Mathematical analysis/Partial differential equations

A sharp weighted anisotropic Poincaré inequality for convex domains

Une inégalité de Poincaré anisotrope pondérée pour les domaines convexes

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1. Introduction

In this paper, we prove a sharp lower bound for the optimal constant \( \mu_{p,\mathcal{H},\omega}(\Omega) \) in the Poincaré-type inequality

\[
\inf_{t \in \mathbb{R}} \| u - t \|_{L^p(\Omega)} \leq \frac{1}{[\mu_{p,\mathcal{H},\omega}(\Omega)]^{\frac{1}{p}}} \| \mathcal{H}(\nabla u) \|_{L^p(\Omega)},
\]

with \( 1 < p < +\infty \); \( \Omega \) is a bounded convex domain of \( \mathbb{R}^n \), \( \mathcal{H} \in \mathcal{H}(\mathbb{R}^n) \), where \( \mathcal{H}(\mathbb{R}^n) \) is the set of lower semicontinuous functions, positive in \( \mathbb{R}^n \setminus \{0\} \) and positively 1-homogeneous, and \( \omega \) is a log-concave function.

If \( \mathcal{H} \) is the Euclidean norm of \( \mathbb{R}^n \) and \( \omega = 1 \), then \( \mu_p(\Omega) = \mu_{p,E,\omega}(\Omega) \) is the first nontrivial eigenvalue of the Neumann \( p \)-Laplacian:

\[
\begin{align*}
-\Delta_p u &= \mu_p |u|^{p-2} u \quad \text{in } \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]
Then, for a convex set $\Omega$, it holds that
\[
\mu_p(\Omega) \geq \left( \frac{\pi_p}{D_\mathcal{E}(\Omega)} \right)^p,
\]
where
\[
\pi_p = 2 \int_0^{+\infty} \frac{1}{1 + \frac{1}{p-1}s^p} ds = 2\pi \left( \frac{p-1}{p}\right)^{\frac{1}{p}},
\]
and $D_\mathcal{E}(\Omega)$ being the Euclidean diameter of $\Omega$.

This estimate, proved in the case $p = 2$ in [13] (see also [3]), has been generalized to the case $p \neq 2$ in [1,10,12,15] and for $p \to \infty$ in [9,14]. Moreover, the constant $\left( \frac{\pi_p}{p^{\frac{1}{p}}} \right)^p$ is the optimal constant of the one-dimensional Poincaré–Wirtinger inequality, with $\omega = 1$, on a segment of length $D_\mathcal{E}(\Omega)$. When $p = 2$ and $\omega = 1$, in [4] an extension of the estimate in the class of suitable non-convex domains has been proved.

The aim of the paper is to prove an analogous sharp lower bound for $\mu_{p, \mathcal{H}, \omega}(\Omega)$, in a general anisotropic case. More precisely, our main result is:

**Theorem 1.** Let $\mathcal{H} \in \mathcal{H}(\mathbb{R}^n)$, $\mathcal{H}^0$ be its polar function. Let us consider a bounded convex domain $\Omega \subset \mathbb{R}^n$, $1 < p < \infty$, and take a positive log-concave function $\omega$ defined in $\Omega$. Then, given that
\[
\mu_{p, \mathcal{H}, \omega}(\Omega) = \inf_{u \in W^{1,p}(\Omega)} \int_{\Omega} \mathcal{H}(\nabla u)^p \omega \, dx
\]
and
\[
\|u\|_{W^{1,p}(\Omega)} = \left( \int_{\Omega} |u|^p \omega \, dx \right)^{\frac{1}{p}}.
\]

This result has been proved in the case $p = 2$ and $\omega = 1$, when $\mathcal{H}$ is a strongly convex, smooth norm of $\mathbb{R}^n$ in [17], with a completely different method than the one presented here.

In Section 2 below, we give the precise definition of $\mathcal{H}^0$ and give some details on the set $\mathcal{H}(\mathbb{R}^n)$. In Section 3, we give the proof of the main result.

**2. Notation and preliminaries**

A function
\[
\xi \in \mathbb{R}^n \mapsto \mathcal{H}(\xi) \in [0, +\infty[.
\]
belongs to the set $\mathcal{H}(\mathbb{R}^n)$ if it verifies the following assumptions:

1. $\mathcal{H}$ is positively 1-homogeneous, that is
   
   if $\xi \in \mathbb{R}^n$ and $t \geq 0$, then $\mathcal{H}(t\xi) = t\mathcal{H}(\xi)$;

2. If $\xi \in \mathbb{R}^n \setminus \{0\}$, then $\mathcal{H}(\xi) > 0$;

3. $\mathcal{H}$ is lower semi-continuous.

If $\mathcal{H} \in \mathcal{H}(\mathbb{R}^n)$, properties (1), (2), (3) give that there exists a positive constant $a$ such that
\[
a|\xi| \leq \mathcal{H}(\xi), \quad \xi \in \mathbb{R}^n.
\]

The polar function $\mathcal{H}^0 : \mathbb{R}^n \to [0, +\infty]$ of $\mathcal{H} \in \mathcal{H}(\mathbb{R}^n)$ is defined as
\[
\mathcal{H}^0(\eta) = \sup_{\xi \neq 0} \frac{\langle \xi, \eta \rangle}{\mathcal{H}(\xi)}.
\]

The function $\mathcal{H}^0$ belongs to $\mathcal{H}(\mathbb{R}^n)$. Moreover, it is convex on $\mathbb{R}^n$, and then continuous. If $\mathcal{H}$ is convex, it holds that
Remark the needed. Moreover, and Hardy–Sobolev norm. If \( H, H^\sigma \) for all \( \xi \in \mathbb{R}^n \), then \( H \) is a norm on \( \mathbb{R}^n \), and the same holds for \( H^\sigma \).

We recall that if \( H \) is a smooth norm of \( \mathbb{R}^n \) such that \( \nabla^2(H^2) \) is positive definite on \( \mathbb{R}^n \setminus \{0\} \), then \( H \) is called a Finsler norm on \( \mathbb{R}^n \).

If \( H \in \mathscr{H}(\mathbb{R}^n) \), by definition, we have
\[
\langle \xi, \eta \rangle \leq H(x)H^\delta(y), \quad \forall \xi, \eta \in \mathbb{R}^n.
\]

**Remark 2.** Let \( H \in \mathscr{H}(\mathbb{R}^n) \), and consider the convex envelope of \( H \), that is the largest convex function \( \overline{H} \) such that \( \overline{H} \leq H \). It holds that \( \overline{H} \) and \( H \) have the same polar function:
\[
\langle \overline{H}, \delta \rangle = H \quad \text{in } \mathbb{R}^n.
\]

Indeed, being \( \overline{H} \leq H \), by definition it holds that \( \langle \overline{H}, \delta \rangle \geq H \). To show the reverse inequality, it is enough to prove that \( \langle H^\sigma, \delta \rangle \leq \overline{H} \). Then, being \( \overline{H} \) the convex envelope of \( H \), it must be \( \langle H^\sigma, \delta \rangle \leq \overline{H} \), that implies \( \langle \overline{H}, \delta \rangle \leq H \). Denoting by \( G(x) = \langle H^\sigma(x), \delta \rangle \), for any \( x \) there exists \( \delta_x \) such that
\[
G(x) = \langle x, \delta_x \rangle \quad \text{and} \quad \langle x, \delta_x \rangle \leq \langle H^\sigma(\delta_x), \delta \rangle = H(\delta_x), \quad \text{that implies} \quad G(x) \leq H(x).
\]

Let \( H \in \mathscr{H}(\mathbb{R}^n) \), and consider a bounded convex domain \( \Omega \) of \( \mathbb{R}^n \). Throughout the paper \( D_H(\Omega) \subset ]0, +\infty[ \) will be
\[
D_H(\Omega) = \sup_{x,y \in \Omega} H^\sigma(y - x).
\]

We explicitly observe that since \( H^\sigma \) is not necessarily even, in general \( H^\sigma(y - x) \neq H^\sigma(x - y) \). When \( H \) is a norm, then \( D_H(\Omega) \) is the so-called anisotropic diameter of \( \Omega \) with respect to \( H^\sigma \). In particular, if \( H = E \) is the Euclidean norm in \( \mathbb{R}^n \), then \( E^\sigma = E \) and \( D_E(\Omega) \) is the standard Euclidean diameter of \( \Omega \). We refer the reader, for example, to \([5, 11]\) for remarkable examples of convex not even functions in \( \mathscr{H}(\mathbb{R}^n) \). On the other hand, in \([16]\) some results on isoperimetric and optimal Hardy–Sobolev inequalities for a general function \( H \in \mathscr{H}(\mathbb{R}^n) \) have been proved, by using a generalization of the so-called convex symmetrization introduced in \([2]\) (see also \([6–8]\)).

**Remark 3.** In general, \( H \) and \( H^\sigma \) are not rotational invariant. Anyway, if \( A \in SO(n) \), defining
\[
H_A(x) = \mathcal{H}(Ax),
\]
and being \( A^T = A^{-1} \), then \( H_A \in \mathscr{H}(\mathbb{R}^n) \) and
\[
\langle H_A, \delta \rangle = \sup_{x \in \mathbb{R}^n} H_A(x) \quad \text{and} \quad \langle H^\sigma, \delta \rangle = \sup_{y \in \mathbb{R}^n} H^\sigma(y) = \langle H^\sigma, \delta \rangle.
\]
Moreover,
\[
D_{H_A}(A^T \Omega) = \sup_{x,y \in A^T \Omega} (H^\sigma)_A(y - x) = \sup_{\tilde{x}, \tilde{y} \in \Omega} H^\sigma(\tilde{y} - \tilde{x}) = D_H(\Omega).
\]

3. Proof of the Payne–Weinberger inequality

In this section, we state and prove Theorem 1. To this aim, the following Wirtinger-type inequality, contained in \([12]\) is needed.

**Proposition 4.** Let \( f \) be a positive log-concave function defined on \([0, L]\) and \( p > 1 \), then
\[
\inf \left\{ \int_0^L |u'|^p f \, dx \right. \left. \quad \text{subject to} \quad \int_0^L |u|^p \, dx \right. \left. \quad \text{and} \quad \int_0^L |u|^{p-2} uf \, dx = 0 \right\} \geq \frac{\pi^p}{L^p}.
\]

The proof of the main result is based on a slicing method introduced in \([13]\) in the Laplacian case. The key ingredient is the following Lemma. For a proof, we refer the reader, for example, to \([13, 3, 12]\).
Lemma 5. Let $\Omega$ be a convex set in $\mathbb{R}^n$ having (Euclidean) diameter $D_\mathcal{E}(\Omega)$, let $\omega$ be a positive log-concave function on $\Omega$, and let $u$ be any function such that $\int_{\Omega} |u|^p \omega \, dx = 0$. Then, for all positive $\varepsilon$, there exists a decomposition of the set $\Omega$ in mutually disjoint convex sets $\Omega_i$ ($i = 1, \ldots, k$) such that

$$\bigcup_{i=1}^k \overline{\Omega}_i = \overline{\Omega}$$

$$\int_{\Omega_i} |u|^{p-2} u \omega \, dx = 0$$

and for each $i$ there exists a rectangular system of coordinates such that

$$\Omega_i \subset \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq d_i, \ |x_i| \leq \varepsilon, \ i = 2, \ldots, n\},$$

where $d_i \leq D_\mathcal{E}(\Omega)$, $i = 1, \ldots, k$.

Proof of Theorem 1. By density, it is sufficient to consider a smooth function $u$ with uniformly continuous first derivatives and $\int_{\Omega} |u|^{p-2} u \omega \, dx = 0$.

Hence, we can decompose the set $\Omega$ in $k$ convex domains $\Omega_i$ as in Lemma 5. In order to prove (1), we will show that, for any $i \in \{1, \ldots, k\}$, it holds that

$$\int_{\Omega_i} H^p(\nabla u) \omega \, dx \geq \frac{\pi_p}{D_H(\Omega)^p} \int_{\Omega_i} |u|^p \omega \, dx. \tag{5}$$

By Lemma 5, for each fixed $i \in \{1, \ldots, k\}$, there exists a rotation $A_i \in SO(n)$ such that

$$A_i \Omega_i \subset \{(x_1, \ldots, x_n) \in \mathbb{R}^n : 0 \leq x_1 \leq d_i, \ |x_i| \leq \varepsilon, \ i = 2, \ldots, n\}.$$

By changing the variable $y = A_i x$, recalling the notation (3) and using (4), it holds that

$$\int_{\Omega_i} H^p(\nabla u(x)) \omega(x) \, dx = \int_{A_i \Omega_i} H^p_{A_i^T}(\nabla u(A_i^T y)) \omega(A_i^T y) \, dy; \quad D_H(\Omega) = D_{H_{A_i^T}}(A_i \Omega).$$

We deduce that it is not restrictive to suppose that for any $i \in \{1, \ldots, n\}$ $A_i$ is the identity matrix, and the decomposition holds with respect to the $x_1$-axis.

Now we may argue as in [12]. For any $t \in [0, d_i]$ let us denote by $v(t) = u(t, 0, \ldots, 0)$, and $f_i(t) = g_i(t)\omega(t, 0, \ldots, 0)$, where $g_i(t)$ will be the $(n-1)$-volume of the intersection of $\Omega_i$ with the hyperplane $x_1 = t$. By the Brunn–Minkowski inequality, $g_i$, and then $f_i$, is a log-concave function in $[0, d_i]$. Since $u$, $u_{x_1}$, and $\omega$ are uniformly continuous in $\Omega$, there exists a modulus of continuity $\eta(\cdot)$ with $\eta(\varepsilon) \searrow 0$ for $\varepsilon \to 0$, independent of the decomposition of $\Omega$ and such that

$$\left| \int_{\Omega_i} |u|_t^p \omega \, dx - \int_{\Omega_i} |v|_t^p f_i \, dt \right| \leq \eta(\varepsilon)|\Omega_i|,$$

$$\left| \int_{\Omega_i} |u|^p \omega \, dx - \int_{\Omega_i} |v|^p f_i \, dt \right| \leq \eta(\varepsilon)|\Omega_i|,$$

and

$$\left| \int_{\Omega_i} |v|^{p-2} v f_i \, dt \right| \leq \eta(\varepsilon)|\Omega_i|.$$
where $C$ is a constant which does not depend on $\varepsilon$. Being $d_i \leq D_{\mathcal{E}}(\Omega)$, and then $d_i M \leq D_{\mathcal{H}}(\Omega)$, by letting $\varepsilon$ to zero, we get (5). Hence, by summing over $i$, we get the thesis.

**Remark 6.** In order to prove an estimate for $\mu_{p,\mathcal{H},\omega}$, we could use directly property (2) with $v = \frac{\nu u}{|\nu u|}$, and the Payne–Weinberger inequality in the Euclidean case, obtaining that

$$
\int_{\Omega} \mathcal{H}^p(\nabla u) \omega \, dx \geq \int_{\Omega} \frac{|\nabla u|^p}{\mathcal{H}^p(v)} \omega \, dx \geq \frac{\pi_p^p}{D_{\mathcal{E}}(\Omega)^p \mathcal{H}^p(v_m)^p} \int_{\Omega} |u|^p \omega \, dx,
$$

where $\mathcal{H}^p(v_m) = \max \mathcal{H}^p(v)$. However, we have a worse estimate than (1) because $D_{\mathcal{E}}(\Omega) \cdot \mathcal{H}^p(v_m)$ is, in general, strictly larger than $D_{\mathcal{H}}(\Omega)$, as shown in the following example.

**Example 1.** Let $\mathcal{H}(x, y) = \sqrt{a^2 x^2 + b^2 y^2}$, with $a < b$. Then $\mathcal{H}$ is a even, smooth norm with $\mathcal{H}^p(x, y) = \sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}$ and the Wulff shapes ($\mathcal{H}^p(x, y) < R$), $R > 0$, are ellipses. Clearly, we have:

$$D_{\mathcal{E}}(\Omega) = 2b \quad \text{and} \quad D_{\mathcal{H}}(\Omega) = 2.$$

Let us compute $\mathcal{H}^p(v_m)$. We have:

$$\max_{|v|=1} \mathcal{H}^p(v) = \max_{\vartheta \in [0, 2\pi]} \sqrt{\frac{(\cos \vartheta)^2}{a^2} + \frac{(\sin \vartheta)^2}{b^2}} = \mathcal{H}^p(0, \pm 1) = \frac{1}{a}.$$

Then $D_{\mathcal{E}}(\Omega) \cdot \mathcal{H}^p(v_m) = 2 \frac{b}{a} > 2$.

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**References**


