



ELSEVIER

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com



Differential geometry

Rigidity of negatively curved geodesic orbit Finsler spaces<sup>☆</sup>

*Rigidité des espaces de Finsler à géodésiques homogènes et à courbure négative*

Ming Xu<sup>a</sup>, Shaoqiang Deng<sup>b</sup><sup>a</sup> School of Mathematical Sciences, Capital Normal University, Beijing 100048, PR China<sup>b</sup> School of Mathematical Sciences and LPMC, Nankai University, Tianjin 300071, PR China

## ARTICLE INFO

## Article history:

Received 16 April 2016

Accepted 13 September 2017

Available online 19 September 2017

Presented by Jean-Michel Bismut

## ABSTRACT

We prove some rigidity results on geodesic orbit Finsler spaces with non-positive curvature. In particular, we show that a geodesic orbit Finsler space with strictly negative flag curvature must be a non-compact Riemannian symmetric space of rank one.

© 2017 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

## R É S U M É

Nous montrons des résultats de rigidité sur les espaces de Finsler homogènes, dont les géodésiques sont des espaces homogènes à courbure non positive. En particulier, nous montrons qu'un tel espace de Finsler dont la courbure de drapeau est strictement négative doit être un espace symétrique riemannien non compact de rang un.

© 2017 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

## 1. Introduction

The main goal of this short note is to prove some rigidity results on geodesic-orbit Finsler spaces (simply g.o. Finsler spaces) with negative curvature. Recall that a homogeneous Finsler space  $(G/H, F)$  is called a  $G$ -geodesic orbit Finsler space if any geodesic of  $(G/H, F)$  is  $G$ -homogeneous, i.e. it is the orbit of a one-parameter subgroup  $g_t$  in  $G$  generated by a vector  $v \in \text{Lie}(G) = \mathfrak{g}$ . This definition is a natural generalization of the Riemannian geodesic-orbit space, which has been extensively studied, see for example [1,7]. Finslerian g.o. spaces and homogeneous normal Finsler spaces have also recently been studied in [14] and [16]. This research shows that there exists many examples of compact Riemannian or non-Riemannian g.o. spaces, e.g., Clifford–Wolf homogeneous spaces, normal homogeneous spaces, generalized normal homogeneous spaces (or  $\delta$ -homogeneous spaces), etc. Some noncompact examples have also been constructed.

Our rigidity results are relevant to curvatures in Finsler geometry, where flag curvature is one of the most important geometric quantities, and most of the research in this field is focused on issues related to it. For example, the classifica-

<sup>☆</sup> Supported by NSFC (no. 11771331, 11671212, 51535008).E-mail addresses: [mngmgxu@163.com](mailto:mngmgxu@163.com) (M. Xu), [dengsq@nankai.edu.cn](mailto:dengsq@nankai.edu.cn) (S. Deng).

tion of Finsler spaces with certain flag curvature conditions is a very interesting problem. Recently, we have made some progress in this consideration, see [6,14,15]. For example, a complete classification of normal homogeneous Finsler spaces with positive flag curvature is established, which generalizes the results of M. Berger [2]. In this process, we found that many positively curved normal homogeneous Riemannian manifolds can be endowed with invariant positively curved non-Riemannian normal Finsler metrics. One may expect that there may also exist non-Riemannian negatively curved examples. However, contrary to the above fact, in the negative case we get the following rigidity theorem.

**Theorem 1.1.** *Any connected negatively curved g.o. Finsler space must be a compact Riemannian symmetric space of rank one.*

This fact is interesting in that any homogeneous normal Finsler space must be a g.o. space [14]. We have also made some efforts to generalize the result to g.o. Finsler spaces with negative Ricci curvature. However, the problem becomes rather hard, and it is still open. Nevertheless, the same argument can also be applied to a slightly general case to get the following rigidity theorem.

**Theorem 1.2.** *Any g.o. Finsler space with negative Ricci scalar and non-positive flag curvature must be a symmetric space of non-compact type.*

Note that a symmetric Finsler space must be Berwald, but can be non-Riemannian if its rank is larger than one [3].

Another goal of this paper is to explore the general properties of negatively curved homogeneous Finsler metrics. We will show that any connected negatively curved homogeneous Finsler space can be viewed as a solvemanifold endowed with a left-invariant Finsler metric. This fact motivates us to consider the classification of left-invariant Finsler metrics on a solvable Lie group with negative curvature. Applying the homogeneous flag curvature formula of L. Huang [9], we prove the theorem below.

**Theorem 1.3.** *Let  $G$  be a simply connected solvable Lie group such that  $[\mathfrak{g}, \mathfrak{g}]$  is Abelian and nonzero, where  $\mathfrak{g}$  is the Lie algebra of  $G$ . Then it admits a negatively curved left-invariant Finsler metric if and only if the following conditions are satisfied:*

- (1)  $\dim \mathfrak{g} = \dim[\mathfrak{g}, \mathfrak{g}] + 1$ ;
- (2) *there exists a vector  $u \in \mathfrak{g}$ , such that all the eigenvalues of  $\text{ad}(u)|_{[\mathfrak{g}, \mathfrak{g}]} : [\mathfrak{g}, \mathfrak{g}] \rightarrow [\mathfrak{g}, \mathfrak{g}]$  have positive real parts.*

Inspired by [8], we conjecture that Theorem 1.3 is valid for general negatively curved homogeneous Finsler spaces.

## 2. A homogeneous flag curvature formula

On a Finsler space  $(M, F)$ , the Riemann curvature  $R^F_y = R^i_k(y)\partial_{x^i} \otimes dx^k : T_xM \rightarrow T_xM$  can be similarly defined as in Riemann geometry, either by the structure equation of the Chern connection, or the Jacobi field equation of the variation of geodesics [11], and it can be used to define the flag curvature and the Ricci curvature. Let  $y \in T_xM$  be a nonzero tangent vector and  $\mathbf{P}$  a tangent plane in  $T_xM$  containing  $y$ , and suppose that  $\mathbf{P}$  is linearly spanned by  $y$  and  $v$ . Then the flag curvature of the triple  $(x, y, y \wedge v)$  or  $(x, y, \mathbf{P})$  is defined as

$$K^F(x, y, y \wedge v) = K^F(x, y, \mathbf{P}) = \frac{\langle R_y v, v \rangle_y}{\langle y, y \rangle_y \langle v, v \rangle_y - \langle y, v \rangle_y^2},$$

and the Ricci scalar of the nonzero vector  $y \in T_xM$  is

$$\text{Ric}^F(x, y) = F(x, y)^2 \sum_{i=1}^{n-1} K^F(x, y, y \wedge e_i)$$

where  $\{e_1, \dots, e_{n-1}, y/F(y)\}$  is a  $g^F_y$ -orthonormal basis of  $T_xM$ .

The following flag curvature formula, which is very convenient in homogeneous Finsler geometry, can be found in [15].

**Theorem 2.1.** *Let  $(G/H, F)$  be a connected homogeneous Finsler space, and  $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$  be an  $\text{Ad}(H)$ -invariant decomposition for  $G/H$ . Then, for any linearly independent commutative pair  $u$  and  $v$  in  $\mathfrak{m}$  satisfying  $\langle [u, \mathfrak{m}], u \rangle_u^F = 0$ , we have*

$$K^F(o, u, u \wedge v) = \frac{\langle U(u, v), U(u, v) \rangle_u^F}{\langle u, u \rangle_u^F \langle v, v \rangle_u^F - \langle u, v \rangle_u^F \langle u, v \rangle_u^F},$$

where  $U$  is a bilinear map from  $\mathfrak{m} \times \mathfrak{m}$  to  $\mathfrak{m}$  defined by

$$\langle U(u, v), w \rangle_u^F = \frac{1}{2}(\langle [w, u]_{\mathfrak{m}}, v \rangle_u^F + \langle [w, v]_{\mathfrak{m}}, u \rangle_u^F), \quad w \in \mathfrak{m};$$

here  $[\cdot, \cdot]_{\mathfrak{m}} = \text{pr}_{\mathfrak{m}} \circ [\cdot, \cdot]$  and  $\text{pr}_{\mathfrak{m}}$  is the projection with respect to the given  $\text{Ad}(H)$ -invariant decomposition.

### 3. Proofs of Theorem 1.1 and Theorem 1.2

**Proof of Theorem 1.1.** Let  $(M, F)$  be a connected negatively curved geodesic orbit Finsler space. Then we can write  $M$  as  $M = G/H$ , where  $G = I_0(M, F)$  and  $H$  is the compact isotropy subgroup at  $o = eH$ .

We first prove that  $G$  is a semisimple Lie group. Assume conversely that it is not. Then we can find an Abelian ideal  $\mathfrak{a}$  of  $\mathfrak{g}$  such that  $\mathfrak{a} \cap \mathfrak{h} = \{0\}$ , where  $\mathfrak{h} = \text{Lie } H$ . Hence we can find an  $\text{Ad}(H)$ -invariant decomposition  $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$  such that  $\mathfrak{a} \subset \mathfrak{m}$ .

**Lemma 3.1.** *Keep the above notation and assumptions. Then any vector  $v$  in  $\mathfrak{a}$  defines a Killing vector field of constant length of  $(G/H, F)$ .*

**Proof.** We first prove that  $o = eH$  is the critical point of the function  $F^2(V(\cdot))$ , where  $V$  is the Killing vector field defined by a fixed nonzero vector  $v \in \mathfrak{a}$ . This is equivalent to  $\langle v, [v, \mathfrak{g}]_{\mathfrak{m}} \rangle_v^F = 0$ . Since  $\langle v, [v, \mathfrak{h}] \rangle_v^F = 0$ , we only need to show that  $\langle v, [v, \mathfrak{m}]_{\mathfrak{m}} \rangle_v^F = 0$ . By Lemma 3.3 in [16], there exists a vector  $v' \in \mathfrak{h}$ , such that  $\langle v, [v, u]_{\mathfrak{m}} \rangle_v^F = \langle v, [v', u] \rangle_v^F, \forall u \in \mathfrak{m}$ . On the other hand, by Theorem 3.1 of [4], we have

$$\langle v, [v', u] \rangle_v^F = \langle [v, v'], u \rangle_v^F + 2C_v^F(v, u, [v, v']) = \langle [v, v'], u \rangle_v^F.$$

If  $u$  is contained in the  $g_v^F$ -orthogonal complement of  $\mathfrak{a}$  in  $\mathfrak{m}$ , then  $\langle [v, v'], u \rangle_v^F \subset \langle \mathfrak{a}, v' \rangle_v^F = 0$ . In case  $u \in \mathfrak{a}$ , we also have  $\langle v, [v, u]_{\mathfrak{m}} \rangle_v^F = 0$ . Therefore, we always have  $\langle v, [v, \mathfrak{m}]_{\mathfrak{m}} \rangle_v^F = 0$ . Thus  $o = eH$  is a critical point of the function  $F^2(V(\cdot))$ .

Now given  $g \in G$ ,  $\text{Ad}(g^{-1})v$  is a nonzero vector in  $\mathfrak{a}$ , which defines another Killing vector field  $V'$  of  $(G/H, F)$ . Moreover, we have  $F(V'(\cdot)) = F(V(g \cdot))$ , and  $F(V'(e)) = F(V(g)) \neq 0$ . Since  $o = eH$  is a critical point of  $F^2(V'(\cdot))$ ,  $x = gH$  is also a critical point of  $F^2(V(\cdot))$ . Consequently each point is a critical point of  $F^2(V(\cdot))$ , i.e.  $V$  is a Killing vector field of constant length.  $\square$

By the above lemma, both the flag curvature  $K^F(o, y, \mathbf{P})$  and the Ricci scalar  $\text{Ric}^F(o, y)$  are non-negative for nonzero  $y \in \mathfrak{a}$  (see the proof of Theorem 5.1 in [14] for more details). This is a contradiction. Thus  $G$  is semisimple.

Next we prove that  $H$  is a maximal compact subgroup of  $G$ . Assume conversely that it is not. Then  $H$  is contained in a connected maximal subgroup  $K$  with  $\dim H < \dim K$ . Correspondingly, we have the Iwasawa decomposition  $G = NAK$ , where the solvable subgroup  $NA$  is diffeomorphic to a Euclidean space. Then  $G/H$  is diffeomorphic to  $NA \times K/H$ , hence it is homotopic to  $K/H$ . By the main theorem in [5],  $G/H$  is diffeomorphic to a Euclidean space, but the compact homogeneous space  $K/H$  can not be topologically trivial. This contradiction proves our assertion.

In summarizing,  $H$  is a maximal compact subgroup of the semisimple Lie group  $G$ , hence  $G/H$  is a simply connected symmetric homogeneous space of non-compact type. The homogeneous metric  $F$  on  $G/H$  is Berwald, so it is negatively curved if and only if the Riemannian symmetric homogeneous space  $G/H$  is negatively curved, i.e.  $G/H$ , is of rank 1 [3]. However, in an irreducible Riemannian symmetric spaces of rank 1, the isotropy group  $H$  acts transitively on the unit sphere of the tangent space at the origin. So  $F$  must be a positive multiple of the canonical Riemannian metric. This completes the proof of the theorem.  $\square$

**Proof of Theorem 1.2.** Note that the arguments in the proof of Theorem 1.1 are still valid under the assumption that  $(G/H, F)$  has non-positive flag curvature and negative Ricci scalar. So we can similarly show that  $G/H$  is a simply connected symmetric space, but not necessarily of rank 1. Then the homogeneous Finsler metric  $F$  must be Berwald, and  $F$  can be non-Riemannian (see [3]).  $\square$

### 4. Negatively curved solvemanifolds

We first show that any connected negatively curved homogeneous Finsler space can be viewed as a solvemanifold endowed with a left-invariant Finsler metric. Let  $G/H$  be a connected negatively curved homogeneous Finsler space, where  $G$  is connected and  $H$  is the compact isotropy subgroup at  $o = eH$ . By [5],  $G/H$  is simply connected. Using Theorem 3.13 in [10], we can also prove that  $H$  is a maximal compact subgroup of  $G$  using a similar argument as in the proof of Theorem 1.1, and this proves our assertion. We remark here that, recently, Wang and Li [12] made some arguments to show that any connected negatively curved homogeneous Finsler space  $(M, F)$  admits a closed solvable transitive subgroup  $G$  of  $I_0(M, F)$ , based on the results of Wolf in [13].

**Proof of Theorem 1.3.** If  $G$  satisfies (1) and (2) in the theorem, then by [8] it admits a left-invariant negatively curved Riemannian metric.

Assume that  $G$  is a simply connected solvable Lie group, with  $\text{Lie}(G) = \mathfrak{g}$  satisfying  $[\mathfrak{g}, \mathfrak{g}] = 0$ , and  $F$  is a left-invariant negatively curved Finsler metric on  $G$ . Then (1) follows from an unpublished proposition of L. Huang. With his permission, we quote it here.

**Proposition 4.1.** *Let  $F$  be a left-invariant negatively curved Finsler metric on a simply connected solvable Lie group  $G$ . Then  $\mathfrak{g} = \text{Lie}(G)$  must satisfy  $\dim \mathfrak{g} = \dim[\mathfrak{g}, \mathfrak{g}] + 1$ .*

**Proof.** Since  $G$  (and  $\mathfrak{g}$ ) is solvable, we have  $\dim[\mathfrak{g}, \mathfrak{g}] < \dim \mathfrak{g}$ . Thus there exists a nonzero vector  $u \in \mathfrak{g} \setminus [\mathfrak{g}, \mathfrak{g}]$ , such that  $\langle u, [\mathfrak{g}, \mathfrak{g}] \rangle_u^F = 0$ . Then a direct calculation shows that  $\eta(u) = 0$ . If there exists a nonzero vector  $v \in \mathfrak{g}$  such that  $u$  and  $v$  are commutative and linearly independent, then we have  $\langle u, [u, \mathfrak{g}] \rangle_u^F = 0$ . Hence, by [Theorem 2.1](#),  $K^F(o, u, u \wedge v) \geq 0$ . Thus  $\text{Ad}(u) : \mathfrak{g} \rightarrow [\mathfrak{g}, \mathfrak{g}]$  has only a one-dimensional kernel  $\mathbb{R}u$ , i.e.  $\dim \mathfrak{g} = \dim[\mathfrak{g}, \mathfrak{g}] + 1$ .  $\square$

Now we prove (2). If  $\dim[\mathfrak{g}, \mathfrak{g}] = 1$ , then either (2) is satisfied or  $\mathfrak{g}$  is Abelian. But in the second case,  $(G, F)$  is flat.

Assume  $\dim[\mathfrak{g}, \mathfrak{g}] > 1$ . We assert that there does not exist a pair of nonzero vectors  $u \in [\mathfrak{g}, \mathfrak{g}]$  and  $u' \in \mathfrak{g} \setminus [\mathfrak{g}, \mathfrak{g}]$  such that  $\langle u, [u, u'] \rangle_u^F = 0$ . Otherwise, we have  $\langle u, [u, \mathfrak{g}] \rangle_u^F = 0$ , since  $[\mathfrak{g}, \mathfrak{g}] = 0$ . We can then find a vector  $v \in [\mathfrak{g}, \mathfrak{g}]$  such that  $u$  and  $v$  are linearly independent. Then by [Theorem 2.1](#),  $K^F(o, u, u \wedge v) \geq 0$ . This is a contradiction.

Consequently, we can find a vector  $u' \in \mathfrak{g} \setminus [\mathfrak{g}, \mathfrak{g}]$ , such that  $\langle u, [u', u] \rangle_u^F > 0$  for any nonzero  $u \in [\mathfrak{g}, \mathfrak{g}]$ . Let  $g_t$  be the one-parameter subgroup of  $G$  generated by  $u'$ . Then

$$\frac{d}{dt} F(\text{Ad}(g_t)u)|_{t=0} > 0, \quad (4.1)$$

for any nonzero  $u \in [\mathfrak{g}, \mathfrak{g}]$ . Now suppose that the restriction of  $\text{Ad}(u')$  to  $[\mathfrak{g}, \mathfrak{g}]$  has an eigenvalue whose real part is non-positive. If this eigenvalue is real, then  $\langle u, [u', u] \rangle_u^F \leq 0$  for the nonzero eigenvalue  $u$ . On the other hand, if this eigenvalue is not real, then we can find a linearly independent pair  $u_1$  and  $u_2$ , such that all  $\text{Ad}(g_t)$  with  $t > 0$  rotate the two-dimensional subspace  $\mathbf{V} = \mathbb{R}u_1 + \mathbb{R}u_2$ , or shrink it at the same time as they rotate  $\mathbf{V}$ . This is a contradiction with (4.1) restricted to  $\mathbf{V}$ . Thus all the eigenvalues of  $\text{Ad}(u')|_{[\mathfrak{g}, \mathfrak{g}]}$  have positive real parts, proving (2).  $\square$

## Acknowledgements

We are grateful to J.A. Wolf and L. Huang for helpful discussions.

## References

- [1] D. Alekseevsky, A. Arvanitoyeorgos, Riemannian flag manifolds with homogeneous geodesics, *Trans. Amer. Math. Soc.* 359 (2007) 3769–3789.
- [2] M. Berger, Les variétés riemanniennes homogènes normales simplement connexes à courbure strictement positive, *Ann. Sc. Norm. Super. Pisa, Cl. Sci.* (3) 15 (1961) 179–246.
- [3] S. Deng, *Homogeneous Finsler Spaces*, Springer, New York, 2012.
- [4] S. Deng, Z. Hou, Invariant Finsler metrics on homogeneous manifolds, *J. Phys. A, Math. Gen.* 37 (2004) 8245–8253.
- [5] S. Deng, Z. Hou, Homogeneous Finsler spaces of negative curvature, *J. Geom. Phys.* 57 (2007) 657–664.
- [6] S. Deng, M. Xu, Clifford–Wolf translations of Finsler spaces, *Forum Math.* 26 (2014) 1413–1428.
- [7] C.S. Gordon, Homogeneous Riemannian manifolds whose geodesics are orbits, in: *Topics in Geometry, in Memory of Joseph D’Atri*, Birkhäuser, 1996, pp. 155–174.
- [8] E. Heintze, On homogeneous manifolds of negative curvature, *Math. Ann.* 211 (1974) 23–34.
- [9] L. Huang, On the fundamental equations of homogeneous Finsler spaces, *Differ. Geom. Appl.* 40 (2015) 187–208.
- [10] K. Iwasawa, On some types of topological groups, *Ann. of Math.* 50 (1949) 507–558.
- [11] Z. Shen, *Lectures on Finsler Geometry*, World Scientific Publishing, 2001.
- [12] H. Wang, J. Li, Cartan’s fixed points theorem for Finsler spaces, preprint.
- [13] J.A. Wolf, Homogeneity and bounded isometries in manifolds of negative curvature, *Ill. J. Math.* 8 (1964) 14–18.
- [14] M. Xu, S. Deng, Normal homogeneous Finsler spaces, *Transform. Groups* (2017), <http://dx.doi.org/10.1007/s00031-017-9428-7> (published online).
- [15] M. Xu, S. Deng, L. Huang, Z. Hu, Even dimensional homogeneous Finsler spaces with positive flag curvature, *Indiana Univ. Math. J.* 66 (2017) 949–972.
- [16] Z. Yan, S. Deng, Finsler spaces whose geodesics are orbits, *Differ. Geom. Appl.* 36 (2014) 1–23.