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Conformal dimension on boundary of right-angled hyperbolic buildings

Dimension conforme du bord d'un immeuble hyperbolique à angles droits

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ABSTRACT

In this note, we use some combinatorial modulus on the boundary of a right-angled hyperbolic building to control its conformal dimension. The lower bound obtained is optimal in the case of Fuchsian buildings.

RÉSUMÉ

Dans cette note, on utilise des modules combinatoires sur le bord d'un immeuble hyperbolique à angles droits pour encadrer sa dimension conforme. La borne inférieure obtenue est optimale dans le cas des immeubles fuchsiens.

1. Introduction

The conformal dimension is a quasi-isometric invariant of Gromov hyperbolic spaces that has been introduced by P. Pansu in [12]. Since then it has become a major tool used to study quasi-conformal properties of boundaries of hyperbolic groups in relation with rigidity phenomena. In particular, in [1], it plays a key role in proving that right-angled Fuchsian buildings satisfy a Mostow type rigidity theorem. We refer to [9] and [8] for surveys concerning the connection between the conformal dimension and Mostow type rigidity results and to [10] for a survey concerning the conformal dimension in the more general context of self-similar spaces.

In this note, we give bounds to the conformal dimension of the boundary of a right-angled hyperbolic building. This is obtained by simple computations not on the boundary of the building, but on the boundary of an apartment.

The motivation of this work is that a good understanding of the conformal dimension of these buildings could lead to new rigidity results. Some of them are indeed highly suspected to be rigid.

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2. Recalls

Let (Z, d) and (Z', d') be two compact arcwise connected metric spaces. The *cross-ratio* of four distinct points $a, b, c, d \in Z$ is

$$[a : b : c : d]_Z = \frac{d(a,b)}{d(a,c)} \cdot \frac{d(c,d)}{d(b,d)}.$$

An homeomorphism $f : Z \longrightarrow Z'$ is *quasi-Moebius* (*QM*) if there exists an homeomorphism $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that for any quadruple of distinct points $a, b, c, d \in Z$

$$[f(a) : f(b) : f(c) : f(d)]_{Z'} \leq \phi([a : b : c : d]_Z).$$

In this case, f^{-1} is also QM and Z and Z' are said *quasi-Moebius equivalent* (*QM equivalent*). Now we assume that Z is a Q-*Ahlfors-regular* (*AR*) for $Q > 1$. This means that there exists a constant $C > 1$ such that for any $0 < R \leq \operatorname{diam} Z$ and any R-ball $B \subset Z$ one has

$$C^{-1} \cdot R^Q \leq \mathcal{H}_d(B) \leq C \cdot R^Q,$$

where $\mathcal{H}_d(\cdot)$ denotes the Hausdorff measure of (Z, d). Notice that, under this assumption, Q is equal to the Hausdorff dimension $\dim_{\mathcal{H}}(Z, d)$ of (Z, d). The *Ahlfors-regular conformal gauge* of (Z, d) is defined as follows:

$$\mathcal{J}_c(Z, d) := \{(Z, \delta) : (Z, \delta) \text{ is AR and is QM equivalent to } (Z, d)\}.$$

Definition 2.1. *The* Ahlfors-regular conformal dimension *of* $\partial\Gamma$ *is*

$$\operatorname{Confdim}(Z, d) := \inf\{\dim_{\mathcal{H}}(Z, \delta) : (Z, \delta) \in \mathcal{J}_c(Z, d)\}.$$

In the rest of the note we will simply call it the *conformal dimension*. As the topological and the Hausdorff dimensions are respectively invariant under homeomorphisms and bi-Lipschitz maps, the conformal dimension is invariant under quasi-Moebius maps. The inclusions between these three classes of maps imply the following inequalities:

$$\dim_T(Z) \leq \operatorname{Confdim}(Z, d) \leq \dim_{\mathcal{H}}(Z, d),$$

where $\dim_T(Z)$ stands for the topological dimension of Z.

We recall that all the visual metrics on the boundary of a hyperbolic space are quasi-Moebius homeomorphic to each other and AR. In particular, the conformal dimension is a quasi-isometric invariant of a hyperbolic space. This will be the context of this note.

Combining ideas of G.D. Mostow, P. Pansu, and M. Bourdon with a theorem of M. Bonk and B. Kleiner (see [1, Theorem 1.3]), it is known that if the conformal dimension of the boundary of a CAT(-1) group is achieved in the conformal gauge, then the underlying CAT(-1) space satisfies a Mostow-type rigidity theorem (see [8, Théorème 5.11]).

Hence, the conformal dimension is a very powerful tool, but, unfortunately, it is also very hard to compute.

3. The result

The goal of this note is to prove the theorem below that relates the conformal dimension of the boundary of a building to the conformal dimension of the boundary of an apartment. We refer to [6] for generalities concerning Coxeter groups and buildings.

Let $\mathcal{G}$ denote a *finite simplicial graph i.e* $\mathcal{G}^{(0)}$ is finite, each edge has two different vertices, and $\mathcal{G}$ contains no double edge. We denote by $\mathcal{G}^{(0)} = \{v_1, \ldots, v_n\}$ the vertices of $\mathcal{G}$ and we set $S = \{s_1, \ldots, s_n\}$. If for $i \neq j$ the corresponding vertices v_i, v_j are connected by an edge, then we write $v_i \sim v_j$. We denote by W the *right-angled Coxeter group* whose relation graph is $\mathcal{G}$, namely

$$W = \left\langle s_i \in S | s_i^2 = 1, s_i s_j = s_j s_i \text{ if } v_i \sim v_j \right\rangle.$$

For $q \geq 2$ we denote by Γ_q the group defined by the following presentation

$$\Gamma_q = \langle s_i \in S | s_i^q = 1, s_i s_j = s_j s_i \text{ if } v_i \sim v_j \rangle.$$

In the rest of this note, we will assume that Γ_q is infinite hyperbolic with arcwise connected boundary $\partial\Gamma_q$. This assumption is in fact only an assumption on the graph $\mathcal{G}$ (see [11] and [7]). We also equip $\partial\Gamma_q$ with a visual metric d.

We recall that Γ_q acts by isometry properly discontinuously and cocompactly on Δ_q the building of type (W, S) and of constant thickness q. As a consequence, the boundaries $\partial\Gamma_q$ and $\partial\Delta_q$ are canonically identified by a Γ_q-equivarient quasi-Moebius homeomorphism. We also recall that Δ_q is a CAT(-1) metric space.

In the following, for $g \in \Gamma_q$, we designate by $|g|$ the distance, for the word metric on Γ_q relative to the generating set S^q, between e and g. Then $\tau(q) = \limsup_k \frac{1}{k} \log(\#\{g \in \Gamma_q : |g| \leq k\})$ is the growth rate of Γ_q. In the rest of the note, we will write $Q(q) = \operatorname{Confdim}(\partial\Gamma_q)$. Notice that, in particular, $Q(2) = \operatorname{Confdim}(\partial W)$.

Theorem 3.1. *There exists a constant $C > 0$ independent of q such that*

$$Q(2) \cdot \big(1 + \frac{\log(q-1)}{\tau(2)}\big) \leq Q(q) \leq C \log(q-1).$$

4. Consequences of Theorem 3.1

In general, the conformal dimension $Q(2)$ of ∂W is unknown. However, the topological dimension of ∂W is easy to read in the graph $\mathcal{G}$ (see [7]). Hence, one can use the inequality $\mathrm{Confdim}(\partial W) \geq \dim_T(\partial W)$, to obtain an explicit lower bound.

The only hyperbolic buildings for which we can compute the conformal dimension of their boundaries are Fuchsian buildings. In this case, the lower bound of Theorem 3.1 is optimal (see [2, Théorème 1.1]). We can wonder if this bound is optimal for other examples. The following example is particularly interesting because it should lead to a new rigidity result.

Example 1. Let D be the regular right-angled dodecahedron in $\mathbb{H}^3$ and W_D be the right-angled Coxeter group generated by the reflections about the faces of D. Then ∂W_D is quasi-Moebius homeomorphic to the Euclidean sphere $\mathbb{S}^2$ and $Q(2) = 2$.

For $q \geq 3$, the corresponding building Δ_q has all chances to be Mostow rigid. Indeed, Fuchsian buildings (the analogues of Δ_q in dimension 2) are rigid, whereas their apartments are not. These are hyperbolic planes $\mathbb{H}^2$. On the other hand, the apartments of Δ_q are rigid, they are hyperbolic 3-spaces $\mathbb{H}^3$. Hence, the rigidity of the apartments should be increased by the building structure. Moreover, in [5] it is proved that $\partial\Delta_q$ satisfies the *combinatorial Loewner property*. This property is conjecturally equivalent to admitting a metric realizing the conformal dimension (see [9, Conjecture 7.5]). A next step towards the rigidity of this building would be to know if the lower bound $Q(q) \geq 2 + \frac{2\log(q-1)}{\tau(2)}$ is sharp.

This example has an analogue in dimension 4 if we replace the dodecahedron in $\mathbb{H}^3$ by the right-angled regular 120-cell in $\mathbb{H}^4$. The boundary of the building obtained also satisfies the combinatorial Loewner property and for the same reasons has also all the chances to be rigid.

Finally, an immediate computation shows that the growth rates $\tau(q)$ and $\tau(2)$ are related by the formula $\tau(q) = \tau(2) + \log(q-1)$. On the other hand, growth rates of Coxeter groups can be computed by hand. For instance, if W_D is the reflection group of Example 1, then $\tau(2) = \log(4 + \sqrt{15})$.

5. Combinatorial modulus

For a complete introduction on combinatorial modulus on boundaries of hyperbolic groups we refer to [3]. In this note, we will restrict to the example given by the group $\partial\Gamma_q$ equipped with a visual metric d.

For $k \geq 0$ and $\kappa > 1$, a *κ-approximation of $(\partial\Gamma_q, d)$ on scale k* is a finite covering G_k by open subsets such that for any $v \in G_k$ there exists $z_v \in v$ satisfying the following properties:

- $B(z_v, \kappa^{-1}2^{-k}) \subset v \subset B(z_v, \kappa 2^{-k})$,
- $\forall v, w \in G_k$ with $v \neq w$ one has $B(z_v, \kappa^{-1}2^{-k}) \cap B(z_w, \kappa^{-1}2^{-k}) = \emptyset$.

A sequence $\{G_k\}_{k\geq 0}$ is called a *κ-approximation of $(\partial\Gamma_q, d)$*.

Now we fix the approximation $\{G_k\}_{k\geq 0}$. Throughout this note, we will call a curve in $\partial\Gamma_q$ a continuous map $\gamma : [0,1] \longrightarrow \partial\Gamma_q$ and we will denote by γ its image. Let $\rho : G_k \longrightarrow [0, +\infty)$ be a positive function and γ be a curve in $\partial\Gamma_q$. The *ρ-length* of γ is

$$L_\rho(\gamma) = \sum_{\substack{\gamma\cap v \neq \emptyset \\ v \in G_k}} \rho(v).$$

For $p \geq 1$, the *p-mass* of ρ is $M_p(\rho) = \sum_{v\in G_k} \rho(v)^p$. Let $\mathcal{F}$ be a non-empty set of curves in $\partial\Gamma_q$. We say that the function ρ is *$\mathcal{F}$-admissible* if $L_\rho(\gamma) \geq 1$ for any curve $\gamma \in \mathcal{F}$. For $p \geq 1$, the *G_k-combinatorial p-modulus of $\mathcal{F}$* is

$$\mathrm{Mod}_p(\mathcal{F}, G_k) = \inf\{M_p(\rho)\},$$

where the infimum is taken over the set of $\mathcal{F}$-admissible functions and with $\mathrm{Mod}_p(\emptyset, G_k) := 0$.

6. Two ingredients for the proof

The first ingredient of the proof is that the conformal dimension is a critical exponent for the combinatorial modulus. Let $d_0 > 0$ be a small constant compared with the geometric constants of $(\partial\Gamma_q, d)$ and let $\mathcal{F}_0$ be the set of all the curves in $\partial\Gamma_q$ of diameter larger than d_0. Under this assumption

$$\mathrm{Confdim}(\partial\Gamma_q) = \inf\{p \in [1, +\infty) : \lim_{k\to+\infty} \mathrm{Mod}_p(\mathcal{F}_0, G_k) = 0\}. \quad (1)$$

This is the immediate application of an unpublished theorem due to S. Keith and B. Kleiner (see [4, Theorem 1.2.] for a proof in a more general context).

The second ingredient of the proof is a control of the combinatorial modulus on the boundary of the building by the combinatorial modulus on the boundary of an apartment established in [5], which we recall here briefly. Let G_k^W and G_k be two κ-approximations of respectively ∂W and $\partial \Gamma$ constructed as in [5, Subsection 8.2]. We recall that by construction, there exists $\lambda \geq 1$ such that, for any $k \geq 0$

$$\#G_k^W \leq \#\{g \in W : |g| \leq k\} \leq \lambda \cdot \#G_k^W.$$

In particular, this implies that $\tau(2) = \limsup_{k\to\infty} \frac{1}{k} \log(\#G_k^W)$. Now we designate by $\mathrm{mod}_p(\cdot, G_k^W)$ and $\mathrm{Mod}_p(\cdot, G_k)$ the combinatorial modulus computed respectively in ∂W and in $\partial \Gamma_q$. Let $d_0 > 0$ be a small constant compared with the geometric parameters of ∂W and $\partial \Gamma_q$ and let $\mathcal{F}_0^W$ and $\mathcal{F}_0$ be the set of all the curves respectively in ∂W and $\partial \Gamma_q$ of diameter larger than d_0. Under these assumptions, according to [5, Theorem 9.1], for any $p \geq 1$, there exists a constant $D \geq 1$ such that, for every $k \geq 1$,

$$D^{-1} \cdot \mathrm{Mod}_p(\mathcal{F}_0, G_k) \leq (q-1)^k \cdot \mathrm{mod}_p(\mathcal{F}_0^W, G_k^W) \leq D \cdot \mathrm{Mod}_p(\mathcal{F}_0, G_k). \tag{2}$$

As an immediate consequence of the relations (1) and (2),

$$Q(q) = \inf\{p \in [1, +\infty) : \lim_{k\to+\infty} (q-1)^k \cdot \mathrm{mod}_p(\mathcal{F}_0^W, G_k^W) = 0\}. \tag{3}$$

7. Right-hand-side inequality

We use the equality (3) in proving Theorem 3.1. Throughout the proof, $\{\rho_k\}_{k\geq 0}$ is a sequence of $\mathcal{F}_0^W$-admissible functions.

Comparing the diameters of the elements of G_k with d_0, one obtains that there exist two constants $K > 0$ and $0 < \lambda < 1$ independent of k and q such that, up to changing the sequence $\{\rho_k\}_{k\geq 0}$, for all $k \geq 1$ and all $w \in G_k^W$

$$\rho_k(w) \leq K \cdot \lambda^k.$$

Hence

$$(q-1)^k \sum_{w \in G_k^W} \rho_k(w)^p \leq K^p \cdot \#G_k^W \cdot [\lambda^p (q-1)]^k.$$

As we recalled, $\tau(2) = \limsup_{k\to\infty} \frac{1}{k} \log(\#G_k^W)$, thus $Q(q) \leq \frac{1}{\log 1/\lambda}(\tau(2) + \log(q-1))$.

8. Left-hand-side inequality

Now we set $G = \bigcup_{k\geq 0} G_k^W$ and for $w \in G$ we write $|w| = k$ such that $w \in G_k^W$. For a sequence $\{\rho_k\}_{k\geq 0}$ of $\mathcal{F}_0^W$-admissible function, let $\rho : w \in G \longrightarrow \rho_{|w|}(w) \in \mathbb{R}^+$. Now we observe that

$$\text{if } \sum_{w\in G} (q-1)^{|w|} \rho(w)^p < \infty \text{ then } p \geq Q(q).$$

We define the function

$$P_\rho(s) := \inf\{p > 0 | \sum_{w\in G} (q-1)^{s|w|} \rho(w)^p < \infty\}$$

and we study the function P_ρ to compute the lower bound for $Q(q)$.

To start, one has $P_\rho(0) \geq Q(2)$ and, up to changing $\{\rho_k\}_{k\geq 0}$, we can choose $P_\rho(0)$ arbitrarily close to $Q(2)$. Then, identically, one has $P_\rho(1) \geq Q(q)$ and, up to changing $\{\rho_k\}_{k\geq 0}$, we can choose $P_\rho(1)$ arbitrarily close to $Q(q)$.

Now we set $s_0 = \sup\{s \in \mathbb{R} | \sum_{w\in G} (q-1)^{s|w|} < \infty\}$. Clearly, $s_0 < 0$ and is such that $P_\rho(s_0) = 0$. Indeed, as we said in the preceding paragraph, there exist two constants $K > 0$ and $0 < \lambda < 1$ independent of k and q such that, up to changing the sequence $\{\rho_k\}_{k\geq 0}$, for all $k \geq 1$ and all $w \in G_k^W$

$$\rho(w) \leq K \cdot \lambda^{|w|}.$$

Hence, if for all $\epsilon > 0$

$$\sum_{w\in G} (q-1)^{-\epsilon|w|} . (q-1)^{s_0|w|} < \infty.$$

Then for all $\epsilon' > 0$ small enough, one has

$$\sum_{w\in G} \rho(w)^{\epsilon'}(q-1)^{s_0|w|} \leq K^{\epsilon'} \cdot \sum_{w\in G}(q-1)^{-\epsilon|w|}.(q-1)^{s_0|w|} < \infty.$$

On the other hand, $s_0 = \sup\{s \in \mathbb{R}| \sum_{k\in\mathbb{N}} \#G_k^W (q-1)^{sk} < \infty\}$. As we recalled, $\tau(2) = \limsup_k \frac{1}{k}\log(\#G_k^W)$ thus

$$s_0 = -\frac{\tau(2)}{\log(q-1)}.$$

Now we check that the function P_ρ is convex on $[s_0, +\infty)$. In other word, we check that for all $t \in [0,1]$, for all $[a,b] \subset [s_0, +\infty)$ and for all $\epsilon > 0$ one has

$$\sum_{w\in G}(q-1)^{(ta+(1-t)b)|w|}\rho(w)^{(tP_\rho(a)+(1-t)P_\rho(b))+\epsilon} < \infty.$$

Indeed, for all $\epsilon > 0$

$$\sum_{w\in G} x_w := \sum_{w\in G}(q-1)^{a|w|}\rho(w)^{P_\rho(a)+\epsilon} < \infty \text{ and } \sum_{w\in G} y_w := \sum_{w\in G}(q-1)^{b|w|}\rho(w)^{P_\rho(b)+\epsilon} < \infty.$$

Hence $\{x_w^{1/p}\} \in \ell^p$ and $\{y_w^{1/q}\} \in \ell^q$ with $p = \frac{1}{t}$ and $q = \frac{1}{1-t}$ and by Hölder's inequality:

$$\begin{aligned}(\sum_{w\in G} x_w)^{1/p}.(\sum_{w\in G} y_w)^{1/q} &\geq \sum_{w\in G}((q-1)^{a|w|}\rho(w)^{P_\rho(a)+\epsilon})^t.((q-1)^{b|w|}\rho(w)^{P_\rho(b)+\epsilon})^{1-t},\\ &\geq \sum_{w\in G}(q-1)^{(ta+(1-t)b)|w|}\rho(w)^{(tP_\rho(a)+(1-t)P_\rho(b))+\epsilon}.\end{aligned}$$

Finally, by convexity, for all $t < 0$ one has $P_\rho(ts_0) \geq tP_\rho(s_0) + (1-t)P_\rho(0)$. In particular, for $t = 1/s_0$ one has

$$Q(q) \geq Q(2) \cdot (1 - \frac{1}{s_0}) = Q(2) \cdot \big(1 + \frac{\log(q-1)}{\tau(2)}\big).$$

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