tag{3.9}$$

In the proof of Theorem 5.1 in Rio [8], the distribution of X_1 is the symmetric distribution with values in the lattice $a\mathbb{Z}$ defined by $\mathbb{P}(X_1 = a) = 1/(2a^2) = \mathbb{P}(X_1 = -a)$ and $\mathbb{P}(X_1 = 0) = 1 - 1/a^2$. The lower bound (3.9) is derived from the fact that μ_n has a distribution with values in the lattice $n^{-1/2}a\mathbb{Z}$, while γ_1 is a continuous law. Unfortunately, this argument cannot be used to provide lower bounds in (3.4) or (3.5).

Let us conclude this section by some open problems. Concerning Corollaries (3.1) and (3.5), one can conjecture that these results are still valid for $p > 2$. Concerning (3.4), it seems to me that the strong moment of order $p+2$ in the upper bound may be replaced by a weak moment of order $p+2$, although I have no idea about the proof of such a result.

4. About the CVaR of stationary martingale differences

Throughout this section, Y is a $N(0,1)$-distributed random variable. We will also use the following notations. Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and $T : \Omega \mapsto \Omega$ be a bijective bimeasurable transformation preserving the probability $\mathbb{P}$. For a σ-algebra $\mathcal{F}_0$ satisfying $\mathcal{F}_0 \subseteq T^{-1}(\mathcal{F}_0)$, we define the nondecreasing filtration $(\mathcal{F}_i)_{i\in\mathbb{Z}}$ by $\mathcal{F}_i = T^{-i}(\mathcal{F}_0)$. Let $\mathcal{F}_{-\infty} = \bigcap_{k\in\mathbb{Z}} \mathcal{F}_k$ and $\mathcal{F}_\infty = \bigvee_{k\in\mathbb{Z}} \mathcal{F}_k$.

Let X_0 be a $\mathcal{F}_0$-measurable real-valued random variable, and define the stationary sequence $(X_i)_{i\in\mathbb{Z}}$ by $X_i = X_0 \circ T^i$. For $n > 0$, set $S_n = X_0 + X_1 + \cdots + X_n$. Then the sequence $(X_i)_{i\in\mathbb{Z}}$ is a stationary martingale differences sequence in L^2 if

$$\sigma^2 := \mathbb{E}(X_0^2) < \infty \text{ and } \mathbb{E}(X_0 \mid |\mathcal{F}_{-1}) = 0. \tag{4.1}$$

From now on, we assume that (4.1) is satisfied for some positive σ^2. Under the additional condition $\mathbb{E}(X_0^2 \mid \mathcal{F}_{-\infty}) = 0$, it is well known that the normalized sums $n^{-1/2}S_n$ converge in distribution to σY. However, the question of the rates of convergence in this central limit theorem is much more intricate. Concerning power transport distances, there are only a few results. Below I recall a result that can be found in Dedecker et al. [3]. Assume that X_0 is in L^{p+2} for some p in $]0,1[$ and that the sequence $(X_i)_{i\in\mathbb{Z}}$ satisfies the projective criterion

$$\sum_{n=1}^{\infty} \frac{1}{n^{1-p/2}} \left\| \mathbb{E}\Big(\frac{S_n^2}{n}\Big|\mathcal{F}_0\Big) - \sigma^2 \right\|_{(p+2)/2} < \infty\,. \tag{4.2}$$

Then, according to Theorem 2.1 and Remark 2.1 in Dedecker et al. [3], for any r in $[1, p+2]$ there exists a constant $C_{p,r}$, depending on p, r and $(X_i)_{i\in\mathbb{Z}}$ such that, for any integer $n > 0$,

$$W_r(P_{n^{-1/2}S_n}, P_{\sigma Y}) \le C_{p,r} n^{(p-2)/2r}. \tag{4.3}$$

From the above upper bound and Theorem 2.2, we immediately get the normal approximation result below.

Theorem 4.1. *Let $(X_i)_{i\in\mathbb{Z}}$ be a stationary sequence, as defined above, satisfying (4.1) for some $\sigma^2 > 0$. Assume furthermore that X_0 is in L^{p+2} for some p in $]0,1[$ and that (4.2) is satisfied. Then, for any r in $[1, p+2]$,*

$$|\tilde{Q}_{n^{-1/2}S_n}(1/z) - \sigma\, \tilde{Q}_Y(1/z)| \le C_{p,r}\, n^{-p/2r} z^{1/r}(1 + (z-1)^{1-r})^{-1/r} \quad \textit{for any } z > 1,$$

where $C_{p,r}$ is the constant in Inequality (4.3).

Remark 4.1. Theorem 4.1 provides rates of convergence under weaker moment assumptions than the conditions required in Inequalities (3.4) and (3.5). It is worth noticing that the rate of convergence appearing here in the case $r = 1$ is the usual rate of convergence appearing in the Berry–Esseen theorem under the moment condition $\mathbb{E}|X_0|^p < \infty$. However, in the case $p \ge 1$, the extension of (3.4) and (3.5) to martingales remains open.

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