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Logic/Dynamical systems

On the classification problem of free ergodic actions of nonamenable groups

Sur le problème de la classification des actions libres ergodiques de groupes discrets non moyennables

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A R T I C L E   I N F O

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A B S T R A C T

We show that, for any countable discrete nonamenable group Γ, the relations of conjugacy, orbit equivalence, stable orbit equivalence, von Neumann equivalence, and stable von Neumann equivalence of free ergodic pmp actions of Γ on the standard atomless probability space are not Borel. This answers a question of Kechris.

R É S U M É

Nous montrons que, pour tout groupe dénombrable discret et non moyennable Γ, les relations de conjugaison, d'équivalence orbitale, d'équivalence orbitale stable, d'équivalence de von Neumann et d'équivalence de von Neumann stable des actions libres ergodiques de Γ sur un espace borélien standard muni d'une mesure de probabilité sans atomes ne sont pas Borel. Cela répond à une question de Kechris.

1. The classification problem in ergodic theory

Ergodic theory studies probability-measure-preserving (pmp) actions of a countable discrete group Γ on a standard probability space (X, μ). The *classification problem* in ergodic theory asks for an explicit method to classify (certain classes of) such actions. This program has been initially championed by Halmos, who asked in his famous ergodic theory lectures [7] whether there exists a *method* to determine whether two given $\mathbb{Z}$-actions on the standard atomless probability space are *conjugate*. By Halmos' own admission, this is a vague question, but it can be made precise in the setting of Borel complexity theory.

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Two pmp actions of Γ on (X,μ) are conjugate if there exists a measure-preserving automorphism of (X,μ) that intertwines the given actions. One can then study the *complexity* of the relation of conjugacy in the sense of Borel complexity theory. Indeed, for a fixed countable discrete group Γ, the space $\mathrm{Act}_\Gamma(X,\mu)$ of pmp actions of Γ on the standard probability space (X,μ) is endowed with a canonical standard Borel structure. In this setting, one of the most basic questions one can ask is whether this relation, or its restriction to the Borel set of free ergodic actions, is *Borel* as a subset of the product space $\mathrm{Act}_\Gamma(X,\mu)\times\mathrm{Act}_\Gamma(X,\mu)$ endowed with the product Borel structure.

Since the influential work of Dye, the notion of *orbit equivalence* has become one of the most relevant notions of equivalence in ergodic theory. Two free pmp actions (of possibly different groups) on a standard probability space (X,μ) are orbit equivalent if there is a measure-preserving transformation of the space that maps orbits of one action to orbits of the other action. Orbit equivalence admits a natural operator-algebraic reformulation: two pmp actions are orbit equivalent if and only if there is an isomorphism of the corresponding (von Neumann algebraic) crossed products that respects the canonical copies of $L^\infty(X,\mu)$. The more generous notion of *stable* orbit equivalence is defined in a similar fashion, where one is moreover allowed to restrict oneself to a non-null Borel subset of the space. Dye showed [3] that any two free ergodic pmp actions of $\mathbb{Z}$ on the standard atomless probability space are orbit equivalent. Later on, Ornstein and Weiss showed [12] that any free, ergodic pmp action of an *amenable* group is orbit equivalent to a free, ergodic pmp action of $\mathbb{Z}$. It follows that any two free ergodic pmp actions of an amenable group are orbit equivalent.

In recent years, the study of orbit equivalence has focused on pmp actions of *nonamenable* groups. In this setting, it is often possible to obtain *rigidity* results, showing that under certain assumptions on the group or on the action, orbit equivalence implies conjugacy, or at least that at most countably many nonconjugate pmp actions can be orbit equivalent. Towards this goal, cocycle superrigidity theory, as pioneered by Zimmer, has played a pivotal role. In recent years, the infusion of techniques from operator algebras, such as Popa's deformation/rigidity theory, has given new impetus to this field. This has led to the solution to many long-standing open problems, including the following strong converse to the results of Dye and Ornstein–Weiss, due to Epstein and Ioana: for any nonamenable group Γ, there exists a continuum of pairwise non-orbit-equivalent free ergodic pmp actions of Γ on the standard atomless probability space [4,9]. This has motivated Kechris to ask in [11] what is the complexity of the relation of orbit equivalence of free ergodic pmp actions of Γ, and in particular whether such a relation is Borel. Our main result is the following answer to Kechris' question.

Theorem 1.1. *Let Γ be a nonamenable countable discrete group. The relations of conjugacy and (stable) orbit equivalence of free ergodic pmp actions of Γ on the standard atomless probability space are not Borel.*

We also show that the same conclusions as in Theorem 1.1 hold if one considers the relation of (stable) von Neumann equivalence, where two actions are (stably) von Neumann equivalent if they have (stably) isomorphic crossed product. In fact, we show that the result applies even to the more restrictive class of free weak mixing actions.

The proof of Theorem 1.1 relies on Popa's cocycle superrigidity theorem for malleable actions of "rigid" groups, and it also builds on previous work by Epstein and Törnquist, who proved it under the additional assumption that Γ contains a copy of the free group $\mathbb{F}_2$ as an almost normal subgroup [5].

The assertion about conjugacy in Theorem 1.1 also holds when Γ is the group of integers. This has been shown by Foreman, Rudolph, and Weiss, with completely different methods, by encoding trees within invertible pmp transformations, in such a way that a given tree has an infinite branch if and only if the corresponding transformation is conjugate to its inverse. It is an open problem whether the assertion about conjugacy in Theorem 1.1 holds for an arbitrary countably infinite group.

2. Cocycle superrigidity

In cocycle superrigidity theory, one of the main sources of rigidity is Kazhdan's property (T) for (pairs of) groups. We consider the following natural generalization to the case of triples of groups.

Definition 2.1. A triple $\Delta\leq\Lambda\leq\Gamma$ of countable discrete groups has property (T) if every unitary representation of Γ with almost invariant vectors *which are also Δ-invariant*, has a Λ-invariant vector.

Consider a triple of groups $\Delta\leq\Lambda\leq\Gamma$. An action ζ of Γ on (X,μ) is *malleable* in the sense of Popa if the flip map of $(X\times X,\mu\times\mu)$ belongs to the connected component of the identity within the *centralizer* of the diagonal product action $\zeta\times\zeta$. The latter is the group of measure-preserving automorphisms of $X\times X$ that intertwine $\zeta\times\zeta$ with itself. (Here and in the following, we canonically identify ζ with an action on $L^\infty(X,\mu)$.) A cocycle for ζ is a function w from Γ to the unitary group of $L^\infty(X,\mu)$ satisfying $w_\gamma\zeta_\gamma(w_\sigma)=w_{\gamma\sigma}$ for all $\gamma,\sigma\in\Gamma$. A cocycle is Δ-invariant if $w_\delta=1$ and $\zeta_\delta(w_\gamma)=w_\gamma$ for all $\delta\in\Delta$ and $\gamma\in\Gamma$. Two cocycles w and w' are Λ-relatively weakly cohomologous if there exists a unitary v in $L^\infty(X,\mu)$ such that w'_γ is a scalar multiple of $v^*w_\gamma\zeta_\gamma(v)$ for every $\gamma\in\Lambda$. The Δ-invariant Λ-relative 1-cohomology group $H^1_{\Delta,\Lambda,\mathrm{w}}(\zeta)$ of ζ is the group of Δ-invariant cocycles modulo the relation of being Λ-relatively weakly cohomologous, endowed with the group operations defined by the pairing $(w,w')\mapsto ww'$ where $(ww')_\gamma=w_\gamma w'_\gamma$.

The methods used in the proof of Popa's cocycle superrigidity theorem for malleable actions [15] can be adapted to show the following.

Theorem 2.2 *(Popa). Let $\Delta \leq \Lambda \leq \Gamma$ be a triple of groups with property (T), and let ζ be a free malleable action of Γ on a standard probability space (X, μ). If $\zeta|_\Lambda$ is weak mixing, then $H^1_{:\ \Delta,\Lambda,\mathrm{w}}(\zeta)$ is trivial.*

Our main application of Theorem 2.2 is in a situation where $\Lambda = \Gamma$ and Δ is a normal subgroup of Λ. Proving this particular case is, however, not any easier than the general statement.

3. Actions of nonamenable groups

Suppose that E and F are equivalence relations on standard Borel spaces X and Y. A *countable-to-one Borel homomorphism* from E to F is a Borel function $f\colon X \to Y$ that maps E-classes to F-classes, and with the property that any collection of pairwise not E-equivalent elements of X that are mapped by f to the same F-class is at most countable. The proof of Theorem 1.1 relies on the following criterion, established by Epstein and Törnquist in [5, Theorem 5.1]: if there is a countable-to-one Borel homomorphism from the relation of isomorphism of countably infinite abelian groups to an equivalence relation F on a standard Borel space Y, then F is not Borel. The proof of Theorem 1.1 then consists in assigning, in a Borel fashion, to each countably infinite abelian group a free weak mixing action of Γ, in such a way that isomorphic groups yield conjugate actions, and at most countably many pairwise nonisomorphic groups yield stably orbit equivalent actions.

To this purpose, in a first instance, one needs a way to *encode* a given countably infinite abelian group in the conjugacy class of a free weak mixing action of $\mathbb{F}_\infty$, through a construction that goes back to Popa [13], and was later used by Törnquist and Epstein–Törnquist. Fix a countably infinite discrete group A, and a normal subgroup Δ of $\mathbb{F}_\infty$ containing one of the free generators such that $\mathbb{F}_\infty/\Delta$ is an infinite group with property (T). One can consider $\mathbb{F}_\infty$ as a subgroup of $\mathrm{SL}_2(\mathbb{Z})$. This gives a canonical action of $\mathbb{F}_\infty$ on $\mathbb{Z}^2$ by group automorphisms, as well as its dual action ρ on the dual group $\mathbb{T}^2$. This is a free weak mixing action preserving the Haar measure of $\mathbb{T}^2$ that satisfies remarkable *rigidity* properties, as established by Popa [14] and Ioana [8]; see also [11, Section 16].

Let A be a countably infinite abelian discrete group, and denote by G its dual group, endowed with its Haar (probability) measure. Consider the Bernoulli action $\beta\colon \mathbb{F}_\infty \curvearrowright G^{\mathbb{F}_\infty/\Delta}$ associated with the left translation action of $\mathbb{F}_\infty$ on $\mathbb{F}_\infty/\Delta$, and then the product action $\beta \times \rho\colon \mathbb{F}_\infty \curvearrowright G^{\mathbb{F}_\infty/\Delta} \times \mathbb{T}^2$. At the same time, one can consider the continuous action of G on itself by left translation. This induces an action δ of G on $G^{\mathbb{F}_\infty/\Delta} \times \mathbb{T}^2$ obtained by letting G act coordinatewise on $G^{\mathbb{F}_\infty/\Delta}$ and trivially on $\mathbb{T}^2$. Since δ and $\beta \times \rho$ commute, the orbit space X_A of $G^{\mathbb{F}_\infty/\Delta} \times \mathbb{T}^2$ by δ is endowed with a canonical action α_A of $\mathbb{F}_\infty$, which is free and weak mixing.

The hypothesis that Γ is nonamenable is used at this point, by applying the Gaboriau–Lyons measurable solution to the von Neumann problem for Bernoulli actions [6]. Let θ^Γ denote the Bernoulli shift of Γ on $[0,1]^\Gamma$, and find a free ergodic action $\theta^{\mathbb{F}_\infty}$ of $\mathbb{F}_\infty$ on $[0,1]^\Gamma$ endowed with the product measure, such that almost every $\theta^{\mathbb{F}_\infty}$-orbit is contained in the corresponding θ^Γ-orbit. By Dye's theorem, one can moreover choose $\theta^{\mathbb{F}_\infty}$ in such a way that one of the free generators of $\mathbb{F}_\infty$ contained in Δ acts in a mixing way. At this point, one can consider the *coinduced action* $\theta_A = \mathrm{CInd}_{\theta^{\mathbb{F}_\infty}}^{\theta^\Gamma}(\alpha_A)$ of α_A modulo $(\theta^{\mathbb{F}_\infty}, \theta^\Gamma)$ in the sense of Epstein [4]; see also [10, Section 2]. This action is automatically free, since it has α_A as a factor. Moreover, an analysis of the Koopman representation of ρ shows that θ_A is weak mixing. To conclude the proof of Theorem 1.1, we must argue that the assignment $A \mapsto \theta_A$ is a countable-to-one Borel homomorphism from the relation of isomorphism of a countably infinite abelian group to the relations of conjugacy, (stable) orbit equivalence, and (stable) von Neumann equivalence of free weak mixing actions of Γ.

The fact that the construction of θ_A is explicit shows that this assignment is given by a Borel map. The functoriality of the construction allows one to conclude that isomorphic groups give rise to conjugate actions. Suppose now that $\mathcal{A}$ is a collection of pairwise nonisomorphic countably infinite abelian groups such that the actions $\{\theta_A\colon A \in \mathcal{A}\}$ are pairwise (stably) orbit equivalent—or even just (stably) von Neumann equivalent. We want to show that $\mathcal{A}$ is countable. In order to reach a contradiction, assume that $\mathcal{A}$ is uncountable. This part takes some work, and we need some additional tools, which we proceed to describe.

Let R and $\hat{R}$ be countable pmp equivalence relations on standard probability spaces X and $\hat{X}$. Then $\hat{R}$ is a *class-bijective extension* of R if there is a Borel factor map $\pi\colon \hat{X} \to X$ that maps, for almost every $x \in \hat{X}$, the $\hat{R}$-class $[x]_{\hat{R}}$ bijectively onto $[\pi(x)]_R$. For example, the orbit equivalence relation R_A of θ_A is a class-bijective pmp extension of the orbit equivalence relation R^Γ of θ^Γ. Since almost every $\theta^{\mathbb{F}_\infty}$-orbit is contained in the corresponding θ^Γ-orbit, the orbit equivalence relation $R^{\mathbb{F}_\infty}$ of $\theta^{\mathbb{F}_\infty}$ is a subequivalence relation of R^Γ. This allows one to identify $\mathbb{F}_\infty$ with a subgroup of the full group of $R^{\mathbb{F}_\infty}$, which is in turn a subgroup of the full group of R^Γ, which is in turn a subgroup of the full group of R_A. Hence $\mathbb{F}_\infty$ is a subgroup of R_A, and we denote by ζ_A the induced action of $\mathbb{F}_\infty$.

By the rigidity properties of ρ [1, Lemma 7.4], after passing to an uncountable subcollection of $\mathcal{A}$, one can assume that the actions $\{\zeta_A\colon A \in \mathcal{A}\}$ are pairwise conjugate. In order to reach a contradiction, it suffices to show that, for any countably infinite abelian group A, one can reconstruct A from the conjugacy class of ζ_A. We do this by showing that $H^1_{:\ \Delta,\mathbb{F}_\infty,\mathrm{w}}(\zeta_A)$ is isomorphic to A. To this purpose, one should notice that, since α_A is a factor of $\beta \times \rho$, the action $\theta_A = \mathrm{CInd}_{\theta^{\mathbb{F}_\infty}}^{\theta^\Gamma}(\alpha_A)$ is a factor of $\mathrm{CInd}_{\theta^{\mathbb{F}_\infty}}^{\theta^\Gamma}(\beta \times \rho)$, which is conjugate to $\mathrm{CInd}_{\theta^{\mathbb{F}_\infty}}^{\theta^\Gamma}(\beta) \times \mathrm{CInd}_{\theta^{\mathbb{F}_\infty}}^{\theta^\Gamma}(\rho)$. Reasoning as above, one can identify $\mathbb{F}_\infty$ with a subgroup of the full groups of the orbit equivalence relations of $\mathrm{CInd}_{\theta^{\mathbb{F}_\infty}}^{\theta^\Gamma}(\beta)$ and $\mathrm{CInd}_{\theta^{\mathbb{F}_\infty}}^{\theta^\Gamma}(\rho)$. This gives actions $\hat{\beta}$ and $\hat{\rho}$ of

$\mathbb{F}_\infty$ such that $\hat{\beta} \times \hat{\rho}$ has ζ_A as a factor. Furthermore, the action δ of G on $G^{\mathbb{F}_\infty/\Delta} \times \mathbb{T}^2$ canonically extends to an action $\hat{\delta}$ commuting with $\hat{\beta} \times \hat{\rho}$.

An analysis of the Koopman representation of ρ shows that $\hat{\rho}|_\Delta$ is weak mixing. Now, a Δ-invariant cocycle w for ζ_A gives rise to a Δ-invariant cocycle $\hat{w}$ for $(\hat{\beta} \times \hat{\rho})|_{\mathbb{F}_\infty}$. Since $\hat{\rho}|_\Delta$ is weak mixing, $\hat{w}$ is necessarily a Δ-invariant cocycle for $\hat{\beta}$. One can show that $\hat{\beta}$ is weak mixing and malleable. Since the quotient group $\mathbb{F}_\infty/\Delta$ has property (T), the triple $\Delta \leq \mathbb{F}_\infty \leq \mathbb{F}_\infty$ has property (T). Therefore by Theorem 2.2, the cocycle $\hat{w}$ is weakly cohomologous to the trivial one. Thus there exists a $\mathbb{T}$-valued function v such that $\hat{w}_\gamma = v_\gamma \hat{\beta}_\gamma(v)$ modulo scalars for $\gamma \in \mathbb{F}_\infty$. One can then see that v is an eigenfunction for $\hat{\delta}$, that is, there exists a character χ_w of G such that $\hat{\delta}_g(v) = \chi_w(g)v$ for all $g \in G$. Since G is the dual group of A, we regard χ_w naturally as an element of A. This gives an assignment $w \mapsto \chi_w$ from Δ-invariant cocycles for $(\hat{\beta} \times \hat{\rho})|_{\mathbb{F}_\infty}$ to A. It can be verified that this induces a group isomorphism from $H^1_{:\,\Delta,\mathbb{F}_\infty,\mathrm{w}}(\zeta_A)$ to A, which completes the proof.

The general idea described above can also be adapted to deal with the more general setting of class-bijective extensions of a given countable pmp equivalence relation on a standard probability space. In this case, one can use the Bowen–Ioana–Hoff measurable solution [1, Theorem A] to the von Neumann problem for Bernoulli extensions of nonamenable countable pmp equivalence relations.

Theorem 3.1. *Let R be a nonamenable ergodic countable pmp equivalence relation. The relations of (stable) isomorphism and (stable) von Neumann equivalence of free ergodic class-bijective pmp extensions of R are not Borel.*

When R is amenable, all the ergodic class-bijective extensions are also amenable, and hence isomorphic, by classical results of Dye [3] and Connes, Feldman, and Weiss [2].

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