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Partial differential equations

On a Liouville-type theorem for the Ginzburg–Landau system

Sur un théorème de type Liouville pour le système de Ginzburg–Landau

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ABSTRACT

We prove that entire, complex valued solutions to the Ginzburg–Landau system with positive real and imaginary parts are constant in any spatial dimension. This property was shown very recently only in the planar case.

RÉSUMÉ

Nous prouvons que des solutions complexes au système de Ginzburg–Landau dans l'espace entier avec des parties réelles et imaginaires positives sont constantes dans toute dimension spatiale. Cette propriété a été démontrée très récemment, mais seulement dans le cas planaire.

1. Introduction and proof of the main result

We consider (classical) solutions $(u, v) : \mathbb{R}^N \to \mathbb{R}^2$, with $u, v > 0$ and $N \geq 2$, to the Ginzburg–Landau system

$$\begin{aligned} -\Delta u &= u - u^3 - uv^2, \\ -\Delta v &= v - v^3 - vu^2. \end{aligned} \tag{1}$$

In this short note, we will establish the following Liouville-type property, which was shown very recently in [4] only for $N \leq 2$ with an argument that does not work in higher dimensions (see Theorem 1.7 therein). More precisely, this property for $N \leq 2$ was derived in the aforementioned reference as a direct consequence of the observation that the first inequality in (2) implies that the components of such solutions are super-harmonic (for any $N \geq 1$), a fact that we will also use to reach (3).

Theorem 1.1. *Under the above assumptions, it holds that u, v are constants such that $u^2 + v^2 = 1$.*

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Proof. We recall from Theorems 1.3 and 1.11 in [4] that

$$u^2+v^2\le 1 \text{ and } u+v>1 \text{ in } \mathbb{R}^N. \tag{2}$$

In passing, we point out that the first inequality in the above relation holds for any entire solution to (1), see also [2].

Arguing as in the proof of Theorem 1.1 in [3] (componentwise, see also [5, Thm. 3.2]), we deduce that there exists a constant $C>0$ such that

$$\int_{B_R}|\nabla u|^2 dx\le CR^{N-2} \text{ and } \int_{B_R}|\nabla v|^2 dx\le CR^{N-2} \text{ for all } R>1, \tag{3}$$

(where B_R denotes the ball with center at the origin and radius R). In the sequel, abusing notation, we will progressively increase the value of the generic constant C.

Next, we will use a family of smooth cutoffs $0\le\phi_R\le 1$ such that

$$\phi_R=1 \text{ in } B_R;\ \phi_R=0 \text{ in } \mathbb{R}^N\setminus B_{2R};\ |\nabla\phi_R|\le CR^{-1}, \tag{4}$$

for $R>1$. We test (1) by $(\phi_R u,\phi_R v)$, and integrate by parts, to arrive at

$$\int_{B_{2R}}(u^2+v^2)(1-u^2-v^2)\phi_R\, dx=\int_{B_{2R}}\left(|\nabla u|^2+|\nabla v|^2\right)\phi_R\, dx+\int_{B_{2R}}(u\nabla u\nabla\phi_R+v\nabla v\nabla\phi_R)\, dx. \tag{5}$$

By virtue of (2), (3), (4) and the Cauchy–Schwartz inequality, we find that

$$\left|\int_{B_{2R}}u\nabla u\nabla\phi_R\, dx\right|\le CR^{N-2} \quad \text{for } R>1.$$

Analogously we can estimate the remaining terms in the righthand side of (5) to deduce that

$$\int_{B_{2R}}(u^2+v^2)(1-u^2-v^2)\phi_R\, dx\le CR^{N-2} \quad \text{for } R>1.$$

In turn, via (2) and (4), we infer that

$$\int_{B_R}(1-u^2-v^2)^2\, dx\le CR^{N-2} \quad \text{for } R>1.$$

Consequently, recalling (3), we have the following upper bound for the local energy:

$$E_R(u,v)=\int_{B_R}\left\{\frac{|\nabla u|^2}{2}+\frac{|\nabla v|^2}{2}+\frac{(1-u^2-v^2)^2}{4}\right\}dx\le CR^{N-2} \text{ for } R>1.$$

The assertion of the theorem now follows readily from the famous η-ellipticity theorem (see Theorem 2 in [1] and Theorem A.1 below), see also the proof of Theorem 1.1 in [6] for a similar argument.

The proof of the theorem is complete.

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Appendix A. The η-ellipticity theorem

For the reader's convenience, we include below the η-ellipticity theorem from [1] in the scaled form that we employed above. This follows at once by setting $\varepsilon=R^{-1}$ in the assertion of [1, Thm. 2] and stretching coordinates.

Theorem A.1. *Suppose that $u,v\in C^2\left(\mathbb{R}^N;\mathbb{R}\right)$, $N\ge 2$, solve (1) and satisfy*

$$E_R(u,v)=\int_{B_R}\left\{\frac{|\nabla u|^2}{2}+\frac{|\nabla v|^2}{2}+\frac{(1-u^2-v^2)^2}{4}\right\}dx\le\eta(\ln R)R^{N-2}$$

for some $R>2$ and $\eta>0$. Then, it holds

$$|(u(0), v(0))| \geq 1 - K\eta^{\alpha}$$

for some $K, \alpha > 0$ *that depend only on the dimension* N.

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