Complex analysis

Second Hankel determinant for close-to-convex functions

Deuxième déterminant de Hankel pour les fonctions presque convexes

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A B S T R A C T

So far, the sharp bound of the expression \(|a_2a_4 - a_3^2|\) for the class \(C\) of close-to-convex functions has remained unknown. In this paper, we obtain the estimation of this expression, called the second Hankel determinant, for \(C_0\), i.e. the subset of \(C\) consisting of functions \(f\) that satisfy in the unit disk the inequality \(\text{Re}\left(|zf'(z)|/g(z)\right) > 0\) with a starlike function \(g\). Moreover, some remarks on the second Hankel determinant for the class \(S\) of univalent functions are made. It is proven that \(\max\{|a_2a_4 - a_3^2|: f \in S\}\) is greater than 1.

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R É S U M É

Aucune estimation précise de l’expression \(|a_2a_4 - a_3^2|\) pour la classe \(C\) des fonctions presque convexes n’était connue jusqu’à présent. Dans cette Note, nous présentons des estimations de cette expression, nommée deuxième déterminant de Hankel pour la classe \(C_0\), c’est-à-dire la sous-classe \(C\), composée des fonctions \(f\) qui vérifient, dans le disque unité, l’inégalité \(\text{Re}\left(|zf'(z)|/g(z)\right) > 0\) avec une fonction étoilée \(g\).

De plus, nous formulons quelques remarques à propos du deuxième déterminant de Hankel pour la classe \(S\) des fonctions univalentes. Nous démontrons que \(\max\{|a_2a_4 - a_3^2|: f \in S\}\) est plus grand que 1.

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1. Introduction

Let \(A\) denote the family of all analytic functions \(f\) in the open unit disk \(\Delta = \{z \in \mathbb{C} : |z| < 1\}\) normalized by \(f(0) = 0\), \(f'(0) = 1\). Hence the functions in \(A\) are of the form

\[ f(z) = z + a_2z^2 + a_3z^3 + \ldots \]  \hspace{1cm} (1)

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The Hankel determinant for a given function \( f \) of the form (1) is defined as follows

\[
H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\
a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \\
\end{vmatrix},
\]

where \( n, q \) are fixed positive integers.

The investigations of Hankel determinants for various classes of analytic functions started in the 1960s. It was Pommerenke [19], [20] who first studied Hankel’s determinant for the class \( S \) of univalent functions given by (1). He proved for functions in \( S \) that \( |H_2(n)| < Kn^{-1/(1+\beta)}q^{3/2} \), where \( n, q \in \mathbb{N}, q \geq 2, \beta > 1/4000 \) and \( K \) depends only on \( q \). Similar findings, but for different classes, were reported by Hayman [6] and Noor [17], [18].

Many recent papers have been devoted to the problem of finding the exact bounds of \( |H_q(n)| \) for various subfamilies of \( \mathcal{A} \). The majority of the results were obtained for \( H_2(2) = a_2a_4 - a_3^2 \), which is called the second Hankel determinant (see, for example, [1], [7], [8], [13], [22], [23]). There are, however, few papers that discuss the third Hankel determinant \( H_3(1) \) (see, for example: [2], [21], [24]). Although many estimates of \( |H_2(2)| \) are sharp, for example for the classes \( S^* \) or \( K \) consisting of starlike or convex functions, respectively, the exact bound of \( |H_3(2)| \) for \( S \) or for the class \( \mathcal{C} \) of close-to-convex functions is still not known.

In this paper, we focus our discussion on \( C \). It is known (see [5]) that \( f \in \mathcal{C} \) if there exist a starlike function \( g \) and a real number \( \beta \in (-\pi/2, \pi/2) \) such that

\[
\Re \left( e^{i\beta}zf'(z)/g(z) \right) > 0.
\]

(2)

We distinguish subclasses of \( \mathcal{C} \) according to a fixed number \( \beta \). Namely, a function \( f \) of the form (1) is called close to convex with argument \( \beta \) if there exists \( g \in S^* \) such that the condition (2) holds. Let \( \mathcal{C}_\beta \) denote the class of all such functions. It is obvious that

\[
\mathcal{C} = \bigcup_{\beta \in (-\pi/2, \pi/2)} \mathcal{C}_\beta.
\]

Taking into account (2), we can write

\[
e^{i\beta}zf'(z)/g(z) = p(z) \cos \beta + i \sin \beta,
\]

(3)

with \( p \in \mathcal{P} \), where \( \mathcal{P} \) is the well-known class of functions with positive real part that are normalized by \( p(0) = 1 \).

If \( g \in S^* \) and \( p \in \mathcal{P} \) in (3) are given by

\[
g(z) = z + b_2z^2 + b_3z^3 + \ldots
\]

(4)

and

\[
p(z) = 1 + p_1z + p_2z^2 + \ldots,
\]

(5)

then

\[
z + \sum_{n=2}^{\infty} n_0^n z^n = \left( z + \sum_{n=2}^{\infty} b_n z^n \right) \left( 1 + e^{-i\beta} \cos \beta \sum_{n=1}^{\infty} p_n z^n \right).
\]

(6)

Therefore,

\[
n_0^n = b_n + e^{-i\beta} \cos \beta \left( p_{n-1} + \sum_{j=2}^{n-1} b_j p_{n-j} \right), \quad n \geq 2.
\]

(7)

If \( n = 2 \), then the sum in the parentheses vanishes.

It is clear that the maximum of \( |H_2(2)| \) while \( f \) varies in the whole class \( S \) or \( \mathcal{C} \) is greater than or equal to 1 because of the result of Janteng et al. [7]. The estimation of \( |H_2(2)| \) for the functions \( f \) given by (1) belonging to \( \mathcal{C} \) is difficult to obtain, because it involves the coefficients of both functions \( g \in S^* \), \( p \in \mathcal{P} \) and a constant \( \beta \) (see, Remark 3 in [15]). For this reason, it is somewhat easier to estimate the second Hankel determinant if \( \beta = 0 \), i.e. in the class \( C_0 \). Even for \( C_0 \), the known bounds of \( |H_2(2)| \) are not sharp. The best known result (excluding erroneous ones) was obtained by Prajapat et al. in [21]. They proved that \( |H_2(2)| \leq 65/36 = 2.361 \ldots \) in \( C_0 \). In Theorem 1, we essentially improve this result. Moreover, we discuss an example of univalent functions that shows that the maximum of \( |H_2(2)| \) for \( S \) is actually greater than 1.
2. Preliminary results

At the beginning, let us discuss the invariance property of the class $C$.
Let $f$ be given by (1) and let
\[ f_\varphi(z) = e^{-i\varphi} f(z e^{i\varphi}), \varphi \in \mathbb{R}. \tag{8} \]
Directly from the definition of a close-to-convex function, it follows that $f \in C$ if and only if $f_\varphi \in C$. The same remains true if we replace $C$ by $S^*$ or $S$. Moreover, we can prove the following lemma.

**Lemma 1.** The equivalence
\[ f \in C_\beta \iff f_\varphi \in C_\beta \]
holds for every $\varphi \in \mathbb{R}$ and a fixed $\beta \in (-\pi/2, \pi/2)$.

**Proof.** If $f_\varphi$ is in $C_\beta$ for every $\varphi \in \mathbb{R}$, so it is true also for $\varphi = 0$. For this reason, it is enough to prove only that $f \in C_\beta \Rightarrow f_\varphi \in C_\beta$. But for $f$ in $C_\beta$, there exists $g \in S^*$ such that (2) holds. Writing $ze^{i\varphi}$ instead of $z$ in (2), we obtain
\[ \Re\left( e^{i\beta}zf'(ze^{i\varphi}) / e^{-i\varphi}g(ze^{i\varphi}) \right) > 0, \]
which means that $f_\varphi \in C_\beta$ with $e^{-i\varphi}g(ze^{i\varphi})$ as a starlike function. 

Suppose that a given class $A$ of analytic functions is invariant under rotation. Let $f \in A$ be given by (1) and $f_\varphi(z) = z + \alpha_2 z^2 + \ldots$ is defined by (8). Hence,
\[ |\alpha_2 \alpha_4 - \mu \alpha_3^2| = |a_2 e^{i\varphi} \cdot a_4 e^{3i\varphi} - \mu \cdot (a_3 e^{2i\varphi})^2| = |a_2 a_4 - \mu a_3^2|. \tag{10} \]
For this reason (or applying a similar argument), we have the following lemma.

**Lemma 2.** If $A$ is one of the classes: $C$, $C_\beta$, $S^*$, $S$ and $\Phi(f)$ is one of the following functionals: $|a_2 \alpha_4 - \mu \alpha_3^2|$, $|a_4 - \mu a_2 \alpha_3|$, $|a_3 - \mu a_2^2|$ defined on $f \in A$ given by (1) with a fixed real number $\mu$. Then $\Phi(f) = \Phi(f_\varphi)$ for every $\varphi \in \mathbb{R}$.

To prove the main results, we need a few lemmas. The first one is by Libera and Złotkiewicz.

**Lemma 3.** [14] Let $p_1 \in [0, 2]$. A function $p$ given by (5) belongs to $\mathcal{P}$ if and only if
\[ 2p_2 = p_1^2 + x(4 - p_1^2) \]
and
\[ 4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z \]
for some $x$ and $z$ such that $|x| \leq 1$, $|z| \leq 1$.

Let $g \in S^*$ be given by (4). Applying the correspondence between functions in $S^*$ and $\mathcal{P}$
\[ z \frac{g'(z)}{g(z)} = q(z), \quad g \in S^*, \quad q \in \mathcal{P} \tag{11} \]
we get
\[ (n - 1)b_n = \sum_{j=1}^{n-1} b_j q_{n-j}, \quad n = 2, 3, \ldots \tag{12} \]
where $q(z) = 1 + q_1 z + q_2 z^2 + \ldots$.
In particular,
\[ b_2 = q_1, \quad b_3 = \frac{1}{2} (q_2 + q_1^2), \quad b_4 = \frac{1}{3} (q_3 + \frac{2}{3} q_1 q_2 + \frac{1}{2} q_1^3). \tag{13} \]
Lemma 4. If \( g \in S^* \) is given by (4) and \( \mu \in \mathbb{R} \), then
\[
|b_3 - \mu b_2^2| \leq \begin{cases} 
1 + (1/2 - \mu)|b_2|^2 & \text{for } \mu \leq 3/4 , \\
1 + (\mu - 1)|b_2|^2 & \text{for } \mu \geq 3/4 .
\end{cases}
\] (14)

Proof. From (13) we get
\[
b_3 - \mu b_2^2 = (1/2 - \mu)q_1^2 + q_2/2 .
\]
By Lemma 2, we can assume that \( q_1 \in [0, 2] \). Applying Lemma 3,
\[
b_3 - \mu b_2^2 = (3/4 - \mu)q_1^2 + (4 - q_1^2)y/4 , \text{ for some } y, |y| \leq 1 ;
\]
hence we obtain (14). \( \square \)

As a simple consequence of Lemma 4, we get the well-known Fekete–Szegö inequality \( |b_3 - \mu b_2^2| \leq \max\{1, \vert 4\mu - 3 \vert \} \) for \( S^* \).

Lemma 5. If \( g \in S^* \) is given by (4), then
\[
|b_4 - \frac{7}{9}b_2b_3| \leq H(|b_2|) .
\] (15)
where
\[
H(b) = \begin{cases} 
\frac{1}{2} \left( 2 + \frac{7}{18}b^2 + \frac{25}{36}b^3 \right) & \text{for } b \in [0, 6/7] , \\
\frac{1}{3} (11b - 2b^2) & \text{for } b \in [6/7, 2] .
\end{cases}
\] (16)

Proof. From (13), we have
\[
b_4 - \frac{7}{9}b_2b_3 = \frac{1}{2} \left( q_3 + \frac{1}{2}q_1q_2 - \frac{2}{3}q_1^3 \right) .
\]
In view of Lemma 2, we write \( q \) instead of \( q_1, q \in [0, 2] \). From Lemma 3,
\[
b_4 - \frac{7}{9}b_2b_3 = \frac{1}{36} \left[ 3q^2 + 8q(4 - q^2)y - 3q(4 - q^2)y^2 + 6(4 - q^2)(1 - |y|)^2z \right] .
\]
Denoting \( |y| = r \) and applying the triangle inequality, we obtain
\[
|b_4 - \frac{7}{9}b_2b_3| \leq \frac{1}{36} \left[ 3q^2 + 8q(4 - q^2)r + 3q(4 - q^2)r^2 + 6(4 - q^2)(1 - r^2) \right] .
\]

Let us denote the expression in square brackets in the above inequality by \( h(r) \). Since \( h'(r) = 0 \) only for \( r_0 = \frac{4q}{3q^2 - q} \), we conclude that \( \max(h(r) : r \in [0, 1]) \) is equal to \( h(r_0) \) if \( q \in [0, 6/7] \) and is equal to \( h(1) \) if \( q \in [6/7, 2] \). This completes the proof. \( \square \)

Lemma 6. If \( g \in S^* \) is given by (4), then
\[
|b_2b_4 - \frac{8}{9}b_3^2| \leq \frac{1}{9} \left( 4 - |b_2|^2 \right) (2 + |b_2|^2) .
\] (17)

Proof. In view of Lemma 2, we assume \( q = q_1 \in [0, 2] \). From (13) and from Lemma 3
\[
b_2b_4 - \frac{8}{9}b_3^2 = \frac{1}{36} (4 - q^2) \left[ 3q^2y - (q^2 + 8)y^2 + 6q(1 - |y|)^2z \right] .
\]
Hence, writing \( r = |y| \),
\[
|b_2b_4 - \frac{8}{9}b_3^2| \leq \frac{1}{9} \left( 4 - q^2 \right) \left[ 3q^2r + (q^2 + 8)r^2 + 6q(1 - r^2) \right] .
\]
The result follows if we take \( r = 1 \). \( \square \)

It is easy to check that \( \max\left\{ \frac{1}{9} (4 - b^2)(2 + b^2) : b \in [0, 2] \right\} = 1 \). Therefore, the result in Lemma 6 generalizes the result obtained in [25] (Theorem 3, for \( \mu = 8/9 \)); according to this paper, if \( g \in S^* \), then \( |b_2b_4 - \frac{8}{9}b_3^2| \leq 1 \).
3. Main results

Taking into account Lemma 1, we can rotate $f \in C_\beta$ in such a way that after this operation the second coefficient of $f$ is real and non-negative. But, in this case, the coefficients $b_2$ and $p_1$ are not necessarily real. From now on, we proceed in a different manner. A function $f \in C_\beta$ is rotated in such a way that $p_1$ in formula (5) is real and non-negative. Under this assumption, we cannot expect that $a_2$ and $b_2$ are real numbers.

Now, we are ready to prove the main theorem of this paper.

**Theorem 1.** If $f \in C_0$ is given by (1), then

$$|a_2 a_4 - a_2^3| \leq 1.242 \ldots$$  \hspace{1cm} (18)

**Proof.** From (7) it follows for $f \in C_0$ that

$$2a_2 = b_2 + p_1$$ \hspace{1cm} (19)
$$3a_3 = b_3 + b_2 p_1 + p_2$$ \hspace{1cm} (20)
$$4a_4 = b_4 + b_3 p_1 + b_2 p_2 + p_3 .$$ \hspace{1cm} (21)

Hence,

$$a_2 a_4 - a_2^3 = \frac{1}{8}(b_2 + p_1)(b_4 + b_3 p_1 + b_2 p_2 + p_3) - \frac{1}{8}(b_3 + b_2 p_1 + p_2)^2$$
$$= \frac{1}{8}(b_2 b_4 - \frac{8}{9}b_3^2) + \frac{1}{8}p_1(b_4 - \frac{7}{9}b_2 b_3) + \frac{1}{8}(p_1 p_3 - \frac{8}{9}p_2^2)
+ \frac{1}{8}p_1^2(b_3 - \frac{8}{9}b_2^2) - \frac{2}{8}p_2(b_3 - \frac{8}{9}b_2 b_3) + \frac{1}{8}b_2(p_3 - \frac{7}{9}p_1 p_2).$$

Taking into account Lemma 1 and formula (9), we can assume that $p_1$ is a non-negative real number; for this reason we write $p$ instead of $p_1$. Applying Lemma 3, we get

$$\frac{1}{8} b_2(p_3 - \frac{7}{9}pp_2) = \frac{1}{72} b_2 \left[ -\frac{5}{8} p^3 + \frac{4}{9} p(4 - p^2) x - p(4 - p^2)x^2 + 2(4 - p^2)(1 - |x|^2)z \right],$$

where $|x| \leq 1$ and $|z| \leq 1$. Therefore,

$$a_2 a_4 - a_2^3 = \frac{1}{8} (b_2 - \frac{8}{9}b_3^2) + \frac{1}{8}p(b_4 - \frac{7}{9}b_2 b_3) + \frac{1}{72} b_2 \left[ -\frac{5}{8} p^3 + \frac{4}{9} p(4 - p^2) x - p(4 - p^2)x^2 + 2(4 - p^2)(1 - |x|^2)z \right].$$

Let us denote $|b_2|$ by $b$ and $|x|$ by $\varrho$; hence, $b \in [0, 2], \varrho \in [0, 1]$. The triangle inequality leads to

$$|a_2 a_4 - a_2^3| \leq \frac{1}{8} \left[ |b_2 b_4 - \frac{8}{9}b_3^2| + |b_4 - \frac{7}{9}b_2 b_3| \right] + \frac{1}{72} b_2 \left[ |b_3 - \frac{7}{9}b_2^2 - \frac{5}{8} b_2 p + \frac{1}{4} p^2| \right] + \frac{1}{72} b_2 \left[ \frac{4}{8} p - \frac{9}{4} b_2 b_3 - \frac{9}{16} b_2 p - \frac{1}{16} p^2 \right] \left\| \varrho \right\| + \frac{1}{128} (4 - p^2)(32 + 9b_2 p + p^2) \left\| \varrho \right\|^2 + \frac{1}{16} (b + p)(4 - p^2)(1 - \varrho^2).$$

Applying Lemmas 4–6, we can write

$$|a_2 a_4 - a_2^3| \leq F(p, b, \varrho),$$

where

$$F(p, b, \varrho) = A + B \varrho + C \varrho^2, \quad p, b \in [0, 2], \varrho \in [0, 1],$$  \hspace{1cm} (22)

$$C = \frac{1}{72} (4 - p^2)(2 - p)(16 - p - 9b)$$
$$B = \frac{1}{72} (4 - p^2)(16 - b^2 + 2bp + p^2)$$
$$A = \frac{1}{8} (4 - b^2)(2 + b^2) + \frac{1}{8} p H(b) + \frac{1}{72} b^2 (4 + 10b^2 + 5bp + p^2) + \frac{1}{16} (b + p)(4 - p^2),$$

and $H(b)$ is defined by (16).
Now we shall show that \( F \) is an increasing function of \( \varrho \in [0, 1] \). We have
\[
\frac{\partial F}{\partial \varrho} = \frac{1}{164} (4 - p^2) \left[ 16 - b^2 + 2bp + p^2 + (2 - p)(16 - p - 9b)\varrho \right].
\]
If \( 16 - p - 9b \geq 0 \), then \( \frac{\partial F}{\partial \varrho} \geq 0 \). For \( 16 - p - 9b < 0 \),
\[
\frac{\partial F}{\partial \varrho} \geq \frac{1}{164} (4 - p^2)h(p, b),
\]
where
\[
h(p, b) = 48 + 2p^2 - 18p - 9b^2 - 15b + 11pb.
\]
It is not a difficult task to prove that \( h(p, b) \geq 0 \) for all \( (p, b) \in [0, 2] \times [0, 2] \). This proves that \( \frac{\partial F}{\partial \varrho} \geq 0 \) in \( [0, 2] \times [0, 2] \).

Therefore,
\[
F(p, b, \varrho) \leq F(p, b, 1) = A + B + C. \quad (23)
\]

Let us denote \( F(p, b, 1) \) by \( G(p, b) \). Hence,
\[
G(p, b) = \frac{1}{288} \left[ (4 - p^2)(64 + 3p^2 + 13pb - 2b^2) + 4(4 - b^2)(2 + b^2) + 36pH(b) + p^2(4 + 10b^2 + 5bp + p^2) \right], \quad p, b \in [0, 2]. \quad (24)
\]

To obtain the declared result, we divide the set of variability of \( (p, b) \), i.e. \( \Omega = [0, 2] \times [0, 2] \) into two subsets: \( \Omega_1 = [0, 2] \times [0, 6/7] \) and \( \Omega_2 = [0, 2] \times [6/7, 2] \).

I. First, assume that \( (p, b) \in \Omega_2 \). Then \( G(p, b) = \frac{1}{288} G_2(p, b) \), where
\[
G_2(p, b) = -2p^4 - 8p^3b + 12p^2b^2 - 48p^2 - 8pb^3 + 96pb - 4b^4 + 288. \quad (25)
\]

Our task is to find
\[
\max\{G_2(p, b) : (p, b) \in \Omega_2\}. \quad (26)
\]

Instead of (26), we shall derive
\[
\max\{G_2(p, b) : (p, b) \in \Omega\}. \quad (27)
\]

Observe that the critical points of \( G_2 \) satisfy the following system of equations
\[
\begin{cases}
-p^3 - 3p^2b + 3pb^2 - 12p - b^3 + 12b = 0 \\
-p^3 + 3p^2b - 3pb^2 + 12p - 2b^3 = 0.
\end{cases} \quad (28)
\]

For the point \((0, 0)\), (28) is fulfilled. Assume now that \( b \neq 0 \). Summing both equations in (28) we obtain
\[
2p^3 = 3b(4 - b^2). \quad (29)
\]

Applying it in one of the equations of (28), we get
\[
6bp^2 + 6(4 - b^2)p - b(12 + b^2) = 0. \quad (30)
\]

Hence,
\[
p = \frac{1}{6b} \left( 3(b^2 - 4) + \sqrt{15b^4 + 144} \right) \quad (31)
\]
is the positive solution to (30).

Combining (29) with (31), and dividing the obtained equation by \( b^3 \), we get
\[
2 \left[ \frac{1}{2} \left( 1 - \frac{4}{t^2} \right) + \frac{1}{6} \sqrt{15 + \left( \frac{12}{t^2} \right)^2} \right]^3 = 3 \left( \frac{4}{t^2} - 1 \right). \quad (32)
\]

Substituting \( t = 3(4/b^2 - 1) \), \( t \geq 0 \), equation (32) takes the form
\[
2 \left( -\frac{1}{6} t + \frac{1}{8} \sqrt{24 + 6t + t^2} \right)^3 = t. \quad (33)
\]
or equivalently,
\[ \sqrt{24 + 6t + t^2} - t = 3\sqrt{4t}. \] (34)

Now, it is not difficult to show that (34) has only one positive solution. Indeed, a function \( f_1(t) = \sqrt{24 + 6t + t^2} - t \) is decreasing and a function \( f_2(t) = 3\sqrt{4t} \) is increasing for \( t \geq 0 \). Moreover, \( f_1(0) = 2\sqrt{6} > 0 = f_2(0) \) and \( f_1(2) = 2(\sqrt{10} - 1) < 6 = f_2(2) \). It means that the only positive solution to (34) belongs to \((0, 2)\). Its numerical value is \( t_0 = 0.899 \ldots \)

For the reason presented above, we know that (28) has exactly one critical point such that \( p > 0 \) and \( b > 0 \); namely,
\[ p_0 = 1.343 \ldots, \quad b_0 = 1.754 \ldots, \] (35)
for which \( G_2(p_0, b_0) = 357.819 \ldots \)

On the boundary of \( \Omega \), we discuss the following cases. For \( b \in [0, 2] \), \( G_2(0, b) = 288 - 4b^4 \leq 288 \). Similarly, for \( p \in [0, 2] \), \( G_2(p, 0) = 288 - 48p^2 - 2p^4 \leq 288 \). If \( p = 2 \), then \( G_2(2, b) = 64 + 128b + 48b^2 - 16b^3 - 4b^4 \) is an increasing function because its derivative \( (4 + b)(1 + b)(2 - b) \) is greater than or equal to 0. For \( p \in [0, 2] \), \( G_2(p, 2) = 224 + 128p - 16p^3 - 2p^4 \). The derivative of this function is equal to \( 8(2 + p)(8 - 4p - p^2) \). Now, we deduce that the greatest value of \( G_2(p, 2) \) for \( p \in [0, 2] \) is equal to \( G_2(2(\sqrt{3} - 1), 2) = 352 \).

Summing up,
\[ \max\{G_2(p, b) : (p, b) \in \Omega\} = G_2(p_0, b_0). \] (36)

But \((p_0, b_0) \in \Omega_2\), so
\[ \max\{G_2(p, b) : (p, b) \in \Omega_2\} = G_2(p_0, b_0). \] (37)

II. Let \((p, b) \in \Omega_1\). Then \( G(p, b) = \frac{1}{288}G_1(p, b) \), where
\[ G_1(p, b) = -2p^4 - 8p^3b - 12(4 - b^2)p^2 + \left(\frac{25}{3}b^3 + \frac{14}{3}b^2 + 52b + 24\right) p - 4b^4 + 288. \] (38)

Let us denote \( f_3(p, b) = -2p^4 - 8p^3b - 4b^4 + 288 \) and \( f_4(p, b) = \left(\frac{25}{3}b^3 + \frac{14}{3}b^2 + 52b + 24\right) p - 12(4 - b^2)p^2 \). Hence, \( f_3(p, b) \leq 288 \) for all \((p, b) \in \Omega_1\).

The quadratic function \( f_4 \) of the variable \( p \) takes the greatest value for
\[ p = \frac{\frac{25}{3}b^3 + \frac{14}{3}b^2 + 52b + 24}{24(4 - b^2)}. \]

Since \( p \in [0, 2] \) for \( b \in [0, 6/7] \), so
\[ f_4(p, b) \leq f_4(p_+, b) = \left(\frac{\frac{25}{3}b^3 + \frac{14}{3}b^2 + 52b + 24}{48(4 - b^2)}\right)^2. \]

If \( b \in [0, 6/7] \), then the last expression is increasing; consequently
\[ f_4(p_+, b) \leq f_4(p_+, 6/7) = 38.072 \ldots. \]

Hence, for \((p, b) \in \Omega_1\),
\[ G_1(p, b) \leq 326.072 \ldots < G_2(p_0, b_0). \] (39)

Comparing the bounds obtained for \( \Omega_1 \) and \( \Omega_2 \), we deduce (18).

The result obtained in Theorem 1 is not sharp. Under the additional assumption that \( b_2 \) is real, this bound can be improved. This improved value is a little bit greater than 1, but still it is not sharp. We can pose the natural conjecture that \( |H_2(2)| \leq 1 \) for all functions in \( C_0 \). The same inequality likely holds also for \( C \).

4. Remarks on the second Hankel determinant for univalent functions

So far, we have not found any result concerning the estimates, even rough, of the expression \( |a_2a_4 - a_3^2| \) for the whole class \( S \) of univalent functions. Can it be true that \( |H_2(2)| \leq 1 \) for \( S \)?

Regarding the results of some coefficients problems one can find cases when the solutions to problems in \( C \) and \( S \) are the same and those when the solutions are different. For example, the bounds of \( |a_0| \) or \( |a_3 - a_2^2| \) are the same for both \( C \) and \( S \) (namely: \( n \) and 1, respectively). However, the Fekete–Szegő functional \( |a_3 - \mu a_2^2|, \mu \in [0, 1] \) is bounded by \( 1 + 2 \exp\left(-\frac{2\mu}{1-\mu}\right) \) in \( S \) (see, [4]) and by \( 3 - 4\mu \) for \( 0 \leq \mu \leq 1/3 \), \( 1/3 + 4/9\mu \) for \( 1/3 \leq \mu \leq 2/3 \) and 1 for \( 2/3 \leq \mu \leq 1 \) in \( C \). The latter was obtained at first by Keogh and Merkes in [10] for \( C_0 \), and next, by Eenigenburg and Silvia in [3] (independently by
Denoting

\[ \mathcal{C} \setminus \left( (-\infty, -d_\epsilon) \cup \{d_\epsilon e^{i\theta}, \theta \in [\theta_\epsilon, \pi] \} \right). \tag{40} \]

Krzyż and Reade proved that the functions \( f_\epsilon \) determine the Koebe set for the class \( \mathcal{Y} \) of circularly symmetric univalent functions, see [12]. In [16], Netanyahu showed that the maximum in \( \mathcal{S} \) of an expression \(|a_2| \cdot d_f\), where \( d_f = \inf(|\gamma| : f(z) \neq \gamma, z \in \Delta) \), is achieved by \( f_\epsilon \) for properly taken \( \epsilon \in (0, 1) \).

Given \( \epsilon \in (0, 1) \), the function \( f_\epsilon \) is obtained as a composition of a function \( s(z) \) satisfying

\[ s \frac{1}{(1+s)^2} = \frac{4\epsilon}{(1+\epsilon)^2} \frac{z}{(1+z)^2} \tag{41} \]

and a function

\[ w(s) = \frac{(1+\epsilon)^2}{4} \cdot \frac{s(1-\epsilon s)}{s-s}. \tag{42} \]

We have \( s(\Delta) = \Delta \setminus [\epsilon, 1) \). The numbers that appear in (40) take values:

\[ d_\epsilon = \frac{(1+\epsilon)^2}{4} \quad \text{and} \quad \theta_\epsilon = 2 \arccos \epsilon. \tag{43} \]

Observe that, in the limiting case, \( f_1 \) is the identity function. Since both \( s(z) \) and \( w(s) \) are univalent, \( f_\epsilon \) is also univalent. From (40) we conclude that \( f_\epsilon \) is not close to convex.

The function \( s(z) \) can be written as \( s(z) = k^{-1}(Ak(z)) \), with \( k(z) = \frac{z}{(1+z)^2} \) and \( A = \frac{4\epsilon}{(1+\epsilon)^2} \). Since \( k^{-1}(\zeta) = \zeta + 2\zeta^2 + 5\zeta^3 + 14\zeta^4 + 42\zeta^5 + \ldots \), we have

\[ \frac{s(z)}{A} = z - (2 - 2A)z^2 + (3 - 8A + 5A^2)z^3 - (4 - 20A + 30A^2 - 14A^3)z^4 \]
\[ \quad + (5 - 40A + 105A^2 - 112A^3 + 42A^4)z^5 + \ldots. \]

In a small neighbourhood of the origin

\[ w(s) = \frac{1}{A} \left[ s + \epsilon (1-\epsilon^2) \sum_{k=2}^\infty \left( \frac{s}{\epsilon} \right)^k \right]. \]

Therefore,

\[ f_\epsilon(z) = z + \frac{2(1-\epsilon)(1+3\epsilon)}{(1+\epsilon)^2} z^2 + \frac{(1-\epsilon)(3 + 15\epsilon + 33\epsilon^2 - 19\epsilon^3)}{(1+\epsilon)^4} z^3 \]
\[ + \frac{4(1-\epsilon)(1+7\epsilon + 18\epsilon^2 + 54\epsilon^3 - 59\epsilon^4 + 11\epsilon^5)}{(1+\epsilon)^6} z^4 + \ldots. \tag{44} \]

From (44) it follows that \( f_1(z) = z \). Moreover, taking \( \epsilon = 0 \) in (44), we obtain \( f_0(z) = z + 2z^2 + \ldots = \frac{z}{(1+z)^2} \). In this case, the set (40) coincides with \( \mathcal{C} \setminus (-\infty, -1/4) \).

For a function (44),

\[ H_2(2) = -F(\epsilon), \]

where

\[ F(\epsilon) = \frac{(1-\epsilon)^4}{(1+\epsilon)^8}(1 + 12\epsilon + 134\epsilon^2 + 268\epsilon^3 + 97\epsilon^4), \quad \epsilon \in [0, 1]. \tag{45} \]

Therefore, \( H_2(2) \leq 0 \) and

\[ \frac{F'(\epsilon)}{F(\epsilon)} = \frac{128\epsilon(1 - 6\epsilon - 20\epsilon^2 - 7\epsilon^3)}{(1-\epsilon^2)(1+12\epsilon + 134\epsilon^2 + 268\epsilon^3 + 97\epsilon^4)}. \]

Denoting by \( \epsilon_0 \) the only solution to \( 1 - 6\epsilon - 20\epsilon^2 - 7\epsilon^3 = 0 \) in \((0, 1)\), i.e. \( \epsilon_0 = 0.118 \ldots \), we can write

\[ \max\{F(\epsilon) : \epsilon \in [0, 1]\} = F(\epsilon_0) = 1.175 \ldots. \tag{46} \]

We have proved the following theorem.

**Theorem 2.** If \( f \) is given by (1), then

\[ \max\{|a_2a_4 - a_3^2| : f \in \mathcal{S}\} \geq 1.175 \ldots. \tag{47} \]
References