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Partial differential equations/Numerical analysis

Eigenvalue problems with sign-changing coefficients

Problèmes aux valeurs propres en présence de coefficients changeant de signe

Camille Carvalho [a], Lucas Chesnel [b], Patrick Ciarlet Jr. [c]

[a] *Applied Mathematics Unit, School of Natural Sciences, University of California, Merced, 5200 North Lake Road, Merced, CA 95343, USA*
[b] *INRIA/Centre de mathématiques appliquées, École polytechnique, Université Paris-Saclay, route de Saclay, 91128 Palaiseau cedex, France*
[c] *POEMS, UMR 7231 CNRS–INRIA–ENSTA, 828, boulevard des Maréchaux, 91762 Palaiseau cedex, France*

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ABSTRACT

We consider a class of eigenvalue problems involving coefficients changing sign on the domain of interest. We describe the main spectral properties of these problems according to the features of the coefficients. Then, under some assumptions on the mesh, we explain how one can use classical finite element methods to approximate the spectrum as well as the eigenfunctions while avoiding spurious modes. We also prove localisation results of the eigenfunctions for certain sets of coefficients.

RÉSUMÉ

Nous étudions, d'un point de vue théorique et numérique, des problèmes aux valeurs propres mettant en jeu des coefficients dont le signe change sur le domaine d'intérêt.

Version française abrégée

Nous nous intéressons au problème aux valeurs propres (1) posé dans un domaine borné Ω partitionné en deux régions Ω_1, Ω_2. Le problème (1) met en jeu des coefficients ς, μ que nous supposons constants non nuls sur Ω_1, Ω_2. Ces constantes peuvent être de signes différents. Nous fournissons d'abord des critères assurant que le spectre de (1) est discret, propriété qui n'est pas toujours satisfaite lorsqu'à la fois ς et μ changent de signe. Lorsqu'un seul coefficient (ς ou μ) change de signe, on peut aussi montrer que le spectre de (1) est réel, constitué de deux suites de valeurs propres $(\lambda_{\pm n})_{n\geq 1}$ telles que $\lim_{n\to+\infty} \lambda_{\pm n} = \pm\infty$. Dans un second temps, moyennant certaines hypothèses sur le maillage, nous expliquons comment on peut utiliser les méthodes éléments finis classiques pour approcher le spectre ainsi que les fonctions propres. Pour ce faire, nous utilisons la théorie d'approximation des opérateurs compacts développée notamment par Babuška et Osborn [2]. Il est à noter que, lorsque ς et/ou μ change(nt) de signe, l'approximation numérique de (1) doit être réalisée avec soin pour éviter

E-mail addresses: ccarvalho3@ucmerced.edu (C. Carvalho), lucas.chesnel@inria.fr (L. Chesnel), patrick.ciarlet@ensta-parsitech.fr (P. Ciarlet).

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la pollution spectrale [11]. Nous établissons ensuite un résultat de localisation des fonctions propres $u_{\pm n}$ associées aux $\lambda_{\pm n}$ dans le cas où un seul coefficient (ς ou μ) change de signe. Plus précisément, nous prouvons que les $u_{\pm n}$ deviennent confinées ou bien dans Ω_1, ou bien dans Ω_2, lorsque $n \to +\infty$. Enfin, nous présentons des tests numériques illustrant ce confinement, ainsi que la convergence des valeurs propres.

1. Introduction

Consider a bounded set $\Omega \subset \mathbb{R}^d$, $d \geq 1$, partitioned into two subsets Ω_1, Ω_2 such that $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$ and $\Omega_1 \cap \Omega_2 = \emptyset$. We assume that Ω, Ω_1, Ω_2 are bounded open sets that have Lipschitz boundaries $\partial\Omega$, $\partial\Omega_1$, $\partial\Omega_2$. Introduce ς, μ two functions such that for $i = 1, 2$, $\varsigma|_{\Omega_i} = \varsigma_i$, $\mu|_{\Omega_i} = \mu_i$, where $\varsigma_i \neq 0$, $\mu_i \neq 0$ are some real constants. The goal of this work is to study the eigenvalue problem

$$\text{Find } (u, \lambda) \in H^1_0(\Omega) \setminus \{0\} \times \mathbb{C} \text{ such that } -\operatorname{div}(\varsigma \nabla u) = \lambda \mu u \quad \text{in } \Omega \tag{1}$$

as well as its Finite Element (FE) approximation. In the present note, we are particularly interested in situations where ς and/or μ change sign over Ω. Such problems appear for instance while considering time-harmonic Maxwell's equations in structures involving negative materials ($\varsigma = \varepsilon^{-1} < 0$ for metals at optical frequencies, $\varsigma = \varepsilon^{-1} < 0$ and $\mu < 0$ for some metamaterials). Set $a(u, v) = (\varsigma \nabla u, \nabla v)$, $b(u, v) = (\mu u, v)$ and using the Riesz representation theorem, define the bounded linear operators $A, B : H^1_0(\Omega) \to H^1_0(\Omega)$ such that, for all $u, v \in H^1_0(\Omega)$, $(\nabla(Au), \nabla v) = a(u, v)$, $(\nabla(Bu), \nabla v) = b(u, v)$. In these definitions, $(\cdot, \cdot)$ stands indistinctly for the usual inner products of $L^2(\Omega)$ or $L^2(\Omega)^d$. With such a notation, (u, λ) is an eigenpair of Problem (1) if and only if

$$a(u, v) = \lambda\, b(u, v) \quad \forall v \in H^1_0(\Omega) \qquad \Leftrightarrow \qquad Au = \lambda Bu. \tag{2}$$

When ς changes sign, properties of the operator A have been extensively studied, in particular using the T-coercivity approach [5,3]. It has been shown that A is a Fredholm operator if and only if the contrast $\kappa_\varsigma := \varsigma_2/\varsigma_1$ lies outside a closed interval I of $(-\infty; 0)$. For $d = 1$, we have $I = \emptyset$, whereas for $d \geq 2$, I always contains the value -1, and its definition depends on the properties of the interface $\Sigma := \partial\Omega_1 \cap \partial\Omega_2$. Note that when $\Sigma \cap \partial\Omega = \emptyset$, it holds $I = \{-1\}$ if and only if Σ is smooth. In the present work, we shall systematically assume that $\kappa_\varsigma \notin I$ so that A is a Fredholm operator. In addition, we shall assume that A is injective. Under these two assumptions, A is an isomorphism and (u, λ) is an eigenpair of Problem (1) if and only if $Ku = \lambda^{-1} u$ with $K = A^{-1}B$. Since B is compact without any assumption on the sign of μ, K is compact. As a consequence, the spectrum of Problem (1) is discrete and made of isolated eigenvalues.

In the particular case where $d = 1$ and $\mu = \varsigma$, an explicit calculation can be done. If $\Omega_1 = (a; 0)$ and $\Omega_2 = (0; b)$, with $a < 0$, $b > 0$, we find that $\lambda \in \mathbb{C} \setminus \{0\}$ is an eigenvalue of Problem (1) if and only if $(\varsigma_1 + \varsigma_2)\sin(\lambda(b - a)) = (\varsigma_2 - \varsigma_1)\sin(\lambda(b + a))$. Moreover, 0 is an eigenvalue if and only $\varsigma_2/b - \varsigma_1/a = 0$. When $\varsigma_2 = -\varsigma_1 \Leftrightarrow \kappa_\varsigma = -1$ and $b = -a$, we see that the spectrum of Problem (1) covers the whole complex plane despite the fact that A is Fredholm. However, this situation shall be discarded in the following analysis because we impose that A is injective (which is not the case in the above particular setting). With these explicit calculations, one can also verify that a complex spectrum can exist when both ς and μ are (real) sign changing.

In any dimension $d \geq 1$ and when only one coefficient (ς or μ) changes sign, one finds that the spectrum of Problem (1) is real by taking the imaginary part of $a(u, u) = \lambda\, b(u, u)$. Moreover, constructing functions $u_n \in H^1_0(\Omega)$ such that $a(u_n, u_n)/b(u_n, u_n) \to \pm\infty$ as $n \to +\infty$, one proves that it coincides with two sequences $(\lambda_{\pm n})_{n \geqslant 1}$ such that:

$$\cdots \leqslant \lambda_{-n} \leqslant \cdots \leqslant \lambda_{-1} < 0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_n \leqslant \ldots \quad \text{and} \quad \lim_{n \to +\infty} \lambda_{\pm n} = \pm\infty. \tag{3}$$

The outline is as follows. In Section 2, we explain how to approximate numerically the spectrum of Problem (1) using classical FE methods. We prove FE convergence under some conditions on the mesh. Note that the numerical approximation has to be handled carefully to avoid spectral pollution [11]. Then, we show some localization results for the eigenfunctions in the case where only one coefficient changes sign. Finally, in Section 4, we present numerical illustrations.

2. Numerical approximation

In this section, we focus on the numerical analysis of the eigenvalue problem (1). We follow the classical approach developed for instance in [2] by Babuška and Osborn. Let $(V^h)_h$ be a sequence of finite-dimensional subspaces of $H^1_0(\Omega)$ indexed by a parameter $h \in (0; h_0]$ tending to zero. We assume that V^h approximates $H^1_0(\Omega)$ as h tends to zero in the sense that for all $u \in H^1_0(\Omega)$, $\lim_{h \to 0} \inf_{u^h \in V^h} \|u - u^h\|_{H^1_0(\Omega)} = 0$. In practice, V^h will be a space of globally continuous, piecewise polynomial functions defined on a mesh of Ω of characteristic size h. Consider the family of discrete eigenvalue problems

$$\text{Find } (u^h, \lambda^h) \in V^h \setminus \{0\} \times \mathbb{C} \text{ such that } a(u^h, v^h) = \lambda^h\, b(u^h, v^h) \quad \forall v^h \in V^h. \tag{4}$$

Next we introduce some definitions. For λ an eigenvalue of Problem (1), we call *ascent* of λ the smallest integer α such that $\ker(\lambda^{-1} - K)^\alpha = \ker(\lambda^{-1} - K)^{\alpha+1}$ and *algebraic multiplicity* of λ the number $m := \dim \ker(\lambda^{-1} - K)^\alpha \geq 1$. The elements

of $\ker(\lambda^{-1} - K)^{\alpha}$ are called the *generalized eigenvectors* of (1) associated with λ. In the following, we will assume that the form $a(\cdot,\cdot)$ satisfies a uniform discrete *inf–sup* condition, *i.e.* we will assume that

$$\exists \beta > 0,\ \forall h \in (0; h_0] \qquad \inf_{u^h \in V^h,\ \|u^h\|_{H_0^1(\Omega)}=1} \ \sup_{v^h \in V^h,\ \|v^h\|_{H_0^1(\Omega)}=1} |a(u^h, v^h)| \geq \beta. \tag{$\mathscr{P}$}$$

Admittedly, when ς changes sign, it is not clear whether or not ($\mathscr{P}$) holds. We address this issue after Proposition 2.1 below. When ($\mathscr{P}$) is true, we can define the operator $K^h : H_0^1(\Omega) \to V^h$ such that $a(K^h u, v^h) = b(u, v^h)$ for all $u \in H_0^1(\Omega)$, $v^h \in V^h$. Note that $K^h = P^h K$ where P^h denotes the projection of $H_0^1(\Omega)$ onto V^h such that $a(P^h u, v^h) = a(u, v^h)$ for all $u \in H_0^1(\Omega)$, $v^h \in V^h$. Using the fact that V^h approximates $H_0^1(\Omega)$ as $h \to 0$, we can prove that K^h converges in norm to K. We deduce that m eigenvalues $\lambda_1^h, \ldots, \lambda_m^h$ converge to λ. The eigenvalues λ_j^h are counted according to the algebraic multiplicities of the $(\lambda_j^h)^{-1}$ as eigenvalues of K^h. Let

$$\begin{aligned} M(\lambda) &= \{u \,|\, u \text{ is a generalized eigenvector of (1) associated with } \lambda\} \\ M^h(\lambda) &= \{u \,|\, u \text{ in the direct sum of the generalized eigenspaces of (4) corresponding} \\ &\qquad \text{to the eigenvalues } \lambda_j^h \text{ that converge to } \lambda\}. \end{aligned}$$

Denote $\delta(M(\lambda), M^h(\lambda)) = \sup_{u \in M(\lambda),\ \|u\|_{H_0^1(\Omega)}=1} \operatorname{dist}(u, M^h(\lambda))$. Finally, define the quantity

$$\varepsilon^h = \sup_{u \in M(\lambda),\ \|u\|_{H_0^1(\Omega)}=1} \ \inf_{v^h \in V^h} \|u - v^h\|_{H_0^1(\Omega)}.$$

The results of [2, Chap. II, Sect. 8] are as follows.

Proposition 2.1. *Assume that $\kappa_\varsigma \notin I$, that A is injective and that property ($\mathscr{P}$) holds. Then, there is a constant $C > 0$ such that for all $h \in (0; h_0]$*

$$\delta(M(\lambda), M^h(\lambda)) \leq C\,\varepsilon^h, \qquad \left|\lambda - \left(\frac{1}{m}\sum_{j=1}^{m} (\lambda_j^h)^{-1}\right)^{-1}\right| \leqslant C\,(\varepsilon^h)^2, \qquad |\lambda - \lambda_j^h| \leq C\,(\varepsilon^h)^{2/\alpha}, \qquad \text{for } 1 \leq j \leq m.$$

From now on, we assume that $(V^h)_h$ are spaces of globally continuous, piecewise polynomial functions of degree at most $\ell \geq 1$, that are defined on a shape-regular family $(\mathcal{T}^h)_h$ of geometrically conformal meshes of $\overline{\Omega}$. In addition, every mesh $\mathcal{T}^h$ is such that if $\tau \in \mathcal{T}^h$, then $\tau \in \overline{\Omega_1}$ or $\tau \in \overline{\Omega_2}$ (if $\mathcal{T}^h$ is a triangulation, no triangle crosses the interface Σ). The question of the verification of ($\mathscr{P}$) has been addressed in [10,3,7,4] where the source term problem associated with Problem (1) has been considered. More precisely, in these works the authors provide sufficient conditions on the mesh of Ω so that ($\mathscr{P}$) holds. Let us try, in short, to give an idea of the results. In [10,3], when ς changes sign and when $\kappa_\varsigma \notin I$, it is shown how to construct bounded linear operators $\mathrm{T} : H_0^1(\Omega) \to H_0^1(\Omega)$, based on geometrical transformations, such that $|a(u, \mathrm{T}u)| \geq \beta \|u\|^2_{H_0^1(\Omega)}$ for all $u \in H_0^1(\Omega)$ where $\beta > 0$. In [7], it is established that this is equivalent to prove that $a(\cdot,\cdot)$ satisfies an *inf–sup condition*. And if the mesh is such that $\mathrm{T}(V^h) \subset V^h$, then we infer that $a(\cdot,\cdot)$ satisfies property ($\mathscr{P}$). Practically, this boils down to assume that the mesh of Ω verifies some geometrical conditions of symmetry with respect to the interface Σ. The general version is presented in [4]. In the latter article, the authors show that for ($\mathscr{P}$) to hold, it is sufficient that the meshes satisfy some symmetry properties in a neighbourhood of Σ.

In order to complement the result of Proposition 2.1, it remains to assess ε^h. As usual in FE methods, the dependence of ε^h with respect to h varies according to the regularity of u. The latter question has been investigated in [6]. To set ideas, we shall assume that Ω_1, Ω_2 are polygons. Define $PH^t(\Omega) := \{v \in H^1(\Omega) \,|\, v_i \in H^t(\Omega_i),\ i = 1, 2\}$. In [6], it is proved that when $\kappa_\varsigma \notin I$, there is some $s > 0$, which can be computed, such that $\{v \in H_0^1(\Omega) \,|\, \operatorname{div}(\varsigma \nabla v) \in L^2(\Omega)\} \subset PH^{1+s}(\Omega)$ (algebraically and topologically). The exponent $s > 0$ depends both on the geometrical setting and on the value of κ_ς. Moreover, it can be arbitrarily close to zero. For a FE approximation of degree ℓ, one has, for h small enough $\varepsilon^h \leq C\,h^{\min(s,\ell)}$ where $C > 0$ is independent of h.

Let us make some comments regarding this analysis. First, a direct application of the Aubin–Nitsche lemma allows one to derive some results of approximation of the eigenfunctions in the $L^2(\Omega)$-norm. Using isoparametric quadrilateral FEs, one can deal with curved interfaces. Recently, in [1], an alternative approach, based on an optimisation method, has been proposed to consider the source term problem associated with (1). It has the advantage of requiring no geometrical assumption on the mesh. It would be interesting to investigate if it can be employed to deal with the eigenvalue problem (1).

3. Localisation results for eigenvalue problems with one sign-changing coefficient

When only one coefficient (ς or μ) changes sign, we said above that the spectrum of Problem (1) is real and coincides with two sequences $(\lambda_{\pm n})_{n\geqslant 1}$ that fulfil (3). Here we prove that the normalized eigenfunctions $u_{\pm n}$ ($\|u_{\pm n}\|_{H^1_0(\Omega)}=1$) associated with $\lambda_{\pm n}$ tend to be localized in one of the subdomains Ω_1, Ω_2 as $n\to+\infty$ (see Fig. 1 (right) for numerical illustrations).

Let us first consider the case where ς changes sign and $\mu\equiv 1$ in Ω, that is we look at the problem $-\operatorname{div}(\varsigma\nabla u)=\lambda u$. To fix ideas, we assume that $\varsigma_1>0$ and $\varsigma_2<0$. Denote d_Σ the distance to the interface such that for $x\in\Omega$, $d_\Sigma(x)=\inf_{z\in\Sigma}|x-z|$. Define the weight functions $\chi_n^\pm$ such that, for $\alpha>0$,

$$\chi_n^+(x)=\begin{cases}1 \text{ in } \Omega_1\\ e^{\alpha d_\Sigma(x)\sqrt{\lambda_n/|\varsigma_2|}} \text{ in } \Omega_2\end{cases}\qquad\text{and}\qquad \chi_n^-(x)=\begin{cases}e^{\alpha d_\Sigma(x)\sqrt{|\lambda_{-n}|/\varsigma_1}} \text{ in } \Omega_1\\ 1 \text{ in } \Omega_2.\end{cases} \tag{5}$$

Observe that χ_n^+ (resp. χ_n^-) blows up in Ω_2 (resp. Ω_1) as $n\to+\infty$.

Proposition 3.1. *For all $\alpha\in(0;1)$, $n\in\mathbb{N}^*$, we have*

$$\varsigma_1\|\nabla u_n\|^2_{L^2(\Omega_1)}\geq\lambda_n\|u_n\|^2_{L^2(\Omega_1)}+(1-\alpha)\left(\lambda_n\|\chi_n^+u_n\|^2_{L^2(\Omega_2)}+|\varsigma_2|\,\|\chi_n^+\nabla u_n\|^2_{L^2(\Omega_2)}\right) \tag{6}$$

$$|\varsigma_2|\,\|\nabla u_{-n}\|^2_{L^2(\Omega_2)}\geq|\lambda_{-n}|\,\|u_{-n}\|^2_{L^2(\Omega_2)}+(1-\alpha)\left(|\lambda_{-n}|\,\|\chi_n^-u_{-n}\|^2_{L^2(\Omega_1)}+\varsigma_1\,\|\chi_n^-\nabla u_{-n}\|^2_{L^2(\Omega_1)}\right). \tag{7}$$

Corollary 3.1. *Let $\omega_1\subset\Omega_1$, $\omega_2\subset\Omega_2$ be two non-empty sets such that $\overline{\omega_1}\cap\Sigma=\overline{\omega_2}\cap\Sigma=\emptyset$. For $\beta_1\in(0;\operatorname{dist}(\omega_1,\Sigma))$, $\beta_2\in(0;\operatorname{dist}(\omega_2,\Sigma))$, there are some constants $C_1,C_2>0$ such that for all $n\in\mathbb{N}^*$*

$$\|u_n\|_{H^1(\omega_2)}\leq C_2\,e^{-\beta_2\sqrt{\lambda_n/|\varsigma_2|}}\qquad\text{and}\qquad\|u_{-n}\|_{H^1(\omega_1)}\leq C_1\,e^{-\beta_1\sqrt{|\lambda_{-n}|/\varsigma_1}}.$$

Thus, the eigenfunctions associated with positive eigenvalues tend to be confined in the positive material Ω_1 whereas the ones associated with negative eigenvalues become confined in the negative material Ω_2.

Proof. In the following, we prove Estimate (6) for u_n. One proceeds similarly with u_{-n} to get (7). For all $v\in H^1_0(\Omega)$, we have $(\varsigma\nabla u_n,\nabla v)=\lambda_n(u_n,v)$. Take $v=(\chi_n^+)^2u_n$. One can show that such a v is indeed an element of $H^1_0(\Omega)$ because χ_n^+ belongs to $W^{1,\infty}(\Omega)$ (since Σ is a Lipschitz manifold, see [8, Chap. 6, Thm. 3.3, (vii)]). Expanding the term $\nabla((\chi_n^+)^2u_n)$ and using that $\nabla\chi_n^+=0$ in Ω_1, we obtain

$$\varsigma_1\|\nabla u_n\|^2_{L^2(\Omega_1)}=\lambda_n\|\chi_n^+u_n\|^2_{L^2(\Omega)}+|\varsigma_2|\,\|\chi_n^+\nabla u_n\|^2_{L^2(\Omega_2)}+2|\varsigma_2|\int_{\Omega_2}\chi_n^+\,u_n\,\nabla\chi_n^+\cdot\nabla u_n\,\mathrm{d}x. \tag{8}$$

Noticing that $\nabla\chi_n^+=\alpha\sqrt{\lambda_n/|\varsigma_2|}\,\chi_n^+\nabla d_\Sigma$ and $|\nabla d_\Sigma|=1$ a.e. in Ω_2, we can write

$$\left|2|\varsigma_2|\int_{\Omega_2}\chi_n^+\,u_n\,\nabla\chi_n^+\cdot\nabla u_n\,\mathrm{d}x\right|\leq\alpha\lambda_n\|\chi_n^+u_n\|^2_{L^2(\Omega_2)}+\alpha|\varsigma_2|\,\|\chi_n^+\nabla u_n\|^2_{L^2(\Omega_2)}. \tag{9}$$

Plugging (9) in (8) leads to (6). Then the first estimate of Corollary 3.1 is a direct consequence of (6) (with $\alpha=\beta_2/\operatorname{dist}(\omega_2,\Sigma)$) because $\chi_n^+(x)\geq e^{\alpha\operatorname{dist}(\omega_2,\Sigma)\sqrt{\lambda_n/|\varsigma_2|}}$ in ω_2. □

Remark 1. A result similar to the one of Proposition 3.1 can be obtained assuming only that $\varsigma\in L^\infty(\Omega)$ is such that $\varsigma\geq C>0$ a.e. in Ω_1 and $\varsigma\leq -C<0$ a.e. in Ω_2 for some $C>0$.

Now, we just state the results in the case where $\varsigma\equiv 1$ in Ω and μ changes sign ($-\Delta u=\lambda\mu u$). To set ideas, we assume that $\mu_1>0$ and $\mu_2<0$. Define the weight functions $\zeta_n^\pm$ such that, for $\alpha>0$,

$$\zeta_n^+(x)=\begin{cases}1 \text{ in } \Omega_1\\ e^{\alpha d_\Sigma(x)\sqrt{\lambda_n|\mu_2|}} \text{ in } \Omega_2\end{cases}\qquad\text{and}\qquad \zeta_n^-(x)=\begin{cases}e^{\alpha d_\Sigma(x)\sqrt{|\lambda_{-n}|\,\mu_1}} \text{ in } \Omega_1\\ 1 \text{ in } \Omega_2.\end{cases}$$

Proposition 3.2. *For all $\alpha\in(0;1)$, $n\in\mathbb{N}^*$, we have*

$$\lambda_n\,\mu_1\,\|u_n\|^2_{L^2(\Omega_1)}\geq\|\nabla u_n\|^2_{L^2(\Omega_1)}+(1-\alpha)\left(\lambda_n|\mu_2|\,\|\zeta_n^+u_n\|^2_{L^2(\Omega_2)}+\|\zeta_n^+\nabla u_n\|^2_{L^2(\Omega_2)}\right)$$

$$|\lambda_{-n}|\,|\mu_2|\,\|u_{-n}\|^2_{L^2(\Omega_2)}\geq\|\nabla u_{-n}\|^2_{L^2(\Omega_2)}+(1-\alpha)\left(|\lambda_{-n}|\,\mu_1\,\|\zeta_n^-u_{-n}\|^2_{L^2(\Omega_1)}+\|\zeta_n^-\nabla u_{-n}\|^2_{L^2(\Omega_1)}\right).$$

Corollary 3.2. *Let $\omega_1 \subset \Omega_1$, $\omega_2 \subset \Omega_2$ be two non-empty sets such that $\overline{\omega_1} \cap \Sigma = \overline{\omega_2} \cap \Sigma = \emptyset$. For $\beta_1 \in (0; \mathrm{dist}(\omega_1, \Sigma))$, $\beta_2 \in (0; \mathrm{dist}(\omega_2, \Sigma))$, there are some constants $C_1, C_2 > 0$ such that for all $n \in \mathbb{N}^*$*

$$\|u_n\|_{H^1(\omega_2)} \leq C_2\, e^{-\beta_2 \sqrt{\lambda_n\, |\mu_2|}} \qquad \text{and} \qquad \|u_{-n}\|_{H^1(\omega_1)} \leq C_1\, e^{-\beta_1 \sqrt{|\lambda_{-n}|\, \mu_1}}.$$

Remark 2. In [9], the authors study the asymptotic behaviour of $(\lambda_{\pm n})$ as $n \to +\infty$ (Weyl formulas) in the case $\varsigma \equiv 1$ and μ changes sign ($-\Delta u = \lambda \mu u$). This question in the situation when $\mu \equiv 1$ and ς changes sign ($-\mathrm{div}(\varsigma \nabla u) = \lambda u$) remains open.

Remark 3. In the case $\varsigma \equiv 1$ and μ changes sign ($-\Delta u = \lambda \mu u$), using min − max formulas, one can show that for $n \geq 1$, $\lambda_n \leq \lambda_n^h$ (as for the usual Dirichlet Laplacian operator) and $\lambda_{-n} \geq \lambda_{-n}^h$. Here $\lambda_{\pm n}^h$ refers to an eigenvalue of (4) which converges to $\lambda_{\pm n}$.

4. Numerical illustrations

Let us illustrate these results on a simple example. We take $\Omega = (-1; 1) \times (-1; 1)$, $\Omega_2 = (0; 1) \times (0; 1)$ and $\Omega_1 = \Omega \setminus \overline{\Omega_2}$. In this geometry, $A : H_0^1(\Omega) \to H_0^1(\Omega)$ is an isomorphism if and only $\kappa_\varsigma = \varsigma_2/\varsigma_1 \notin I = [-3; -1/3]$ (see [3]). Set $\varsigma_1 = 1$, $\varsigma_2 = -4$ and $\mu \equiv 1$. We use meshes with symmetries like the one in Fig. 1 (left) for which we know that the uniform discrete inf–sup condition ($\mathscr{P}$) above holds. In Fig. 1 (centre), we display the errors $|\lambda_1^h - \lambda_1^{\mathrm{ref}}|/|\lambda_1^{\mathrm{ref}}|$ versus the mesh size for three FE orders ($\ell = 1, 2, 3$). Here λ_1^{ref} refers to the approximation λ_1^h of λ_1 obtained with a very refined mesh and $\ell = 4$. From [6] and Section 2, one can expect a convergence order equal to $2 \min(s, \ell)$ where, for this interface with a right angle, $s = \min(\eta, 2 - \eta) \leq 1$, $\eta = 2 \arccos((\varsigma_1 - \varsigma_2)/(2(\varsigma_1 + \varsigma_2)))/\pi$. For $\kappa_\varsigma = -4$, we find $s \approx 0.37$ so that for $\ell = 1, 2, 3$, we have $2 \min(s, \ell) \approx 0.74$. In Fig. 1 (right), we display the eigenfunctions associated with $\lambda_{\pm 1}^h, \ldots, \lambda_{\pm 4}^h$. In accordance with the result of Proposition 3.1, we observe that the eigenfunctions associated with λ_n (resp. λ_{-n}) tend to be more and more localized in Ω_1 (resp. Ω_2) as n grows.

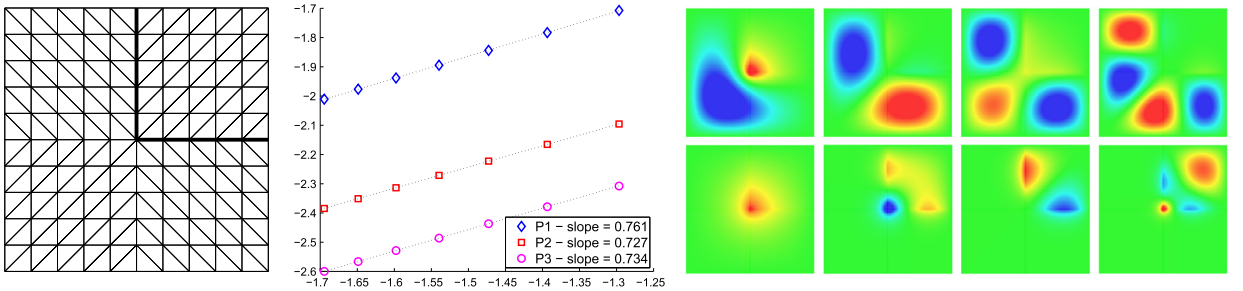

Fig. 1. Left: coarse symmetric mesh. Centre: relative error vs. h (log-log scale) for $\lambda_1 \approx 13.2391$. Right: eigenfunctions associated with $\lambda_1^h, \ldots, \lambda_4^h$ (top) and $\lambda_{-1}^h, \ldots, \lambda_{-4}^h$ (bottom).

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