Group theory/Differential geometry

On the irreducible action of PSL(2, \( \mathbb{R} \)) on the 3-dimensional Einstein universe

*Sur l’action irréductible de PSL(2, \( \mathbb{R} \)) sur l’univers d’Einstein de dimension 3*

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**Abstract**

In this paper, we study the irreducible representation of PSL(2, \( \mathbb{R} \)) in PSL(5, \( \mathbb{R} \)). This action preserves a quadratic form with signature (2, 3). Thus, it acts conformally on the 3-dimensional Einstein universe \( \text{Ein}^{1, 2} \). We describe the orbits induced in \( \text{Ein}^{1, 2} \) and its complement in \( \mathbb{R}^4 \). This work completes the study in [2], and is one element of the classification of cohomogeneity one actions on \( \text{Ein}^{1, 2} \) [5].

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**RÉSUMÉ**

Dans cet article, nous étudions l’action irréductible de PSL(2, \( \mathbb{R} \)) dans PSL(5, \( \mathbb{R} \)). Cette action préserve une forme quadratique de signature (2, 3). Elle agit donc conformément sur l’univers d’Einstein \( \text{Ein}^{1, 2} \) de dimension 3, ainsi que sur son complément dans \( \mathbb{R}^4 \). Ce travail complète l’étude préliminaire dans [2], et est un élément de la classification des actions sur \( \text{Ein}^{1, 2} \) de cohomogénéité un [5].

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1. Introduction

1.1. The irreducible representation of PSL(2, \( \mathbb{R} \))

Let \( V \) denote an \( n \)-dimensional vector space. A subgroup of GL(V) is irreducible if it preserves no proper subspace of \( V \).

It is well known that, for every integer \( n \), up to isomorphism, there is only one \( n \)-dimensional irreducible representation of PSL(2, \( \mathbb{R} \)). For \( n = 5 \), this representation is the natural action of PSL(2, \( \mathbb{R} \)) on the vector space \( V = \mathbb{R}_4[X, Y] \) of homogeneous polynomials of degree 4 in two variables \( X \) and \( Y \). This action induces three types of orbits in the 4-dimensional...
projective space $\mathbb{RP}^4 = \mathbb{P}(V)$: an 1-dimensional orbit, three 2-dimensional orbits, and the orbits on which $\text{PSL}(2, \mathbb{R})$ acts freely.

The irreducible action of $\text{PSL}(2, \mathbb{R})$ on $V$ preserves the following quadratic form

$$q(a_4 x^4 + a_3 x^3 y + a_2 x^2 y^2 + a_1 xy^3 + a_0 y^4) = 2a_4 a_0 - \frac{1}{2} a_1 a_3 + \frac{1}{6} a_2^2.$$ 

The quadratic form $q$ is non-degenerate and has signature $(2, 3)$. This induces an irreducible representation $\text{PSL}(2, \mathbb{R}) \to O(2, 3) \subset \text{PSL}(5, \mathbb{R})$ [2]. On the other hand, by [3, Theorem 1], up to conjugacy, $SO_+(1, 2) \simeq \text{PSL}(2, \mathbb{R})$ is the only irreducible connected Lie subgroup of $O(2, 3)$.

1.2. Einstein’s universe

Let $\mathbb{R}^{2,3}$ denote a 5-dimensional real vector space equipped with a non-degenerate symmetric bilinear form $q$ with signature $(2, 3)$. The null cone of $\mathbb{R}^{2,3}$ is

$$\mathfrak{N} = \{v \in \mathbb{R}^{2,3} \setminus \{0\} : q(v) = 0\}.$$ 

The 3-dimensional Einstein universe $\text{Ein}^{1,2}$ is the image of the null cone $\mathfrak{N}$ under the projectivization:

$$\mathbb{P} : \mathbb{R}^{2,3} \setminus \{0\} \longrightarrow \mathbb{RP}^4.$$ 

The degenerate metric on $\mathfrak{N}$ induces a $O(2, 3)$-invariant conformal Lorentzian structure on the Einstein universe. The group of conformal transformations on $\text{Ein}^{1,2}$ is $O(2, 3)$ [4].

A light-like geodesic in Einstein’s universe is a photon. A photon is the projectivization of an isotropic 2-plane in $\mathbb{R}^{2,3}$. The set of photons through a point $p \in \text{Ein}^{1,2}$ denoted by $L(p)$ is the lightcone at $p$. The complement of a lightcone $L(p)$ in Einstein’s universe is the Minkowski patch at $p$ and we denote it by $\text{Mink}(p)$. A Minkowski patch is conformally equivalent to the 3-dimensional Minkoski space $\mathbb{R}^{1,2}$ [1].

The complement to the Einstein universe in $\mathbb{RP}^4$ has two connected components: the 3-dimensional Anti de-Sitter space $\text{AdS}^{1,2}$ and the generalized hyperbolic space $\mathbb{H}^{2,2}$: the first (respectively the second) is the projection of the domain $\mathbb{R}^{2,3}$ defined by $\{q < 0\}$ (respectively $\{q > 0\}$).

An immersed submanifold $S$ of $\text{AdS}^{1,2}$ or $\mathbb{H}^{2,2}$ is of signature $(p, q, r)$ (respectively $\text{Ein}^{1,2}$) if the restriction of the ambient pseudo-Riemannian metric (respectively the conformal Lorentzian metric) is of signature $(p, q, r)$, meaning that the radical has dimension $r$, and that maximal definite negative and positive subspaces have dimensions $p$ and $q$, respectively. If $S$ is nondegenerate, we forgot $r$ and simply denote its signature by $(p, q)$.

**Theorem 1.** The irreducible action of $\text{PSL}(2, \mathbb{R})$ on the 3-dimensional Einstein universe $\text{Ein}^{1,2}$ admits three orbits:

- a 1-dimensional light-like orbit, i.e. of signature $(0, 0, 1)$,
- a 2-dimensional orbit of signature $(0, 1, 1)$,
- an open orbit (hence of signature $(1, 2)$) on which the action is free.

The 1-dimensional orbit is light-like, homeomorphic to $\mathbb{RP}^1$, but not a photon. The union of the 1-dimensional orbit and the 2-dimensional orbit is an algebraic surface, whose singular locus is precisely the 1-dimensional orbit. It is the union of all projective lines tangent to the 1-dimensional orbit. Fig. 1 describes a part of the 1 and 2-dimensional orbits in the Minkowski patch $\text{Mink}((Y^4))$.  

Fig. 1. Two partial views of the intersection of the 1 and 2-dimensional orbits in Einstein’s universe with $\text{Mink}((Y^4))$. **Red:** Part of the 1-dimensional orbit in Minkowski patch. **Green:** Part of the 2-dimensional orbit in Minkowski patch.
We will also describe the actions on the Anti de-Sitter space and the generalized hyperbolic space $\mathbb{H}^{2,2}$:

**Theorem 1.2.** The orbits of $\text{PSL}(2, \mathbb{R})$ in the Anti de-Sitter component $\text{AdS}^{1,3}$ are Lorentzian, i.e. of signature $(1, 2)$. They are the leaves of a codimension-1 foliation. In addition, $\text{PSL}(2, \mathbb{R})$ induces three types of orbits in $\mathbb{H}^{2,2}$: a 2-dimensional space-like orbit (of signature $(2,0)$) homeomorphic to the hyperbolic plane $\mathbb{H}^2$, a 2-dimensional Lorentzian orbit (i.e. of signature $(1,1)$) homeomorphic to the de-Sitter space $\text{dS}^{1,1}$, and four kinds of 3-dimensional orbits where the action is free:

- a one-parameter family of orbits of signature $(2,1)$, consisting of elements with four distinct non-real roots,
- a one-parameter family of Lorentzian (i.e. of signature $(1,2)$) orbits consisting of elements with four distinct real roots,
- two orbits of signature $(1,1,1)$,
- a one-parameter family of Lorentzian (i.e. of signature $(1,2)$) orbits consisting of elements with two distinct real roots, and two distinct complex conjugate roots so that the cross-ratio of the four roots has an argument strictly between $-\pi/3$ and $\pi/3$.

2. Proofs of the theorems

Let $f$ be an element in $\mathbb{C}[V]$. We consider it as a polynomial function from $\mathbb{C}^2$ into $\mathbb{C}$. Actually, by specifying $Y = 1$, we consider $f$ as a polynomial of degree at most 4. Such a polynomial is determined, up to a scalar, by its roots $z_1, z_2, z_3, z_4$ in $\mathbb{C}^1$ (some of these roots can be $\infty$ if $f$ can be divided by $Y$). It provides a natural identification between $\mathbb{P}(V)$ and the set of 4-tuples (up to permutation) $(z_1, z_2, z_3, z_4)$ of $\mathbb{C}^1$ such that if some $z_i$ is not in $\mathbb{R}^1$, then its conjugate $\bar{z}_i$ is one of the $z_j$’s. This identification is $\text{PSL}(2, \mathbb{R})$-equivariant, where the action of $\text{PSL}(2, \mathbb{R})$ on $\mathbb{C}^1$ is simply the one induced by the diagonal action on $\mathbb{P}(\mathbb{C}^1)^4$.

Actually, the complement of $\mathbb{R}^1$ in $\mathbb{C}^1$ is the union of the upper half-plane model $\mathbb{H}^2$ of the hyperbolic plane, and the lower half-plane. We can represent every element of $\mathbb{C}^1$ by a 4-tuple (up to permutation) $(z_1, z_2, z_3, z_4)$ such that:

- either every $z_i$ lies in $\mathbb{R}^1$,
- or $z_1, z_2$ lies in $\mathbb{R}^1$, $z_3$ lies in $\mathbb{H}^2$ and $z_4 = \bar{z}_3$,
- or $z_1, z_2$ lies in $\mathbb{H}^2$ and $z_3 = \bar{z}_1, z_4 = \bar{z}_2$.

Theorems 1.1 and 1.2 will follow from the proposition below.

**Proposition 2.1.** Let $[f]$ be an element of $\mathbb{P}(V)$. Then:

- it lies in $\mathbb{E}^{1,2}$ if and only if it has a root of multiplicity at least 3, or two distinct real roots $z_1, z_2$, and two complex roots $z_3, z_4 = \bar{z}_3$, with $z_3$ in $\mathbb{H}^2$ and such that the argument of the cross-ratio of $z_1, z_2, z_3, z_4$ is $\pm \pi/3$;
- it lies in $\text{AdS}^{1,3}$ if and only if it has two distinct real roots $z_1, z_2$, and two complex roots $z_3, z_4 = \bar{z}_3$, with $z_3$ in $\mathbb{H}^2$ and such that the argument of the cross-ratio of $z_1, z_2, z_3, z_4$ has absolute value $> \pi/3$;
- it lies in $\mathbb{H}^{2,2}$ if and only if it has no real roots, or four distinct real roots, or a root of multiplicity exactly 2, or it has two distinct real roots $z_1, z_2$, and two complex roots $z_3, z_4 = \bar{z}_3$, with $z_3$ in $\mathbb{H}^2$ and such that the argument of the cross-ratio of $z_1, z_2, z_3, z_4$ has absolute value $< \pi/3$.

**Proof of Proposition 2.1.** Assume that $f$ has no real root. Hence we are in the situation where $z_1, z_2$ lie in $\mathbb{H}^2$ and $z_3 = z_1, z_4 = z_2$. By applying a suitable element of $\text{PSL}(2, \mathbb{R})$, we can assume $z_1 = i$, and $z_2 = ri$ for some $r > 0$. In other words, $f$ is in the $\text{PSL}(2, \mathbb{R})$-orbit of $(X^2 + Y^2)(X^2 + r^2Y^2)$. The value of $q$ on this polynomial is $2 \times 1 \times r^2 + \frac{1}{6}(1 + r^2)^2 > 0$, hence $[f]$ lies in $\mathbb{H}^{2,2}$.

Hence, we can assume that $f$ admits at least one root in $\mathbb{R}^1$, and by applying a suitable element of $\text{PSL}(2, \mathbb{R})$, one can assume that this root is $\infty$, i.e. that $f$ is a multiple of $Y$.

We first consider the case where this real root has multiplicity at least 2:

$$f = Y^2(aX^2 + bXY + cY^2)$$

Then, $q(f) = \frac{1}{6}r^2$: it follows that if $f$ has a root of multiplicity at least 3, it lies in $\mathbb{E}^{1,2}$, and if it has a real root of multiplicity 2, it lies in $\mathbb{H}^{2,2}$.

We assume from now on that the real roots of $f$ have multiplicity 1. Assume that all roots are real. Up to $\text{PSL}(2, \mathbb{R})$, one can assume that these roots are 0, 1, $r$ and $\infty$ with $0 < r < 1$.

$$f(X, Y) = XY(X - Y)(X - rY) = X^3Y - (r + 1)X^2Y^2 + rXY^3.$$ 

Then, $q(f) = -\frac{1}{2}r + \frac{1}{6}(r + 1)^2 = \frac{1}{6}(r^2 - r + 1) > 0$. Therefore, $f$ lies in $\mathbb{H}^{2,2}$ once more.

The only remaining case is the case where $f$ has two distinct real roots, and two complex conjugate roots $z, \bar{z}$ with $z \in \mathbb{H}^2$. Up to $\text{PSL}(2, \mathbb{R})$, one can assume that the real roots are $0, \infty$, hence:
where \( z = |z|e^{i\theta} \). Then:

\[
q(f) = \frac{2|z|^2}{3} (\cos^2 \theta - \frac{3}{4}).
\]

Hence \( f \) lies in \( \text{Ein}^{1,2} \) if and only if \( \theta = \pi/6 \) or \( 5\pi/6 \). The proposition follows easily.

**Remark 1.** F. Fillastre indicated to us that our description of the open orbit in \( \text{Ein}^{1,2} \) appearing in the first item of Proposition 2.1 has an alternative and more elegant description: this orbit corresponds to polynomials whose roots in \( \mathbb{C}P^1 \) are ideal vertices of a regular ideal tetraedra in \( \mathbb{H}^3 \).

**Remark 2.** In order to determine the signature of the orbits induced by \( \text{PSL}(2, \mathbb{R}) \) in \( \mathbb{P}(\mathbb{V}) \), we consider the tangent vectors induced by the action of 1-parameter subgroups of \( \text{PSL}(2, \mathbb{R}) \). We denote by \( E \), \( P \), and \( H \), the 1-parameter elliptic, parabolic and hyperbolic subgroups stabilizing \( i \), \( \infty \) and \( (0, \infty) \), respectively.

**Proof of Theorem 1.1.** It follows from Proposition 2.1 that there are precisely three \( \text{PSL}(2, \mathbb{R}) \)-orbits in \( \text{Ein}^{1,2} \):

- one orbit \( \mathcal{N} \) comprising polynomials with a root of multiplicity 4, i.e. of the form \( (sY - tX)^4 \) with \( s, t \in \mathbb{R} \). It is clearly 1-dimensional, and equivariantly homeomorphic to \( \mathbb{R}P^1 \) with the usual projective action of \( \text{PSL}(2, \mathbb{R}) \). Since

\[
\frac{d}{d|t|=0}(Y - tX)^4 = -4XY^3
\]

is a q-null vector, this orbit is light-like (but cannot be a photon since the action is irreducible);

- one orbit \( \mathcal{L} \) comprising polynomials with a real root of multiplicity 3, and another real root. These are the polynomials of the form \( (sY - tX)^3 (s'Y - t'X) \) with \( s, t, s', t' \in \mathbb{R} \). It is 2-dimensional, and it is easy to see that it is the union of the projective lines tangent to \( \mathcal{N} \). The vectors tangent to \( \mathcal{L} \) induced by the 1-parameter subgroups \( P \) and \( E \) at \( [XY^3] \in \mathcal{L} \) are \( v_P = -3Y^2 + Y^4 \) and \( v_E = 3X^2Y^2 + Y^4 \). Obviously, \( v_P \) is orthogonal to \( v_E \) and \( v_P + v_E \) is space-like. Hence \( \mathcal{L} \) is of signature \( (0, 1, 1) \);

- one open orbit comprising polynomials admitting two distinct real roots and a root in \( \mathbb{H}^2 \) such that the action of the cross-ratio of the four roots is \( \pi/3 \). The stabilizers of points in this orbit are trivial, since an isometry of \( \mathbb{H}^2 \) preserving a point in \( \mathbb{H}^2 \) and one point in \( \partial \mathbb{H}^2 \) is necessarily the identity. □

**Proof of Theorem 1.2.** According to Proposition 2.1, the polynomials in AdS\(^{1,3} \) have two distinct real roots, and a complex root \( \mathbb{H}^2 \) (and its conjugate) such that the action of the cross-ratio of the four roots has absolute value \( > \pi/3 \). It follows that the action in AdS\(^{1,3} \) is free, and that the orbits are the level sets of the function \( \theta \). Suppose that \( M \) is a \( \text{PSL}(2, \mathbb{R}) \)-orbit in AdS\(^{1,3} \). There exists \( r \in \mathbb{R} \) such that \( [f] = [XY(X^2 + Y^2)(X - rY)] \in M \). The orbit induced by the 1-parameter subgroup \( E \) at \( [f] \) is:

\[
\gamma(t) = [(X^2 + Y^2)((\sin t \cos t - r \sin^2 t)X^2 - (\sin t \cos t + r \cos^2 t)Y^2 + (\cos^2 t - \sin^2 t + 2r \sin t \cos t)XY)].
\]

Then

\[
\frac{d\gamma}{dt}\bigg|_{t=0} = -2 - 2r^2 < 0.
\]

This implies, as for any submanifold of a Lorentzian manifold admitting a time-like vector, that \( M \) is Lorentzian, i.e., of signature \( (1, 2) \).

The case of \( \mathbb{H}^2 \) is the richest one. According to Proposition 2.1, there are four cases to consider.

- **No real roots.** Then \( f \) has two complex roots \( z_1, z_2 \) in \( \mathbb{H}^2 \) (and their conjugates). It corresponds to two orbits: one orbit corresponding to the case \( z_1 = z_2 \); it is space-like and has dimension 2. It is the only maximal \( \text{PSL}(2, \mathbb{R}) \)-invariant surface in \( \mathbb{H}^2 \) described in [2, Section 5.3]. The case \( z_1 \neq z_2 \) provides a one-parameter family of 3-dimensional orbits on which the action is free (the parameter being the hyperbolic distance between \( z_1 \) and \( z_2 \)). One may assume that \( z_1 = i \) and \( z_2 = ri \) for some \( r > 0 \). Denote by \( M \) the orbit induced by \( \text{PSL}(2, \mathbb{R}) \) at \( [f] = [(X^2 + Y^2)(X^2 + 2r^2Y^2)] \). The vectors tangent to \( M \) at \( [f] \) induced by the 1-parameter subgroups \( H, P \) and \( E \) are:

\[
\begin{align*}
\nu_H &= -4X^4 + 4r^2Y^4, \\
\nu_P &= -4X^3Y - 2(r^2 + 1)XY^3, \\
\nu_E &= 2(r^2 - 1)X^3Y + 2(r^2 - 1)Y^3,
\end{align*}
\]

respectively. The time-like vector \( \nu_H \) is orthogonal to both \( \nu_P \) and \( \nu_E \). It is easy to see that the 2-plane generated by \( \nu_P, \nu_E \) is of signature \( (1, 1) \). Therefore, the tangent space \( T_{[f]}M \) is of signature \( (2, 1) \).

- **Four distinct real roots.** This case provides a one-parameter family of 3-dimensional orbits on which the action is free – the parameter being the cross-ratio between the roots in \( \mathbb{R}P^1 \). Denote by \( M \) the \( \text{PSL}(2, \mathbb{R}) \)-orbit at \( [f] = [XY(X - Y)(X - rY)] \) (here as explained in the proof of Proposition 2.1, we can restrict ourselves to the case \( 0 < r < 1 \). The vectors tangent to \( M \) at \( [f] \) induced by the 1-parameter subgroups \( H, P \), and \( E \) are:

\[
\begin{align*}
\nu_H &= -rY^4 + 2(r + 1)XY^3 - 3X^2Y^2, \\
\nu_P &= -2X^3Y + 2rXY^3, \\
\nu_E &= X^4 - rY^4 + 3(r - 1)X^2Y^2 + 2(r + 1)XY^3 - 2(r + 1)X^3Y,
\end{align*}
\]
respectively. A vector \( x = a v_H + b v_P + c v_E \) is orthogonal to \( v_P \) if and only if \( 2ra + b(r + 1) + c(r + 1)^2 = 0 \). Set \( a = \frac{(b(r + 1) + c(r + 1)^2)}{-2r} \)

\[
q(x) = 2ra^2 + \frac{3}{2}b^2 + \left( \frac{7}{2}(r^2 + 1) - r \right)c^2 + 2(r + 1)ab + 2(r + 1)^2 + ac(2r^2 - r + 5).
\]

Consider \( q(x) = 0 \) as a quadratic polynomial \( F \) in \( b \). Since \( 0 < r < 1 \), the discriminant of \( F \) is non-negative and it is positive when \( c \neq 0 \). Thus, the intersection of the orthogonal complement of the space-like vector \( v_P \) with the tangent space \( T_{(f)}M \) is a 2-plane of signature \((1, 1)\). This implies that \( M \) is Lorentzian, i.e. of signature \((1, 2)\).

- **A root of multiplicity 2**. Observe that if there is no non-real root of multiplicity 2, when we are in the first “no real root” case. Hence we consider here only the case where the root of multiplicity 2 lies in \( \mathbb{R}P^1 \). Then, we have three subcases to consider:
  - two distinct real roots of multiplicity 2: The orbit induced at \( X^2Y^2 \) is the image of the \( \text{PSL}(2, \mathbb{R}) \)-equivariant map
    \[
    dS^{1,1} \subset \mathbb{P}(\mathbb{R}_2(X, Y)) \rightarrow \mathbb{H}^{2,2}, \quad [L] \mapsto [L^2],
    \]
    where \( \mathbb{R}_2(X, Y) \) is the vector space of homogeneous polynomials of degree 2 in two variables \( X \) and \( Y \), endowed with discriminant as a \( \text{PSL}(2, \mathbb{R}) \)-invariant bilinear form of signature \((1, 2)\) [2, Section 5.3]. (Here, \( L \) is the projective class of a Lorentzian bilinear form on \( \mathbb{R}^2 \).) The vectors tangent to the orbit at \( X^2Y^2 \) induced by the 1-parameter subgroups \( P \) and \( E \) are \( v_P = -2XY^3 \) and \( v_E = 2X^3 - 2XY^3 \), respectively. It is easy to see that the 2-plane generated by \( \{v_P, v_E\} \) is of signature \((1, 1)\). Hence, the orbit induced at \( X^2Y^2 \) is Lorentzian.
  - three distinct real roots, one of them being of multiplicity 2; denote by \( M \) the orbit induced by \( \text{PSL}(2, \mathbb{R}) \) at \([f] = [XY^2(X - Y)]\). The vectors tangent to \( M \) at \([f] \) induced by the 1-parameter subgroups \( H, P \) and \( E \) are:
    \[
    v_H = -2XY^3, \quad v_P = Y^4 - 2XY^3, \quad v_E = Y^4 - 2X^2Y^2 + X^3Y - XY^3,
    \]
    respectively. Obviously, the light-like vector \( v_H + v_P \) is orthogonal to \( T_{(f)}M \). Therefore, the restriction of the metric on \( T_{(f)}M \) is degenerate. It is easy to see that the quotient of \( T_{(f)}M \) by the action of the isotropic line \( \mathbb{R}(v_H + v_P) \) is of signature \((1, 1)\). Thus, \( M \) is of signature \((1, 1, 1)\).
  - one real root of multiplicity 2, and one root in \( \mathbb{H}^2 \): Denote by \( M \) the orbit induced by \( \text{PSL}(2, \mathbb{R}) \) at \([f] = [Y^2(X^2 + Y^2)]\). The vectors tangent to \( M \) at \([f] \) induced by the 1-parameter subgroups \( H, P \) and \( E \) are \( v_H = 4Y^4, v_P = -2XY^3 \), and \( v_E = 2X^3 + 2XY^3 \), respectively. Obviously, the light-like vector \( v_H \) is orthogonal \( T_{(f)}M \). Therefore, the restriction of the metric on \( T_{(f)}M \) is degenerate. It is easy to see that the quotient of \( T_{(f)}M \) by the action of the isotropic line \( \mathbb{R}(v_H) \) is of signature \((1, 1)\). Thus \( M \) is of signature \((1, 1, 1)\).
  - **Two distinct real roots, and a complex root in \( \mathbb{H}^2 \) (and its conjugate) such that the argument of the cross-ratio of the four roots has absolute value \( < \pi/3 \).** Denote by \( M \) the orbit induced by \( \text{PSL}(2, \mathbb{R}) \) at \([f] = [Y(X^2 + Y^2)(X - rY)]\). The vectors tangent to \( M \) at \([f] \) induced by the 1-parameter subgroups \( H, P \) and \( E \) are:
    \[
    v_H = -4rY^4 - 2X^3Y + 2XY^3, \quad v_P = -3X^2Y^2 + 2rXY^3 - Y^4, \quad v_E = X^4 - Y^4 - 2rX^3Y - 2rXY^3,
    \]
    respectively. The following set of vectors is an orthogonal basis for \( T_{(f)}M \) where the first vector is time-like and the two others are space-like.
    \[
    \{(7r + 3r^2)v_H + (6 - 2r^2)v_P + (5 + r^2)v_E, 4v_P + v_E, v_H\}.
    \]
    Therefore, \( M \) is Lorentzian, i.e. of signature \((1, 2)\). □

References