C. R. Acad. Sci. Paris, Ser. I 355 (2017) 956–959

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Mathematical analysis/Dynamical systems

A note on singularity of a recently introduced family of Minkowski's question-mark functions

Note sur la singularité d'une famille de fonctions « Minkowski's question-mark » récemment introduite

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ARTICLE INFO

Article history:
Received 16 September 2016
Accepted after revision 5 September 2017
Available online 19 September 2017

Presented by the Editorial Board

ABSTRACT

We point out a mistake in the proof of the main theorem in a recent article on a family of generalized Minkowski's question-mark functions, saying that each member of the family is a singular homeomorphism, and provide two alternative proofs, one based on the ergodicity of the Gauss map G and the α-Lüroth map L_α, and another one focusing more on classical properties of continued fraction expansions.

RÉSUMÉ

Nous mettons en évidence une erreur dans la démonstration du théroème principal dans un article récent traitant d'une famille de fonctions « Minkowski's question-mark » généralisées, stipulant que chaque membre de la famille est un homéomorphisme singulier, et nous produisons deux preuves alternatives, l'une basée sur l'ergodicité de l'application de Gauss G et de l'application α-Lüroth L_α, l'autre se focalisant davantage sur des propriétés classiques des décompositions de fractions continues.

1. Introduction

Based on continued fraction- and generalized Lüroth expansions, a new family of Minkowski's question-mark functions was recently introduced in [1]. When proving the main theorem of the paper (Theorem 1.3), the author correctly shows that each member $?_\alpha$ of the family is a strictly increasing homeomorphism of the unit interval $[0, 1]$ and then tackles proving that $?_\alpha$ is singular (in the sense that $?_\alpha$ has derivative zero λ-a.e.). Unfortunately, the presented proof of singularity is

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http://dx.doi.org/10.1016/j.crma.2017.09.009
1631-073X/

incorrect, implying that the theorem is still formally unproven. The main objective of this note is to provide two correct proofs of the singularity of $?_\alpha$ for every partition α.

The rest of this note is organized as follows: we first introduce some notation (essentially following [1] with some slight modifications) in Section 2, point out what exactly went wrong in [1], and then prove Theorem 1.3 in [1] using two different methods in Section 3.

2. Notation

In the sequel, $\mathcal{B}([0,1])$ will denote the Borel σ-field on $[0,1]$, and λ will denote the Lebesgue measure on $\mathcal{B}([0,1])$. We will write $\mathbb{N}_\infty = \{1,2,3,\ldots\} \cup \{\infty\}$ and will refer to $\Sigma := (\mathbb{N}_\infty)^{\mathbb{N}}$ as a code-space. As usual, $G:[0,1] \to [0,1]$ will denote the Gauss map, defined by $G(0)=0$ and $G(x) = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$ for $x \in (0,1]$. Set $s_i = \frac{1}{i}$ for every $i \in \mathbb{N}$ and $s_\infty = 0$. Then the intervals $I_\infty = \{s_\infty\}, I_1 = (s_2, s_1], I_2 = (s_3, s_2], \ldots$ form a partition γ_G of $[0,1]$. Coding orbits of G via γ_G, the continued fraction expansion $\mathrm{cf}: [0,1] \to \Sigma$ is defined by setting $\mathrm{cf}(x) = \underline{a} = (a_1, a_2, a_3, \ldots) \in \Sigma$ if and only if $G^{i-1}(x) \in I_{a_i}$ holds for every $i \in \mathbb{N}$. It is well known that G is strongly mixing (hence ergodic) w.r.t. the absolutely continuous probability measure μ_G with density $\frac{1}{\ln 2}\frac{1}{1+x}$ for $x \in [0,1]$ (see [3]).

In the sequel, we will let $\alpha = \{J_\infty, J_1, J_2, J_3, \ldots\}$ denote partitions of the unit interval induced by strictly decreasing sequences $(t_i)_{i=1}^\infty$ converging to $0 =: t_\infty$ and fulfilling $t_1 = 1$, i.e. we have $J_\infty = \{t_\infty\}, J_1 = (t_2, t_1], J_2 = (t_3, t_2], \ldots$ For each such partition α, the α-Lüroth map L_α is defined by $L_\alpha(0) = 0$ as well as $L_\alpha(x) = \frac{t_j - x}{t_j - t_{j+1}}$ for $x \in (t_{j+1}, t_j]$ and $j \in \mathbb{N}$. Coding orbits of L_α via α, the α-Lüroth expansion $\mathrm{Lür}_\alpha : [0,1] \to \Sigma$ is defined by setting $\mathrm{Lür}_\alpha(x) = \underline{a} = (a_1, a_2, a_3, \ldots) \in \Sigma$ if and only if $L_\alpha^{i-1}(x) \in J_{a_i}$ holds for every $i \in \mathbb{N}$. Additionally to being strongly mixing w.r.t. λ, the α-Lüroth map is even (isomorphic to) a Bernoulli shift (see [2]). Moreover (again see [2]) the transformation $\Phi_\alpha : \Sigma \to [0,1]$, defined by

$$\Phi_\alpha(\underline{a}) = t_{a_1} + \sum_{m=2}^{\infty} (-1)^{m+1} t_{a_m} \prod_{j=1}^{m-1} (t_{a_j} - t_{a_j+1}), \tag{1}$$

with the convention $t_{\infty+1} = t_\infty$, fulfills $\Phi_\alpha \circ \mathrm{Lür}_\alpha = id_{[0,1]}$.

3. Two proofs of singularity of $?_\alpha$

Based on $\mathrm{cf}: [0,1] \to \Sigma$ and $\Phi_\alpha : \Sigma \to [0,1]$, in [1] the author introduces the (generalized) question-mark function $?_\alpha$ by setting $?_\alpha(x) = \Phi_\alpha \circ \mathrm{cf}(x)$ for every $x \in [0,1]$, and then states the following theorem.

Theorem 3.1 *([1]). Given a partition α as above, the map $?_\alpha : [0,1] \to [0,1]$ is an increasing singular homeomorphism fulfilling $L_\alpha \circ ?_\alpha = ?_\alpha \circ G$. Moreover, if $t_1, t_2, \ldots \in \mathbb{Q}$, then $?_\alpha$ maps the set $\mathbb{A}_2$ of all quadratic surds into $\mathbb{Q}$.*

In [1], a correct proof for the fact that $?_\alpha$ is an increasing homeomorphism and for the assertion concerning $\mathbb{A}_2$ is given. It is well known that singularity can be shown by establishing the existence of a Borel set $\tilde{B} \subseteq [0,1]$ fulfilling $\lambda(\tilde{B}) = 0$ and $\lambda(?_\alpha(\tilde{B})) = 1$. Letting μ_α denote the pull-back of λ via $?_\alpha$ (or, equivalently, the push-forward of λ via $?_\alpha^{-1}$), defined by $\mu_\alpha(B) = \lambda(?_\alpha(B))$ for every Borel set $B \in \mathcal{B}([0,1])$, the author then uses the identity

$$\int_{?_\alpha(B)} \mathrm{d}x = \int_B ?_\alpha^{-1}(x)\,\mathrm{d}x \tag{2}$$

to prove that μ_α and μ_G are singular with respect to each other. Eq. (2), however, is easily seen to be wrong and should be $\int_{?_\alpha(B)} 1\,\mathrm{d}x = \int_B 1\,\mathrm{d}\mu_\alpha = \mu_\alpha(B)$ instead. In fact, considering, for instance, $B = [0,1]$, we get $\int_{?_\alpha([0,1])} \mathrm{d}x = \int_{[0,1]} \mathrm{d}x = 1$, whereas the right-hand side of Eq. (2) obviously fulfills $\int_{[0,1]} ?_\alpha^{-1}(x)\,\mathrm{d}x < 1$ since $?_\alpha^{-1}$ is a homeomorphism of $[0,1]$ too. Since the rest of the proof in [1] builds upon Eq. (2), an alternative method is needed to show the singularity of $?_\alpha$.

We will now provide two proofs of the singularity of $?_\alpha$ for every partition α – the first one uses the ergodicity of G and L_α and explicitly constructs a Borel set $\tilde{B}$ with the afore-mentioned properties, the second one is more elementary and directly derives the fact that $?_\alpha'(x) = 0$ for λ-a.e. $x \in [0,1]$ via some properties of continued fraction expansions.

Proof a. We distinguish two different types of partitions α.

Type I: There exists $k \in \mathbb{N}$ such that $t_k - t_{k+1} \neq \frac{1}{\ln 2} \ln \frac{(k+1)^2}{k(k+2)}$.

Let the Borel sets Λ and Γ be defined by

$$\Lambda = \left\{ x \in [0,1] : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_{(\frac{1}{k+1}, \frac{1}{k}]} \circ G^i(x) = \frac{1}{\ln 2} \ln \frac{(k+1)^2}{k(k+2)} \text{ holds for every } k \in \mathbb{N} \right\} \tag{3}$$

$$\Gamma = \left\{ x \in [0,1] : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{1}_{(t_{k+1}, t_k]} \circ L_\alpha^i(x) = t_k - t_{k+1} \text{ holds for every } k \in \mathbb{N} \right\}. \tag{4}$$

The ergodicity of G w.r.t. μ_G and of L_α w.r.t. λ (see [2,3]) and the fact that μ_G is absolutely continuous with strictly positive density implies that $\lambda(\Lambda)=\lambda(\Gamma)=1$. Considering that $?_\alpha(x)=\Phi_\alpha\circ \mathrm{cf}(x)$ holds for every $x\in(0,1]$ and that for every $x\in\Lambda$, we have $\mathrm{cf}(x)\notin \mathrm{Lür}_\alpha(\Gamma)$, $?_\alpha(\Lambda)\subseteq\Gamma^c$ follows. Hence, choosing $\tilde{B}=\Lambda^c$ directly yields $\lambda(\tilde{B})=0$ as well as $1\geq\lambda(?_\alpha(\tilde{B}))\geq\lambda(\Gamma)=1$, which completes the proof for all partitions α of Type I.

Type II: For every $k\in\mathbb{N}$, we have $t_k-t_{k+1}=\frac{1}{\ln 2}\ln\frac{(k+1)^2}{k(k+2)}$.

Taking into account $t_1=1$, we get that there is only one partition α of Type II, namely the one fulfilling $t_k=\frac{1}{\ln 2}\ln\frac{k+2}{k+1}$ for every $k\in\mathbb{N}$. Instead of considering asymptotic frequencies of single 'digits' of the Lüroth- and continued-fraction expansions, we now consider the asymptotic frequency of the 'block' $(1,1)$ and set

$$\Lambda=\left\{x\in[0,1]:\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbf{1}_{(\frac{1}{2},1]^2}\circ(G^i(x),G^{i+1}(x))=\frac{1}{\ln 2}\ln\frac{10}{9}\right\} \tag{5}$$

$$\Gamma=\left\{x\in[0,1]:\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbf{1}_{(\frac{1}{2},1]^2}\circ(L_\alpha^i(x),L_\alpha^{i+1}(x))=\left(\frac{\ln\frac{4}{3}}{\ln 2}\right)^2\right\}. \tag{6}$$

According to Proposition 4.1.2 in [3], we have $\lambda(\Lambda)=1$. Moreover, using the fact that L_α is (isomorphic to) a Bernoulli shift and that $(t_1-t_2)^2=\left(\frac{\ln\frac{4}{3}}{\ln 2}\right)^2$ holds, $\lambda(\Gamma)=1$ follows. Considering $\left(\frac{\ln\frac{4}{3}}{\ln 2}\right)^2\neq\frac{\ln\frac{10}{9}}{\ln 2}$ and proceeding as in the second part of the first case and setting $\tilde{B}=\Lambda^c$ completes the proof. □

Remark 1. Instead of using Proposition 4.1.2 in [3] and the fact that L_α is a Bernoulli shift in order to prove $\lambda(\Lambda)=\lambda(\Gamma)=1$ for α of Type II, we could as well consider the maps $\varepsilon_G,\varepsilon_{L_\alpha}:[0,1]\to[0,1]^2$, defined by $\varepsilon_G(x)=(x,G(x))$ and $\varepsilon_{L_\alpha}(x)=(x,L_\alpha(x))$, and directly work with the ergodicity of G and L_α. In fact, applying Birkhoff's ergodic theorem ([4]) to the indicator function $f:x\mapsto\mathbf{1}_{(\frac{1}{2},1]^2}\circ\varepsilon_G(x)$ directly yields that, for μ_G-a.e. $x\in[0,1]$ (hence for λ-a.e. $x\in[0,1]$), we get

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbf{1}_{(\frac{1}{2},1]^2}\circ(G^i(x),G^{i+1}(x))=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}f\circ G^i(x)=\frac{1}{\ln 2}\int_{[0,1]}f(x)\frac{1}{1+x}\mathrm{d}x=\frac{1}{\ln 2}\ln\frac{10}{9}.$$

Proceeding analogously with the function $f:x\mapsto\mathbf{1}_{(\frac{1}{2},1]^2}\circ\varepsilon_{L_\alpha}(x)$ shows that $\lambda(\Gamma)=1$.

Proof b. Let Λ' denote the set of all points $x\in(0,1)$ at which $?_\alpha$ is differentiable and define Λ according to Eq. (3). For every $k\in\mathbb{N}$, define two new Borel sets Λ_1^k,Λ_3^k by

$$\Lambda_1^k=\left\{x\in[0,1]:\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbf{1}_{(\frac{1}{2},1]\times(\frac{1}{k+1},\frac{1}{k}]}\circ(G^i(x),G^{i+1}(x))=\frac{1}{\ln 2}\ln\frac{1+\frac{k+1}{k+2}}{1+\frac{k}{k+1}}\right\}$$

$$\Lambda_3^k=\left\{x\in[0,1]:\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mathbf{1}_{(\frac{1}{4},\frac{1}{3}]\times(\frac{1}{k+1},\frac{1}{k}]}\circ(G^i(x),G^{i+1}(x))=\frac{1}{\ln 2}\ln\frac{1+\frac{k+1}{3k+4}}{1+\frac{k}{3k+1}}\right\}.$$

Following the same reasoning as in Remark 1 (or using Proposition 4.1.2 in [3]), we get $\lambda(\Lambda_1^k\cap\Lambda_3^k)=1$, implying that $\Omega:=\Lambda'\cap\Lambda\cap\bigcap_{k=1}^\infty(\Lambda_1^k\cap\Lambda_3^k)$ fulfills $\lambda(\Omega)=1$. Fix $x\in\Omega$ and set $\underline{a}:=\mathrm{cf}(x)\in\Sigma$. Additionally, for every $n\geq 3$, let $x_n,y_n\in\mathbb{Q}\cap[0,1]$ be defined by

$$x_n=[a_1,a_2,\ldots,a_n]:=\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\cdots+\frac{1}{a_n}}}},\quad y_n=[a_1,a_2,\ldots,a_n+1]:=\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{\cdots+\frac{1}{a_n+1}}}}. \tag{7}$$

Then we have $\lim_{n\to\infty}x_n=\lim_{n\to\infty}y_n=x$ as well as $x\in(x_n,y_n)$ for n even and $x\in(y_n,x_n)$ for n odd. Setting $x_n=\frac{p_n}{q_n}$ with p_n,q_n relatively prime and using the recurrence relations (1.12) in [2], we get $y_n=\frac{(a_n+1)p_{n-1}+p_{n-2}}{(a_n+1)q_{n-1}+q_{n-2}}=\frac{p_n+p_{n-1}}{q_n-q_{n-1}}$, which implies $x_n-y_n=(-1)^{n+1}\frac{1}{q_n(q_n+q_{n-1})}$. On the other hand, considering $?_\alpha(x_n)=\Phi_\alpha((a_1,a_2,\ldots,a_n,\infty,\infty,\ldots))$ and $?_\alpha(y_n)=\Phi_\alpha((a_1,a_2,\ldots,a_n+1,\infty,\infty,\ldots))$, using Eq. (1) yields:

$$\delta_n:=\frac{?_\alpha(x_n)-?_\alpha(y_n)}{x_n-y_n}=\frac{(-1)^{n+1}\prod_{j=1}^n(t_{a_j}-t_{a_j+1})}{(-1)^{n+1}\frac{1}{q_n(q_n+q_{n-1})}}=q_n(q_n+q_{n-1})\prod_{j=1}^n(t_{a_j}-t_{a_j+1})\geq 0. \tag{8}$$

Since, by construction, $?_\alpha$ is differentiable at x, obviously $\lim_{n\to\infty}\delta_n=?'_\alpha(x)\geq 0$ holds.

Suppose now that $?'_\alpha(x) > 0$. Then $\lim_{n\to\infty} \frac{\delta_n}{\delta_{n-1}} = 1$ follows, from which, again using the recurrence relations (1.12) in [2], we get:

$$\begin{aligned} 1 &= \lim_{n\to\infty} \frac{\delta_n}{\delta_{n-1}} = \lim_{n\to\infty} (t_{a_n} - t_{a_n+1}) \frac{q_n(q_n + q_{n-1})}{q_{n-1}(q_{n-1} + q_{n-2})} = \lim_{n\to\infty} (t_{a_n} - t_{a_n+1}) \frac{\frac{q_n^2}{q_{n-1}^2} + \frac{q_n}{q_{n-1}}}{1 + \frac{q_{n-2}}{q_{n-1}}} \\ &= \lim_{n\to\infty} (t_{a_n} - t_{a_n+1}) \frac{\big(a_n + \frac{q_{n-2}}{q_{n-1}}\big)^2 + (a_n + \frac{q_{n-2}}{q_{n-1}})}{1 + \frac{q_{n-2}}{q_{n-1}}}. \end{aligned} \tag{9}$$

Fix an arbitrary $k \in \mathbb{N}$. Letting let $(n_j)_{j\in\mathbb{N}}$ denote the subsequence of all indices with $a_{n_j} = k$, Eq. (9) simplifies into

$$1 = (t_k - t_{k+1}) \lim_{j\to\infty} \frac{\big(k + \frac{q_{n_j-2}}{q_{n_j-1}}\big)^2 + (k + \frac{q_{n_j-2}}{q_{n_j-1}})}{1 + \frac{q_{n_j-2}}{q_{n_j-1}}}. \tag{10}$$

By construction of Ω, we have that $(a_{n_j-1}, a_{n_j}) = (1, k)$ is fulfilled infinitely often and that the same is true for $(a_{n_j-1}, a_{n_j}) = (3, k)$. Using the same notation as in Eq. (7), according to [2], $\frac{q_{n_j-2}}{q_{n_j-1}} = [a_{n_j-1}, \ldots, a_2, a_1]$ holds, from which we conclude that $\frac{q_{n_j-2}}{q_{n_j-1}}$ lies infinitely often in $(\frac{1}{2}, 1]$ and infinitely often in $(\frac{1}{4}, \frac{1}{3}]$. This contradicts Eq. (10), implying that we can not have $?'_\alpha(x) > 0$ if $x \in \Omega$. Since $x \in \Omega$ was arbitrary, we have shown that $?'_\alpha = 0$ holds λ-a.e., which completes the proof. □

Acknowledgements

The authors have been supported by the Ministerio de Economía y Competitividad (Spain) under research project MTM2014-60594.

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