



Mathematical analysis/Dynamical systems

Fully oscillating sequences and weighted multiple ergodic limit



Suites pleinement oscillantes et limite des moyennes multi-ergodiques pondérées

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ABSTRACT

We prove that fully oscillating sequences are orthogonal to multiple ergodic realizations of affine maps of zero entropy on compact Abelian groups. It is more than what Sarnak's conjecture requires for these dynamical systems.

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R É S U M É

Nous montrons que les suites pleinement oscillantes sont orthogonales aux réalisations d'une application affine d'entropie nulle sur un groupe abélien compact. Ceci est plus que ce que demande la conjecture de Sarnak à ces systèmes dynamiques.

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1. Introduction and results

A sequence of complex numbers $c = (c_n)_{n \geq 0}$ is said to be *oscillating of order d* ($d \geq 1$) if for any real polynomial $P \in \mathbb{R}_d[x]$ of degree less than or equal to d , we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} c_n e^{2\pi i P(n)} = 0. \quad (1)$$

It is said to be *fully oscillating* if it is oscillating of all orders. This notion of oscillation of higher order was introduced in [7]. The oscillation of order 1 was earlier considered in [8] in order to formulate some results related to Sarnak's conjecture. See [19], [20] for Sarnak's conjecture. See [1], [2], [6], [9], [14], [17], [15], [23] for some related recent works. The Möbius sequence $(\mu(n))$ is a typical example of fully oscillating sequences ([5], [13]). Recall that $\mu(1) = 1$, $\mu(n) = (-1)^k$ if n is

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square free and has k distinct prime factors, and $\mu(n) = 0$ for other integers n . The random subnormal sequence is almost surely fully oscillating ([7]) and the sequence $(\beta^n \bmod 1)$ is fully oscillating for almost all $\beta > 1$ ([3]). The oscillating sequences of orders d are characterized by their orthogonality to different classes of dynamical systems ([21]).

Sarnak’s conjecture states that for any topological dynamical system (X, T) of zero entropy, for any continuous function $f \in C(X)$ and any point $x \in X$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \mu(n) f(T^n x) = 0.$$

This conjecture remains open in its generality. The above equality is referred to as the orthogonality of Möbius sequence to the realization $f(T^n x)$ of the system (X, T) , or as the disjointness of $(\mu(n))$ to the system (X, T) . For $\ell (\geq 1)$ functions $f_1, \dots, f_\ell \in C(X)$, the sequence $f_1(T^n x) f_2(T^{2n} x) \dots f_\ell(T^{\ell n} x)$ could be referred to as a multiple ergodic realization.

Following Liu and Sarnak [18], we can prove the following orthogonality of fully oscillating sequences to the multiple ergodic realizations of affine linear maps on a compact Abelian group that are of zero entropy.

Theorem 1. *Let $\ell \geq 1$ be an integer. Let G be a compact Abelian group. Assume that*

- (i) $T : G \rightarrow G$ is an affine linear map of zero entropy;
- (ii) (c_n) is a fully oscillating sequence;
- (iii) $q_1, \dots, q_\ell \in \mathbb{Z}[x]$ are ℓ polynomials such that $q_j(\mathbb{N}) \subset \mathbb{N}$ for all j .

Then, for any continuous function $F \in C(G^\ell)$ and any point $x \in G$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} c_n F(T^{q_1(n)} x, \dots, T^{q_\ell(n)} x) = 0. \tag{2}$$

Recall that an affine map on a compact Abelian group G is defined by

$$Tx = Ax + b$$

where $A : G \rightarrow G$ is an automorphism of G and $b \in G$.

Theorem 1 generalizes the following theorem due to Liu and Sarnak, which holds for the Möbius sequence to fully oscillating sequences.

Theorem 2 (Liu and Sarnak [18]). *The Möbius sequence $(\mu(n))$ is linearly disjoint from any affine linear map on a compact Abelian group that is of zero entropy.*

The result of **Theorem 1** was proved in [7], based on [10], [11], [12], for the class of topological systems of quasi-discrete spectrum in the sense of Hahn–Parry [10], including minimal affine linear maps on a connected compact Abelian group. The proof of **Theorem 1** in this note will be based on ideas of Liu and Sarnak and on the fact that arithmetic subsequences of oscillating sequences are oscillating. One of the ideas of Liu and Sarnak is stated as follows. It is drawn from the proof of their first theorem in [18].

Theorem 3 (Liu–Sarnak [18]). *Let $Tx = Ax + b$ be an affine linear map of zero entropy on $X := \mathbb{T}^d \times F$ where $d \geq 1$ and F is a finite Abelian group. Consider the automorphism W on the product group $X \times X$ defined by $W(x_1, x_2) := (Ax_1 + x_2, x_2)$. Then*

- (i) $(T^n x, b) = W^n(x, b)$ for all $x \in X$;
- (ii) there exist integers $\nu \geq 1$ and $\kappa \geq 0$ such that $W^\nu = I + N$ where N is nilpotent in the sense $N^{\kappa+1} = 0$.

The following fact, which has its own interest, will also be useful in the proof of **Theorem 1**.

Theorem 4. *Let $a \geq 2$ be an integer. A sequence (w_n) is oscillating of order d if and only if the arithmetic subsequence (w_{an+b}) is oscillating of order d for any integer $b \geq 0$.*

Before proving **Theorem 1**, we prove **Theorem 4**.

2. Proof of Theorem 4

If (w_{an+b}) are oscillating of order d for $0 \leq b \leq a - 1$, it is obvious that (w_n) is oscillating of order d .

Now assume that (w_n) is oscillating of order d . First observe that from the definition, it is clear that any shifted sequence (w_{n+b}) ($b \geq 0$) is oscillating of order d . So, it suffices to prove that (w_{an}) is oscillating of order d . Since a can be decomposed into a product of primes, we have only to prove that (w_{pn}) is oscillating of order d for any prime $p \geq 2$.

Let $P \in \mathbb{R}_d[t]$. For any $N \geq p$, denote

$$S_N = \sum_{0 \leq n < N} w_n e^{2\pi i P(n)},$$

$$S_{N,j} = \sum_{m: 0 \leq pm+j < N} w_{pm+j} e^{2\pi i P(pm+j)} \quad (0 \leq j < p).$$

We have the trivial decomposition

$$S_N = S_{N,0} + S_{N,1} + \dots + S_{N,p-1}. \tag{3}$$

For any integer $0 \leq u \leq p - 1$, denote

$$S_N^u = \sum_{0 \leq n < N} w_n e^{2\pi i (P(n) + \frac{un}{p})}$$

where $x \mapsto P(x) + ux/p$ is a real polynomial. Write $\omega = e^{2\pi i/p}$. We have the following decomposition

$$S_N^u = S_{N,0} + \omega^u S_{N,1} + \omega^{2u} S_{N,2} \dots + \omega^{(p-1)u} S_{N,p-1}, \tag{4}$$

which is similar to (3) and which contains (3) as a particular case corresponding to $u = 0$. Taking sum over u , we get

$$\sum_{u=0}^{p-1} S_N^u = p S_{N,0} + S_{N,1} \sum_{u=0}^{p-1} \omega^u + S_{N,2} \sum_{u=0}^{p-1} \omega^{2u} + \dots + S_{N,p-1} \sum_{u=0}^{p-1} \omega^{(p-1)u}.$$

Since p is prime, any j with $1 \leq j \leq p - 1$ is invertible in the ring $\mathbb{Z}/p\mathbb{Z}$ so that $\sum_{u=0}^{p-1} \omega^{ju} = 0$ for all $1 \leq j \leq p - 1$. Thus

$$S_{N,0} = \frac{1}{p} \sum_{u=1}^{p-1} S_N^u.$$

Since the sequence $\{w_n\}$ is oscillating of order d , we have $S_N^u = o(N)$ for all u as $N \rightarrow \infty$, so that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{m=0}^{[N/p]} w_{pm} e^{2\pi i P(pm)} = \lim_{N \rightarrow \infty} \frac{S_{N,0}}{N} = 0.$$

This implies that $m \mapsto w_{pm}$ is oscillating of order d .

3. Proof of Theorem 1

Let \widehat{G} be the dual group of G . We have $\widehat{G^\ell} = \widehat{G}^\ell$. Any continuous function $F \in C(G^\ell)$ can be uniformly approximated by trigonometric polynomials on G^ℓ , which are finite linear combinations of functions of the form $\phi_1(x_1) \dots \phi_\ell(x_\ell)$ where $\phi_1, \dots, \phi_\ell \in \widehat{G}$. So, it suffices to prove that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} c_n \phi_1(T^{q_1(n)}x) \dots \phi_\ell(T^{q_\ell(n)}x) = 0 \tag{5}$$

holds for all $\phi_1, \dots, \phi_\ell \in \widehat{G}$ and all $x \in G$ (see [7] for details).

Now, we mimic Liu and Sarnak [18]. First remark that the problem can be reduced to a torus. In fact, let $\Phi = \{\phi_1, \dots, \phi_\ell\} \subset \widehat{G}$. Recall that the action of A on \widehat{G} is defined by $(A\gamma)(x) = \gamma(Ax)$ for $\gamma \in \widehat{G}$ and $x \in G$. Let $\langle \Phi \rangle$ be the smallest A -invariant closed subgroup of \widehat{G} which contains Φ and

$$\langle \Phi \rangle^\perp := \{x \in G : \gamma(x) = 1 \ \forall \gamma \in \langle \Phi \rangle\}$$

be the annihilator of $\langle \Phi \rangle$, a closed subgroup of G . Let

$$G_\Phi := G / \langle \Phi \rangle^\perp$$

be the quotient group. The A -invariance of $\langle \Phi \rangle$ implies that

$$\forall x \in G, \forall y \in \langle \Phi \rangle^\perp, T(x + y) = Tx \pmod{\langle \Phi \rangle^\perp}.$$

Thus T induces an affine map, which will be denoted by T_Φ , on the quotient group G_Φ . Being a factor of (G, T) , the system (G_Φ, T_Φ) has zero entropy. By Aoki’s Theorem ([4], a statement in the proof on p. 13), $\langle \Phi \rangle$ is finitely generated. Remark that

$\widehat{G}_\phi = \langle \Phi \rangle$. Then G_ϕ , as dual group of $\langle \Phi \rangle$, is isomorphic to $\mathbb{T}^d \times F$ for some $d \geq 1$ and some finite Abelian group F . On the other hand, for any $\phi \in \Phi$ and any $x \in G$, we have

$$\phi(T^n x) = \widetilde{\phi}(T_\phi^n \widetilde{x})$$

where $\widetilde{x} := x + \langle \Phi \rangle^\perp$ is the projection of x onto G_ϕ and $\widetilde{\phi}$ is the character in \widetilde{G}_ϕ induced by ϕ . Thus the proof of (5) is reduced to the dynamics $(\mathbb{T}^d \times F, T_\phi)$.

By Theorem 3, it suffices to treat the automorphism W appearing in Theorem 3. Recall that $W^\nu = 1 + N$ with $N^{\kappa+1} = 0$. For any integer $n \geq 1$, write $n = m\nu + r$ with $m \geq 0$ and $0 \leq r \leq \nu - 1$. The following expression was obtained in [18]

$$W^{m\nu+r} x = \sum_{j=0}^{\min(m,\kappa)} \binom{m}{j} N^j y_{r,j} = \sum_{j=0}^{\kappa} \binom{m}{j} N^j y_{r,j} \tag{6}$$

when $m \geq \kappa$, where $y_{r,j} = W^r N^j x$. Notice that these $y_{r,j}$ with $0 \leq r \leq \nu - 1$ and $0 \leq j \leq \kappa$ are independent of m . For any polynomial q (a typical polynomial of q_1, \dots, q_ℓ), we have $q(m\nu + r) = q'(m)\nu + r'$ where $r' = q(r) \pmod{\nu}$ ($0 \leq r' \leq \nu - 1$) is independent of m too, and $q' \in \mathbb{Z}[z]$ is a polynomial having the same degree as q . It follows from (6) that

$$W^{q(m\nu+r)} x = W^{q'(m)\nu+r'} x = \sum_{j=0}^{\kappa} \binom{q'(m)}{j} N^j y_{r',j}. \tag{7}$$

This holds for m sufficiently large so that $q'(m) \geq \kappa$. Apply (7) to each $q := q_s$ ($1 \leq s \leq \ell$). Then

$$\phi_s(W^{q_s(m\nu+r)}) = \prod_{j=0}^{\kappa} \phi_s(N^j y_{r'_s,j})^{q'_s(m)}$$

Let $t_{s,r,j}$ be the argument of the complex number $\phi_s(N^j y_{r'_s,j})$. Then we get

$$\prod_{s=1}^{\ell} \phi_s(W^{q_s(m\nu+r)}) = e^{2\pi i P(m)}$$

where $P \in \mathbb{R}[x]$ is the real polynomial

$$P(x) = \sum_{s=1}^{\ell} \sum_{j=0}^{\kappa} t_{s,r,j} \binom{q'_s(m)}{j}.$$

By Theorem 4, for any $0 \leq r < \nu$, we get

$$\lim_{M \rightarrow \infty} \frac{1}{M} \sum_{m=1}^M w_{m\nu+r} \prod_{s=1}^{\ell} \phi_s(W^{q_s(m\nu+r)} x) = \lim_{M \rightarrow \infty} \frac{1}{M} \sum_{m=1}^M w_{m\nu+r} e^{2\pi i P(m)} = 0.$$

Finally

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} w_n \prod_{s=1}^{\ell} \phi_s(W^{q_s(n)} x) = 0.$$

Addendum. The oscillation of order d is strongly related to the control of the $(d + 1)$ -th Gowers uniformity norm (see [22], [16]). We can use Gowers uniformity norms to study oscillating properties.

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References

[1] E.H. El Abdalaoui, M. Lemańczyk, T. de la Rue, Automorphisms with quasi-discrete spectrum, multiplicative functions and average orthogonality along short intervals, *Int. Math. Res. Not.* 2017 (14) (2017) 4350–4368.
 [2] E.H. El Abdalaoui, S. Kasjan, M. Lemańczyk, 0–1 sequences of the Thue–Morse type and Sarnak’s conjecture, *Proc. Amer. Math. Soc.* 144 (1) (2016) 161–176.
 [3] S. Akiyama, Y.P. Jiang, Higher order oscillation and uniform distribution, preprint.
 [4] N. Aoki, Topological entropy of distal affine transformations on compact abelian groups, *J. Math. Soc. Jpn.* 23 (1971) 11–17.

- [5] H. Davenport, On some infinite series involving arithmetical functions (II), *Q. J. Math.* 8 (1937) 313–320.
- [6] T. Downarowicz, E. Glasner, Isomorphic extensions and applications, *Topol. Methods Nonlinear Anal.* 48 (1) (2016) 321–338.
- [7] A.H. Fan, Oscillating sequences of higher orders and topological systems of quasi-discrete spectrum, preprint.
- [8] A.H. Fan, Y.P. Jiang, Oscillating sequences, minimal mean attractability and minimal mean-Lyapunov stability, *Ergod. Theory Dyn. Syst.* doi: <http://dx.doi.org/10.1017/etds.2016.121>.
- [9] S. Ferenzi, J. Kulaga-Przymus, M. Lemańczyk, C. Mauduit, Substitutions and Möbius disjointness, preprint, 2015.
- [10] F. Hahn, W. Parry, Minimal dynamical systems with quasi-discrete spectrum, *J. Lond. Math. Soc.* 40 (1965) 309–323.
- [11] H. Hoare, W. Parry, Affine transformations with quasi-discrete spectrum (I), *J. Lond. Math. Soc.* 41 (1966) 88–96.
- [12] H. Hoare, W. Parry, Affine transformations with quasi-discrete spectrum (II), *J. Lond. Math. Soc.* 41 (1966) 529–530.
- [13] L.G. Hua, *Additive Theory of Prime Numbers*, Transl. Math. Monogr., vol. 13, American Mathematical Society, Providence, RI, USA, 1966.
- [14] W. Huang, Z. Lian, S. Shao, X. Ye, Sequences from zero entropy noncommutative toral automorphisms and Sarnak conjecture, preprint.
- [15] W. Huang, Z.R. Wang, G.H. Zhang, Möbius disjointness for topological models of ergodic systems with discrete spectrum, preprint.
- [16] J. Konieczny, Gowers norms for the Thue–Morse and Rudin–Schapiro sequences, preprint.
- [17] J. Kulaga-Przymus, M. Lemańczyk, The Möbius function and continuous extensions of rotations, *Monatshefte Math.* 178 (4) (2015) 553–582.
- [18] J.Y. Liu, P. Sarnak, The Möbius function and distal flows, *Duke Math. J.* 164 (7) (2015) 1353–1399.
- [19] P. Sarnak, Three Lectures on the Möbius Function, Randomness and Dynamics, IAS Lect. Notes, 2009, [http://publications.ias.edu/sites/default/files/MobiusFunctionsLectures\(2\).pdf](http://publications.ias.edu/sites/default/files/MobiusFunctionsLectures(2).pdf).
- [20] P. Sarnak, Möbius randomness and dynamics, *Not. S. Afr. Math. Soc.* 43 (2012) 89–97.
- [21] R.X. Shi, Equivalent definitions of oscillating sequences of higher orders, preprint.
- [22] T. Tao, *Higher Order Fourier Analysis*, Grad. Stud. Math., vol. 142, American Mathematical Society, Providence, RI, USA, 2012.
- [23] Z.R. Wang, Möbius disjointness for analytic skew products, preprint.