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Density of disk algebra functions in de Branges–Rovnyak spaces

Densité des fonctions dans l'algèbre du disque dans les espaces de de Branges–Rovnyak

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A B S T R A C T

We prove that functions analytic in the unit disk and continuous up to the boundary are dense in the de Branges–Rovnyak spaces induced by the extreme points of the unit ball of H^∞. Together with previous theorems, it follows that this class of functions is dense in any de Branges–Rovnyak space.

R É S U M É

On démontre que les fonctions analytiques dans le disque unité et continues dans le disque fermé sont denses dans l'espace de Branges–Rovnyak généré par un point extrémal de la boule unité de H^∞. En utilisant aussi des théorèmes précédents, il résulte que cette classe de fonctions est dense dans un espace de Branges–Rovnyak quelconque.

1. Introduction

Let H^∞ be the algebra of bounded analytic functions in the unit disk $\mathbb{D}$ in the complex plane, and denote by $\mathcal{A}$ the disk algebra, i.e. the subalgebra of H^∞ consisting of functions that extend continuously to the closed disk. The Hardy space H^2 consists of power series in $\mathbb{D}$ with square-summable coefficients. If $\mathbb{T}$ denotes the unit circle, we identify as usual H^2 with the closed subspace of $L^2(\mathbb{T})$ consisting of functions whose negative Fourier coefficients vanish. The orthogonal projection from $L^2(\mathbb{T})$ onto H^2 is denoted by P_+.

For $\phi \in L^\infty(\mathbb{T})$, let T_ϕ denote the Toeplitz operator on H^2 defined by $T_\phi f = P_+ \phi f$. Given $b \in H^\infty$ with $\|b\|_\infty \le 1$, we define the corresponding *de Branges–Rovnyak space* $\mathcal{H}(b)$ as

$$\mathcal{H}(b) = (1 - T_b T_{\bar{b}})^{1/2} H^2.$$

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$\mathcal{H}(b)$ is endowed with the unique norm which makes the operator $(1-T_bT_{\overline{b}})^{1/2}$ a partial isometry from H^2 onto $\mathcal{H}(b)$. Alternatively, $\mathcal{H}(b)$ is defined as the reproducing kernel Hilbert space with kernel

$$k_b(z,\lambda) = \frac{1-\overline{b(\lambda)}b(z)}{1-\overline{\lambda}z}.$$

$\mathcal{H}(b)$-spaces are naturally split into two classes with fairly different structures according to whether the quantity $\int_{\mathbb{T}} \log(1-|b|)\,\mathrm{d}m$ is finite or not. Here m denotes the normalized arc-length measure on $\mathbb{T}$. The present note concerns the approximation of $\mathcal{H}(b)$-functions by functions in $\mathcal{A}\cap\mathcal{H}(b)$ and from the technical point of view there is a major difference between the two classes, which we shall briefly explain.

If $\int_{\mathbb{T}} \log(1-|b|)\,\mathrm{d}m < -\infty$, or equivalently, if b is a non-extreme point of the unit ball of H^∞, then $\mathcal{H}(b)$ contains all functions analytic in a neighborhood of the closed unit disk (see section (IV-6) of [9]). By a theorem of Sarason, the polynomials form a norm-dense subset of the space (see section (IV-3) of [9]). An interesting feature of the proofs of density of polynomials in an $\mathcal{H}(b)$-space is that the usual approach of approximating a function f first by its dilations $f_r(z) = f(rz)$, and then by their truncated Taylor series, or by their Cesàro means, does not work. Sarason's initial proof of density of polynomials is based on a duality argument. In recent years, a more involved constructive polynomial approximation scheme has been obtained in [7].

The picture changes dramatically in the case when $\int_{\mathbb{T}} \log(1-|b|)\,\mathrm{d}m = -\infty$, or equivalently when b is an extreme point of the unit ball of H^∞. Then it is in general a difficult task to identify any functions in the space other than the reproducing kernels, and it might happen that $\mathcal{H}(b)$ contains no non-zero function analytic in a neighborhood of the closed disk. A special class of extreme points are the inner functions. If b is inner, then $\mathcal{H}(b) = H^2 \ominus bH^2$ with equality of norms, and it is a consequence of a celebrated theorem of Aleksandrov [1] that in this case the intersection $\mathcal{A}\cap\mathcal{H}(b)$ is dense in the space. The result is surprising since, as pointed out above, in most cases it is not obvious at all that $\mathcal{H}(b)$ contains any non-zero function in the disk algebra $\mathcal{A}$.

Motivated by the situation described here, E. Fricain [5] raised the natural question of whether Aleksandrov's result extends to all other $\mathcal{H}(b)$-spaces induced by extreme points b of the unit ball of H^∞. It is the purpose of this note to provide an affirmative answer to this question, contained in the main result below.

Theorem 1. *If b is an extreme point of the unit ball of H^∞, then $\mathcal{A}\cap\mathcal{H}(b)$ is a dense subset of $\mathcal{H}(b)$.*

Together with Sarason's result [9] on the density of polynomials in the non-extreme case, it follows that the intersection $\mathcal{A}\cap\mathcal{H}(b)$ is dense in the space $\mathcal{H}(b)$ for any b in the unit ball of H^∞. Our proof of Theorem 1 is deferred to Section 3 and relies on a duality argument. Therefore, just as the earlier proofs of Sarason and Aleksandrov, our approach is non-constructive. Section 2 serves to establish some preliminary results.

2. Preliminaries

2.1. The norm on $\mathcal{H}(b)$

An essential step is the following useful representation of the norm in $\mathcal{H}(b)$. The authors have originally deduced the result using the techniques in [3] (see also [2, Chapter 3]), but once the goal is identified, several available techniques provide simpler proofs. For example, the proposition below can be deduced from results in [9]. For the sake of completeness, we include a new shorter proof.

Proposition 2. *Let b be an extreme point of the unit ball of H^∞ and let*

$$E = \{\zeta \in \mathbb{T} : |b(\zeta)| < 1\}.$$

Then for $f \in \mathcal{H}(b)$ the equation

$$P_+\overline{b}f = -P_+\sqrt{1-|b|^2}g$$

has a unique solution $g \in L^2(E)$, and the map $J : \mathcal{H}(b) \to H^2 \oplus L^2(E)$ defined by

$$Jf = (f, g),$$

is an isometry. Moreover,

$$J(\mathcal{H}(b))^\perp = \left\{(bh, \sqrt{1-|b|^2}h) : h \in H^2\right\}.$$

Proof. Let

$$K = \left\{(bh, \sqrt{1-|b|^2}h) : h \in H^2\right\} \subset H^2 \oplus L^2(E)$$

and let P_1 be the projection from $H^2 \oplus L^2(E)$ onto the first coordinate H^2, i.e. $P_1(f,g) = f$. We observe first that $P_1|K^\perp$ is injective. Indeed, if $K^\perp$ contains a tuple of the form $(0,g) \in H^2 \oplus L^2(E)$, it follows that

$$\int_{\mathbb{T}} \overline{\zeta}^n g(\zeta)\sqrt{1-|b(\zeta)|^2}\,\mathrm{d}m(\zeta) = 0,\ n \geq 0,$$

and consequently the function $g\sqrt{1-|b|^2}$ coincides a.e. with the boundary values of the complex conjugate of a function $f \in H^2_0$. But the assumption that b is an extreme point then implies that $\int_{\mathbb{T}} \log|f|\,\mathrm{d}m = -\infty$, and since $f \in H^2$, we conclude that $f = 0$, i.e. $g = 0$. Thus, the space $\mathcal{H} = P_1 K^\perp$ with the norm $\|f\|_{\mathcal{H}} = \|P_1^{-1} f\|_{H^2 \oplus L^2(E)}$ is a Hilbert space of analytic functions on $\mathbb{D}$, contractively contained in H^2; in particular, it is a reproducing kernel Hilbert space. We now show that $\mathcal{H}$ equals $\mathcal{H}(b)$ by verifying that the reproducing kernels of the two spaces coincide. This follows from a simple computation. For $\lambda \in \mathbb{D}$, the tuple

$$\begin{aligned}(f_\lambda, g_\lambda) &= \Big(\frac{1-\overline{b(\lambda)}b(z)}{1-\overline{\lambda}z}, -\frac{\overline{b(\lambda)}\sqrt{1-|b(z)|^2}}{1-\overline{\lambda}z}\Big) \\ &= \Big(\frac{1}{1-\overline{\lambda}z}, 0\Big) - \Big(\frac{\overline{b(\lambda)}b(z)}{1-\overline{\lambda}z}, \frac{\overline{b(\lambda)}\sqrt{1-|b(z)|^2}}{1-\overline{\lambda}z}\Big)\end{aligned}$$

is obviously orthogonal to K, while the last tuple on the right hand side is in K, so that f_λ is the reproducing kernel in $\mathcal{H}$, which obviously equals the reproducing kernel in $\mathcal{H}(b)$. The first assertion in the statement is now self-explanatory. □

2.2. Cauchy transforms and two classical theorems

The dual $\mathcal{A}'$ of the disk algebra $\mathcal{A}$ can be identified with the space $\mathcal{C}$ of Cauchy transforms of finite Borel measures on $\mathbb{T}$. The Cauchy transform $C\mu$ of a measure μ is given by

$$C\mu(z) = \int_{\mathbb{T}} \frac{1}{1-z\overline{\zeta}}\mathrm{d}\mu(\zeta),$$

and the duality between $\mathcal{A}$ and $\mathcal{C}$ is given by the pairing

$$\langle f, C\mu\rangle = \lim_{r\to 1^-} \int_{\mathbb{T}} f(\zeta)\overline{C\mu(r\zeta)}\mathrm{d}m(\zeta) = \int_{\mathbb{T}} f\,\mathrm{d}\overline{\mu}.$$

A proof of this fact can be found, for example, in Section 4.2 of [6]. The space $\mathcal{C}$ is endowed with the obvious quotient norm and is continuously contained in all H^p spaces for $0 < p < 1$.

Recall that analytic functions f in $\mathbb{D}$ satisfy $\sup_{0<r<1} \int_{\mathbb{T}} \log^+|f_r|\,\mathrm{d}m < \infty$ if and only if they are quotients of H^∞-functions, in particular they have finite nontangential limits a.e. on $\mathbb{T}$ which define a boundary function denoted also by f. The class $N^+(\mathbb{D})$ consists of quotients of H^∞-functions such that the denominator can be chosen to be outer. It contains in particular all Hardy spaces $H^p,\ p > 0$.

Two classical theorems will play an important role in the proof of the Theorem 1. The first is the following theorem of Vinogradov, which also plays a crucial role in the proof of Aleksandrov's result. A proof of the below theorem the can be found in [10].

Theorem 3. *Let $f \in \mathcal{C}$. If I is an inner function such that $f/I \in N^+(\mathbb{D})$, then $f/I \in \mathcal{C}$ and $\|f/I\| \leq \|f\|$.*

The second is the Khintchin–Ostrowski theorem, and it reads as follows. A proof can be found in Section 3.2 of [8].

Theorem 4. *Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions analytic in the unit disk satisfying the following conditions:*

(i) There exists a constant $C > 0$ such that, for all $n \geq 1$, we have

$$\sup_{0<r<1} \int_{\mathbb{T}} \log^+(|f_n(r\zeta)|)\,\mathrm{d}m(\zeta) \leq C.$$

(ii) On some set $E \subseteq \mathbb{T}$ of positive Lebesgue measure, the sequence f_n converges in measure to a function ϕ.

Then the sequence f_n converges uniformly on compact subsets of the unit disk to a function $f \in N^+(\mathbb{D})$, and moreover $f = \phi$ a.e. on E.

3. Proof of the main result

Due to Proposition 2, we can now implement Aleksandrov's strategy from [1], which will then be combined with Theorem 3 and Theorem 4. The following result extends Aleksandrov's approach to the context of $\mathcal{H}(b)$-spaces, when b is extremal in the unit ball of H^∞.

Lemma 5. *Let $b : \mathbb{D} \to \mathbb{D}$ be analytic, $E = \{\zeta \in \mathbb{T} : |b(\zeta)| < 1\}$, $B = \mathcal{A} \oplus L^2(E)$ and $B' = \mathcal{C} \oplus L^2(E)$. Then the set*

$$S = \{(C\mu, h) : C\mu/b \in N^+(\mathbb{D}), C\mu/b = h/\sqrt{1-|b|^2} \text{ a.e. on } E\}$$

is weak-$*$ closed in B'.

Proof. Since $\mathcal{A} \oplus L^2(E)$ is separable and S is a linear subspace, it will be sufficient to show that S is weak-$*$ sequentially closed (see Theorem 5, p. 76 of [4]). Let $(C\mu_n, h_n)$ converge weak-$*$ to $(C\mu, h)$, where $(C\mu_n, h_n) \in S$ for $n \geq 1$. Equivalently, $h_n \to h$ weakly in $L^2(E)$, and

$$\sup_n \|C\mu_n\| < \infty. \quad \lim_{n\to\infty} C\mu_n(z) = C\mu(z), \; z \in \mathbb{D}.$$

Now by passing to a subsequence and the Cesàro means of that subsequence we can assume that $h_n \to h$ in the L^2-norm. Finally, using another subsequence we may also assume that $h_n \to h$ pointwise a.e. on E. Let I_b be the inner factor of b. Since $C\mu_n/I_b \in N^+(\mathbb{D})$, it follows by Theorem 3 that $\{C\mu_n/I_b\}_{n=1}^\infty$ is a bounded sequence in $\mathcal{C}$ converging pointwise on $\mathbb{D}$ to $C\mu/I_b$. This implies weak-$*$ convergence in $\mathcal{C}$, in particular, $C\mu/I_b \in \mathcal{C} \subset N^+(\mathbb{D})$, and consequently, $C\mu/b \in N^+(\mathbb{D})$. Moreover, we have a.e. on E that $C\mu_n/b = h_n/\sqrt{1-|b|^2}$ which converges pointwise to $h/\sqrt{1-|b|^2}$, hence we conclude that the sequence $C\mu_n$ converges in measure to some function ϕ on E. Fix any $p \in (0,1)$. Then

$$\int_{\mathbb{T}} \log^+(|C\mu_n(r\zeta)|)\,\mathrm{d}m(\zeta) \lesssim \int_{\mathbb{T}} |C\mu_n(r\zeta)|^p\,\mathrm{d}m(\zeta) \lesssim \sup_n \|C\mu_n\|^p < \infty.$$

Thus the assumptions of Theorem 4 are satisfied, and so (a subsequence of) $C\mu_n$ converges a.e. on E to $C\mu$. This clearly implies that $C\mu/b = h/\sqrt{1-|b|^2}$ a.e. on E, i.e. $(C\mu, h) \in S$. □

We are now ready to complete the proof of the main theorem.

Proof of Theorem 1. Let J denote the embedding in Proposition 2. Based on the pairing described at the beginning of Section 2.2, a direct application of Proposition 2 gives

$$J(\mathcal{A} \cap \mathcal{H}(b)) = \cap_{h\in H^2} \ker l_h,$$

where the functionals l_h are identified with elements of $\mathcal{C} \oplus L^2(E)$ as

$$l_h = \left(hb, h\sqrt{1-|b|^2}\right).$$

It is a consequence of the Hahn–Banach theorem that the annihilator $J(\mathcal{A} \cap \mathcal{H}(b))^\perp$ is the weak-$*$ closure of the set of the functionals l_h. Since for all $h \in H^2$ we have $l_h \in S$, the set considered in Lemma 5, by the lemma we conclude that $J(\mathcal{A} \cap \mathcal{H}(b))^\perp \subset S$. Thus, if $f \in \mathcal{H}(b)$ is orthogonal to $\mathcal{A} \cap \mathcal{H}(b)$, we must have $Jf \in S$, that is

$$Jf = (hb, h\sqrt{1-|b|^2})$$

for some $h \in N^+(\mathbb{D})$. The boundary values of h satisfy

$$\int_{\mathbb{T}} |h|^2\,\mathrm{d}m = \int_{\mathbb{T}} |bh|^2\,\mathrm{d}m + \int_{\mathbb{T}} (1-|b|^2)|h|^2\,\mathrm{d}m = \|f\|^2$$

and hence by the Smirnov maximum principle we have $h \in H^2$. But then by Proposition 2 $Jf \in J(\mathcal{H}(b))^\perp$, which gives $Jf = 0$ and the proof is complete. □

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