C. R. Acad. Sci. Paris, Ser. I 355 (2017) 876–880

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Complex analysis

A note on the coefficient estimates of bi-close-to-convex functions

Une note sur les estimations des coefficients des fonctions bi-presque convexes

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ARTICLE INFO

Article history:
Received 6 June 2017
Accepted 24 July 2017
Available online 4 August 2017

Presented by the Editorial Board

ABSTRACT

In a recent paper, Hamidi and Jahangiri [C. R. Acad. Sci. Paris, Ser. I 352 (2014) 17–20] introduced and investigated the class of bi-close-to-convex functions, and determined the estimates for the general Taylor–Maclaurin coefficients of the functions therein. This note mainly aims to point out and correct the errors of the main result in the above-mentioned paper.

RÉSUMÉ

Hamidi et Jahangiri [C. R. Acad. Sci. Paris, Ser. I 352 (2014) 17–20] ont introduit et étudié la classe des fonctions bi-presque convexes. Ils majorent les coefficients de Taylor–MacLaurin de ces fonctions bi-presque convexes. Toutefois, la note citée ci-dessus contient des erreurs, que nous mettons en évidence et corrigeons ici.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* in the *open* unit disk

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http://dx.doi.org/10.1016/j.crma.2017.07.014

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

We also denote $\mathcal{S}$ by the subclass of $\mathcal{A}$ whose members are univalent in $\mathbb{U}$.

A function $f \in \mathcal{S}$ is said to be starlike of order α $(0 \le \alpha < 1)$ if it satisfies the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U});$$

we denote this class by $\mathcal{S}^*(\alpha)$, and for simplicity, $\mathcal{S}^*(0) =: \mathcal{S}^*$.

Also, a function $f \in \mathcal{S}$ is said to be close-to-convex of order α $(0 \le \alpha < 1)$ if there exists a function $g \in \mathcal{S}^*$ such that

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha \quad (z \in \mathbb{U}),$$

this class is denoted by $\mathcal{C}(\alpha)$, and for convenience, $\mathcal{C}(0) =: \mathcal{C}$.

We observe that $\mathcal{S}^*(\alpha) \subset \mathcal{C}(\alpha) \subset \mathcal{S}$ (see [3]). Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. Indeed, the Koebe one-quarter theorem ensures that the image of $\mathbb{U}$ under every univalent function f contains a disk with radius $1/4$. Thus, every function $f \in \mathcal{A}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f);\ r_0(f) \ge \frac{1}{4}\right).$$

The inverse function $F = f^{-1}$ is given by

$$\begin{aligned} F(w) &= f^{-1}(w) \\ &= w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \\ &= w + \sum_{n=2}^{\infty} A_n w^n. \end{aligned} \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$, in the sense that f^{-1} has a univalent analytic continuation to $\mathbb{U}$.

In a recent paper, Hamidi and Jahangiri [4] introduced and investigated the class of bi-close-to-convex functions as follows.

Definition 1. A function $f \in \mathcal{A}$ is said to be bi-close-to-convex of order α $(0 \le \alpha < 1)$ if both f and its inverse map $F = f^{-1}$ are close-to-convex of order α in $\mathbb{U}$.

They determined the estimates for the general Taylor–Maclaurin coefficients of functions in the class. The main purpose of this note is to point out and correct the errors in the above-mentioned paper.

2. Main result

Making use of the Faber polynomial expansion of function $f \in \mathcal{A}$ with the form (1.1), the coefficients of its inverse map $F = f^{-1}$ may be expressed as follows (see [1,2]):

$$F(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots, a_n) w^n.$$

In particular, the first three terms of K_{n-1}^{-n} are

$$K_1^{-2} = -2a_2, \quad K_2^{-3} = 3(2a_2^2 - a_3) \text{ and } K_3^{-4} = -4(5a_2^3 - 5a_2a_3 + a_4).$$

In general, for any $p \in \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$, an expansion of K_{n-1}^{p} is given by (see [1])

$$K_{n-1}^{p} = pa_n + \frac{p(p-1)}{2} D_{n-1}^{2} + \frac{p!}{(p-3)!\,3!} D_{n-1}^{3} + \cdots + \frac{p!}{(p-n+1)!\,(n-1)!} D_{n-1}^{n-1}, \tag{2.1}$$

where $D_{n-1}^{p} = D_{n-1}^{p}(a_2, a_3, \ldots, a_n)$. In view of [7], we see that

$$D_{n-1}^{m}(a_2,\ldots,a_n)=\sum\frac{m!}{j_1!\ldots j_{n-1}!}a_2^{j_1}\ldots a_n^{j_{n-1}},$$

and the sum is taken over all non-negative integers $j_1,\ldots,j_{n-1}$ satisfying

$$\begin{cases} j_1+j_2+\cdots+j_{n-1}=m,\\ j_1+2j_2+\cdots+(n-1)j_{n-1}=n-1.\end{cases}$$

It is clear that $D_{n-1}^{n-1}(a_2,\ldots,a_n)=a_2^{n-1}$.

Recently, Hamidi and Jahangiri [4] proved their main result by making use of the assertion: if an analytic function f of the form (1.1) is close-to-convex of order α with $a_k=0$ $(2\le k\le n-1)$, then the coefficients $b_k=0$ $(2\le k\le n-1)$ for the function

$$g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in\mathcal{S}^*,$$

where g satisfies the condition

$$\Re\left(\frac{zf'(z)}{g(z)}\right)>\alpha\quad(0\le\alpha<1).$$

But we can provide a simple counterexample to illuminate that the assertion above is wrong. For example, we choose the functions f and g as

$$f(z)=z\text{ and }g(z)=z-\frac{z^2}{2};$$

clearly, we see that $g\in\mathcal{S}^*$ and $f\in\mathcal{C}$, $a_2=0$ but $b_2=-1/2\neq0$.

Let us recall the following result involving coefficient estimates of the class $\mathcal{S}^*(\alpha)$ due to Robertson [5], which will be used in the proof of our main result.

Lemma 1. *If $f\in\mathcal{S}^*(\alpha)$, then*

$$|a_n|\le\frac{\prod_{j=0}^{n-2}[j+2(1-\alpha)]}{(n-1)!}\qquad\left(n\in\mathbb{N}^*:=\mathbb{N}\backslash\{1\}=\{2,3,\ldots\}\right).$$

Now, we will give the following result.

Theorem 1. *Let the function f and its inverse $F=f^{-1}$ satisfy the conditions*

$$\Re\left(\frac{zf'(z)}{g(z)}\right)>\alpha\text{ and }\Re\left(\frac{wF'(w)}{G(w)}\right)>\alpha\quad(z,w\in\mathbb{U};\ 0\le\alpha<1),\tag{2.2}$$

where f is given by (1.1), $F=f^{-1}$ is defined by (1.2),

$$g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in\mathcal{S}^*\text{ and }G(w)=w+\sum_{n=2}^{\infty}B_nw^n\in\mathcal{S}^*,\tag{2.3}$$

that is, the function $f\in\mathcal{S}$ is bi-close-to-convex of order α in $\mathbb{U}$. If $a_k=0$ $(2\le k\le n-1)$, then for $n\ge3$, we have

$$|a_n|\le\min\left\{1+\frac{2(1-\alpha)+\sum_{l=1}^{n-2}|b_{n-l}K_l^{-1}(b_2,\ldots,b_{l+1})|}{n},\right.$$
$$\left.1+\frac{2(1-\alpha)+\sum_{l=1}^{n-2}|B_{n-l}K_l^{-1}(B_2,\ldots,B_{l+1})|}{n}\right\}.$$

Proof. For $0\le\alpha<1$, let the function $f\in\mathcal{S}$ given by (1.1) be bi-close-to-convex of order α in $\mathbb{U}$. By the hypothesis, the functions f and $F=f^{-1}$ given by (1.2) are close-to-convex of order α in $\mathbb{U}$. Therefore, there exist two functions g and G defined by (2.3) satisfying (2.2). The Faber polynomial expansions for $zf'(z)/g(z)$ and for $wF'(w)/G(w)$ are given by

$$\frac{zf'(z)}{g(z)}=1+\sum_{n=2}^{\infty}\left((na_n-b_n)+\sum_{l=1}^{n-2}K_l^{-1}(b_2,\ldots,b_{l+1})\big[(n-l)a_{n-l}-b_{n-l}\big]\right)z^{n-1},\tag{2.4}$$

and

$$\frac{wF'(w)}{G(w)} = 1 + \sum_{n=2}^{\infty}\left((nA_n - B_n) + \sum_{l=1}^{n-2} K_l^{-1}(B_2, \dots, B_{l+1})\big[(n-l)A_{n-l} - B_{n-l}\big]\right)w^{n-1}, \tag{2.5}$$

respectively. On the other hand, by (2.2), we see that there exist two positive real-part functions

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \text{ and } q(w) = 1 + \sum_{n=1}^{\infty} d_n w^n$$

in $\mathbb{U}$ such that

$$\frac{zf'(z)}{g(z)} = \alpha + (1-\alpha)p(z) = 1 + (1-\alpha)\sum_{n=1}^{\infty} c_n z^n, \tag{2.6}$$

and

$$\frac{wF'(w)}{G(w)} = \alpha + (1-\alpha)q(w) = 1 + (1-\alpha)\sum_{n=1}^{\infty} d_n w^n. \tag{2.7}$$

By the Carathéodory lemma (see [3]), we know that $|c_n| \le 2$ and $|d_n| \le 2$ $(n \in \mathbb{N})$. Comparing the corresponding coefficients of (2.4) and (2.6), for any $n \ge 2$, yields

$$(na_n - b_n) + \sum_{l=1}^{n-2} K_l^{-1}(b_2, \dots, b_{l+1})\big[(n-l)a_{n-l} - b_{n-l}\big] = (1-\alpha)c_{n-1}. \tag{2.8}$$

Similarly, it follows from (2.5) and (2.7) that

$$(nA_n - B_n) + \sum_{l=1}^{n-2} K_l^{-1}(B_2, \dots, B_{l+1})\big[(n-l)A_{n-l} - B_{n-l}\big] = (1-\alpha)d_{n-1}. \tag{2.9}$$

By the hypothesis $a_k = 0$ $(2 \le k \le n-1)$, we find from (2.8) and (2.9) that

$$(na_n - b_n) - \sum_{l=1}^{n-2} b_{n-l}K_l^{-1}(b_2, \dots, b_{l+1}) = (1-\alpha)c_{n-1}, \tag{2.10}$$

and

$$(nA_n - B_n) - \sum_{l=1}^{n-2} B_{n-l}K_l^{-1}(B_2, \dots, B_{l+1}) = (1-\alpha)d_{n-1}. \tag{2.11}$$

Since $a_k = 0$ $(2 \le k \le n-1)$, we find that $A_n = -a_n$. It follows from Lemma 1, (2.10) and (2.11) that the assertion of Theorem 1 holds. □

By setting $b_k = B_k = 0$ $(2 \le k \le n-1)$ in Theorem 1, we easily get the following result. It corrects the errors of [4, Theorem 2.1]. More precisely, Theorem 2.1 in [4] holds only with the additional conditions $b_k = B_k = 0$ $(2 \le k \le n-1)$.

Corollary 1. *For $0 \le \alpha < 1$, let the function f and its inverse $F = f^{-1}$ satisfy the conditions*

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > \alpha \text{ and } \Re\left(\frac{wF'(w)}{G(w)}\right) > \alpha,$$

where f is given by (1.1), $F = f^{-1}$ is defined by (1.2),

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^* \text{ and } G(w) = w + \sum_{n=2}^{\infty} B_n w^n \in \mathcal{S}^*,$$

that is, the function $f \in \mathcal{S}$ is bi-close-to-convex of order α in $\mathbb{U}$. If $a_k = b_k = B_k = 0$ $(2 \le k \le n-1)$, then

$$|a_n| \le 1 + \frac{2(1-\alpha)}{n} \quad (n \ge 3).$$

Remark 1. By the observation of Theorem 1 and Corollary 1, [6, Theorem 2.1] should also be corrected using our method. We here choose to omit the analogous details.

Acknowledgements

The first author was supported by the *Natural Science Foundation of Hunan Province* under Grant no. 2016JJ2036 of the People's Republic of China. The authors would like to thank the editors for their kind help.

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